The proof of Dirichlet's assertion on celestial mechanics

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Abstract
Since the proof given by Wang Qiu-Dong on Dirichlet's assertion is based on the successive approximations objections to assumptions of the global solution of this problem have to be raised.

1 Introduction

The manuscript "The global solution of the n-body problem" by Wang Qiu-Dong from 1991 attracted attention only as recent as in 2013.
The paper contains the claim that the equations of the n-body problem of celestial mechanics can be integrated. The system takes the form

$$\begin{aligned} \frac{d^2}{dt^2}\left(m_k \mathbf{r}_k\right) = \mathbf{K}_k = \sum_{l=1}^{n} \mathbf{K}_{kl}, \quad \mathbf{K}_{kk} = 0, \quad k = 1, \ldots, n \\ \mathbf{K}_{kl} = -\mathbf{K}_{lk} = -\gamma \frac{m_k m_k}{r_{kl}^3} \mathbf{r}_{kl}, \quad \mathbf{r}_{kl} = \mathbf{r}_k - \mathbf{r}_l, \quad r_{kl} = |\mathbf{r}_{kl}| \end{aligned} \tag{1}$$

where γ is the gravitational constant and $k = 1$ stands for the sun and $k = 2, \ldots, n$ stands for the planets or moons. The paper was praised in [2] and [3].
But nevertheless it contains a contradiction since it is wrongely assumed that n equations of the system (1) are linearly independent. This assumption is needed for mathematical reasons to reduce (1) to $n - 1$ changes of the positions of the bodies relative to the sun

$$\begin{aligned} \frac{d^2}{dt^2}(\mathbf{r}_{k1}) = \mathbf{F}_{k1} + \frac{1}{M} \sum_{l=2}^{n} m_l (\mathbf{F}_{l1} - \mathbf{F}_{k1} - \mathbf{F}_{kl}), \quad k = 2, \ldots, n \\ \mathbf{F}_{kl} = \frac{M}{m_k m_l} \mathbf{K}_{kl}, \quad M = \sum_{k=1}^{n} m_k, \quad m_l \ll M, \quad l > 1 \end{aligned} \tag{2}$$

This reduction [4] was already presented at the GAMM meeting 1986 in Dortmund, but it has not attracted any attention.
With regard to the reduction from (1) to (2), we shall first look at the situation for the two-body problem.

2 The two-body problem

For $n = 2$ system (1) reduces to two equations

$$\frac{d^2}{dt^2}(m_1\mathbf{r}_1) = \mathbf{K}_1, \quad \frac{d^2}{dt^2}(m_2\mathbf{r}_2) = \mathbf{K}_2 = -\mathbf{K}_1 \tag{3}$$

Adding both yields

$$\frac{d^2}{dt^2}(M\mathbf{r}_M) = \mathbf{K}_1 + \mathbf{K}_2 = \mathbf{K}_1 - \mathbf{K}_1 = 0, \quad \mathbf{r}_M = \frac{1}{M}\sum_{l=1}^{n=2} m_l\mathbf{r}_l \tag{4}$$

From difference between the second and the first we obtain the determining equation for the two-body problem

$$\frac{d^2}{dt^2}(\mathbf{r}_{21}) = \frac{M}{m_1 m_2}\mathbf{K}_2 = \mathbf{F}_{21} = -\frac{\Gamma}{r_{21}^3}\mathbf{r}_{21}, \quad \Gamma = \gamma M \tag{5}$$

It can be integrated in closed form. From (3) and (5) follow the identities

$$\left[\frac{d^2}{dt^2}(m_2\mathbf{r}_2) = \mathbf{K}_2\right] = \frac{d^2}{dt^2}\left(\frac{m_1 m_2}{M}\mathbf{r}_{21}\right) = \mathbf{K}_2 = -\left[\mathbf{K}_1 = \frac{d^2}{dt^2}(m_1\mathbf{r}_1)\right] \tag{6}$$

This shows that the two equations (3) are linearly dependent.
We take now from (6) the equations

$$m_2\mathbf{r}_2 = \frac{m_1 m_2}{M}\mathbf{r}_{21}, \quad m_1\mathbf{r}_1 = \frac{m_1 m_2}{M}\mathbf{r}_{12} \quad \Rightarrow \quad M\mathbf{r}_M = \frac{m_1 m_2}{M}(\mathbf{r}_{21} + \mathbf{r}_{12}) = 0 \tag{7}$$

which satisfy (4) in the identities

$$\frac{d^2}{dt^2}(M\mathbf{r}_M) = \frac{d^2}{dt^2}(0) = 0 = \sum_k \mathbf{K}_k. \tag{8}$$

This shows that there is no contradiction in (8).
Rather than interpreting (4) in the identity (8), one often misunderstands (4) as postulate

$$\frac{d^2}{dt^2}(m_M\mathbf{r}_M) \overset{!}{=} 0 \quad \Rightarrow \quad \mathbf{r}_M = \mathbf{v}_{M;0}t + \mathbf{r}_{M;0} \neq 0 \tag{9}$$

Thus one might assume that the coordinates $\mathbf{r}_k$ are linearly independent. But this is wrong because it contradicts the derivation of (8) from (4) with $\mathbf{r}_M = 0 \neq 0$. For $n > 2$ the representation of (7) in the form

$$\mathbf{r}_k = \mathbf{r}_k - \mathbf{r}_M = \frac{m_l}{M}\mathbf{r}_{kl}, \quad k \neq l = 1, 2 \tag{10}$$

will be important.

3 Extension to $n > 2$

Naturally, the conditions extended to $n > 2$ must contain for the first indices those of $n = 2$.
The conditions for $k = 1, 2$ from (10) are contained within the general form

$$\mathbf{r}_k = \mathbf{r}_k - \mathbf{r}_M = \frac{1}{M} \sum_{l=1}^{n} m_l \mathbf{r}_{kl} \tag{11}$$

The sum of the equations (1) is then

$$\frac{d^2}{dt^2} \left(\sum_{k=1}^{n} m_k \mathbf{r}_k \right) = \sum_{k=1}^{n} \mathbf{K}_k \tag{12}$$

and (8) satisfies the identities

$$\frac{d^2}{dt^2} \left(\frac{1}{2} \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{m_k m_l}{M} (\mathbf{r}_{kl} + \mathbf{r}_{lk}) \right) = 0 = \frac{1}{2} \sum_{k=1}^{n} \sum_{l=1}^{n} \left(-\gamma \frac{m_k m_l}{r_{kl}^3} \right) (\mathbf{r}_{kl} + \mathbf{r}_{lk}) \tag{13}$$

as the double sums of $\mathbf{r}_{kl} + \mathbf{r}_{lk} = 0$ vanish.

4 Reduction of the original system (1)

By taking differences

$$\mathbf{r}_{kl} = \mathbf{r}_{k1} - \mathbf{r}_{l1} \tag{14}$$

one obtains from (11) equations

$$\begin{aligned} \mathbf{r}_1 &= -\frac{1}{M} \sum_{l=2}^{n} m_l \mathbf{r}_{l1} \quad , \\ \mathbf{r}_k &= \mathbf{r}_{k1} - \frac{1}{M} \sum_{l=2}^{n} m_l \mathbf{r}_{l1}, \quad k = 2, \ldots, n \end{aligned} \tag{15}$$

to derive $\mathbf{r}_k$ from the relative coordinates $\mathbf{r}_{k1}$.
Using those, the original system (1) takes the form

$$\frac{d^2}{dt^2} \left[\sum_{l=2}^{n} m_l \mathbf{r}_{l1} \right] = \sum_{l=2}^{n} m_l \mathbf{F}_{l1}, \quad k = 1 \tag{16a}$$

$$\frac{d^2}{dt^2} \left[M \mathbf{r}_{k1} - \sum_{l=2}^{n} m_l \mathbf{r}_{l1} \right] = M \mathbf{F}_{k1} + \sum_{l=2}^{n} m_l (-\mathbf{F}_{k1} + \mathbf{F}_{kl}), \quad k = 2, \ldots, n \tag{16b}$$

The compatibility of the equations (16) is seen as follows: Equation (16b) can be written in the form

$$\sum_{l=1}^{n} \frac{d^2}{dt^2} (m_l \mathbf{r}_{kl}) = \sum_{l=1}^{n} m_l \mathbf{F}_{kl}, k = 2, \ldots, n. \tag{17}$$

Multiplying with m_k/M and summing over k yields the result

$$\frac{d^2}{dt^2}(M\mathbf{r}_M) - \frac{m_1}{M}\sum_{l=2}^{n}\frac{d^2}{dt^2}(m_l\mathbf{r}_{1l}) = \mathbf{K} - \frac{m_1}{M}\sum_{l=2}^{n} m_l\mathbf{F}_{1l}. \tag{18}$$

With (13) and $d^2/dt^2(M\mathbf{r}_M) = 0 = \mathbf{K}$ this is identical to (16a). This shows the compatibility and the linear independence of the equations (16). This requires the reduction of the n equations (1) to the $n-1$ equations (2), because if one adds (16a) to (16b) one obtains the $n-1$ equations (2) to determine the $(n-1)$ coordinates $\mathbf{r}_{k1}$.
This is the extension of the difference (5) as (5) is for $n = 2$ contained within (2) as required.
The mathematical impossibility in [1] is thus to assume that the linearly dependent equations (1) are linearly independent.

5 Remarks about the reduced system

The term

$$(\mathbf{F}_{l1} - \mathbf{F}_{k1} + \mathbf{F}_{kl})|_{l=k} = 0 \tag{19}$$

vanishes at the right hand side of Equation (2) which means that there is no action of the $l = k$ onto itself.
Note that the relative coordinates $\mathbf{r}_{kl}$ are invariant under the transformation

$$\mathbf{r}_k = \mathbf{r}'_k + \mathbf{a} \rightarrow \mathbf{r}_{kl} = \mathbf{r}_k - \mathbf{r}_l = \mathbf{r}'_k - \mathbf{r}'_l \tag{20}$$

Consequently, on can prescribe the initial conditions for (2) for $\mathbf{r}_{k1;0}$, $\mathbf{v}_{k1;0}$ from differences of geocentric measurements $\mathbf{r}'_{k;0}$, $\mathbf{v}'_{k;0}$.

6 Contradiction and uniqueness

Adding the equations (1) for $k = 2, \ldots, n$ yields

$$\frac{d^2}{dt^2}(M\mathbf{r}_M) - \frac{d^2}{dt^2}(m_1\mathbf{r}_1) = \mathbf{K} - \mathbf{K}_1, \tag{21}$$

which is only under the condition

$$\frac{d^2}{dt^2}(M\mathbf{r}_M) = \mathbf{K} \neq 0 \tag{22}$$

different from $d^2(m_1\mathbf{r}_1)/dt^2 = \mathbf{K}_1$ making hence (1) linearly independent. But this leads, by Newtons equation

$$\mathbf{K} = \sum_{k=1}\mathbf{K}_k = \frac{1}{2}\sum_{k=1}^{n}\sum_{l=1}^{n}(\mathbf{K}_{kl} + \mathbf{K}_{lk}) = 0 \neq 0 = \mathbf{K}. \tag{23}$$

But in [1] is assumed that the n equations (1) are linearly independent and hence the contradiction in (23) applies to [1]. If [1] were mathematically correct then there would

be no contradiction in (23). But this is impossible. Since the n equations (1) are linearly independent they are to be reduced to a system of $n-1$ equations.
This reduction is applied by subtracting the term $d^2(M\mathbf{r}_m)/dt^2$ (which is zero) on the left hand sides of the equations (1), cf. [4]. This reduces (1) to the form

$$\frac{d^2}{dt^2}\left(\sum_{l=1}^{n} m_l \mathbf{r}_{kl}\right) = \mathbf{K}_k(\mathbf{r}_{k1}, \ldots, \mathbf{r}_{kn}), \quad k = 1, \ldots, n \tag{24}$$

using the mutual relative quantities $\mathbf{r}_{kl}$. By the equation (14) and $\mathbf{r}_{kl} = \mathbf{r}_{k1} - \mathbf{r}_{l1}$ one reduces the n linearly dependent equations (24) respectively (1) to the $n-1$ equations (2) for the $n-1$ distances $\mathbf{r}_{k1}$ relative to the sun. This reduction from (1) to (2) is mathematically needed and unique.

7 Integration by successive approximation

It is crucial that the reduced system (2) can be integrated by successive approximation

$$a) \quad \frac{d^2}{dt^2}\left(\mathbf{r}_{k1}^{(1)}\right) = \mathbf{F}_{k1}(\mathbf{r}_{k1}^{(1)}) = \frac{\Gamma}{(r_{k1}^{(1)})^3}\mathbf{r}_{k1}^{(1)}, \quad k = 2, \ldots, n \tag{25}$$

$$b) \quad \frac{d^2}{dt^2}\left(\mathbf{r}_{k1}^{(s+1)}\right) = \mathbf{F}_{k1}(\mathbf{r}_{k1}^{(s)}) + \sum_{l=2}^{n} \frac{m_l}{M}\left[\mathbf{F}_{l1}(\mathbf{r}_{l1}^{(s)}) - \mathbf{F}_{k1}(\mathbf{r}_{k1}^{(s)}) + \mathbf{F}_{kl}(\mathbf{r}_{kl}^{(s)})\right], s = 1, 2, \ldots$$

since the first approximations a) are the solutions of the two-body problem which can be integrated in closed form

$$\mathbf{r}_{k1}^{(1)} = \mathbf{r}_{k1}^{(1)}(t), \quad k = 2, \ldots, n \tag{26}$$

The iterates $s > 1$ can be determined then by using only time integration.
In this context, we shall quote from the book "Vorlesungen über Himmelsmechanik" from C. L. Siegel [5]:

> *"The n-body problem is defined as the problem of describing the all solution of the equations of motion for arbitrary initial values. The problem is unsolved for $n > 2$ until today despite intensive effort of outstanding mathematicians since 200 years.*
>
> *In 1858, Dirichlet [1] claimed in a conversation with this friend Kronecker that he has found a general method to treat problems of mechanics and that this method is not based on direct integration of the differential equations of motion but successive approximation to the solution. He remarked additionally in a different conversation that he succeeded in proving the stability of the planetary system. He died shortly after* (in 1859) *without leaving any notes. Thus we don't know any details of his method."*

(Translated from [1] Lejeune Dirichlet, G.: Werke Bd. 2, S. 344 Berlin, 1897)
Dirichlet's method can only be the method demonstrated with (25) since its derivation from (1) is unique and leaves no room for other possibilities than (25). It is therefore

justified to regard the reduction from (1) to (2) with the possibility (25) as being the proof of Dirichlet's assertion.

The sole purpose of this manuscript is to demonstrate this.

By restricting to planets $k = p$, one obtains from $\mathbf{r}_{p1}^{(1)}(t)$ Kepler's three laws. The first iterates $\mathbf{r}_{p1}^{(1)}(t)$ approximate $\mathbf{r}_{p1}(t)$ within the quality of Kelper's laws. The orbits of the moons with indices q around planets with indices $k = p$ will, according to

$$\mathbf{r}_{q1} = \mathbf{r}_{p1} + \mathbf{r}_{qp} \longrightarrow \mathbf{r}_{q1}^{(1)} = \mathbf{r}_{p1}^{(1)} + \mathbf{r}_{qp}^{(1)}, \qquad r_{p1} \gg r_{qp} \tag{27}$$

be $\mathbf{r}_{qp}^{(1)}$ the approximate solution of the two-body problem (5) for planet and moon. According to this second assertion, Dirichlet must have proofed the stability of the planetary system for his method of successive approximations. His second claim shall not be doubted as shall his first claim. We refrain to also prove this for certain reasons (the age of the author). It shall be noted that Dirichlet based his proof of the stability of the planetary system presumably on the reduced system (2). To assumes that this proof is wrong cannot be said without demonstrating this for (2).

In contrast to Dirichlet's method we quote from [1] (page 87):

> *"One does not obtain a useful solution in series expansion. The reason for this is because the speed of convergence of the resulting solution is terribly slow. One has to sum, for example, an incredible number of terms, even for an approximate solution."*

Note that Lagrange's elementary solutions can be found in a particularly simple way from (2) (cf. [6], pages 28–30).

In addition to the aforementioned purely mathematical reasons something about the beginnings of classical mechanics can be said. For this, we refer to [6], [7].

References

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Associate Editor:

> We propose to follow the opinion of the referee. Even if the reduction process proposed by the author is correct, this does not affect the validity of Wang's argument. Wang works with unreduced equations, which is well-founded.

Reviewer No. 1:

> This paper contains elementary remarks on the n-body problem, which are correct, but not new.
>
> The reduction of the center of mass which the author proposes was standard in the 18th century. (See Lagrange, oeuvres VI, p. 231, Euler, Considerations sur le probleme des trois corps, p. 197).
>
> There are many iterative processes which approximate the solutions of the n-body problem. From such process (or in another way) one can form either Taylor series or some kind of Fourier series or Sundman's type series. The paper by Wang is about the last type. If Dirichlet claimed to get the stability from his series, then his iterative process should probably converge to some kind of Fourier series. This could be some kind of KAM theory. But, anyway, the planetary systems are not stable, so Dirichlet could not have a correct proof. It is difficult to guess what he had and what was his mistake.