Introducing Spin

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Abstract

Imaginary numbers were introduced into our real number system with only a vague and ambiguous definition ($i = \sqrt{-1}$). But what are imaginary numbers? They are not found on the number line, so where do they "live"? How do they interface with real numbers? Trying to find answers to these questions has led us in three very different directions:

1. One says i is a shorthand symbol for working with negative square roots.
2. Another view is that i is a constant of nature, and that imaginary numbers form a linear dimension orthogonal to the real number line.
3. A different view is that i is a constant of nature, but it is limited to one of four points on a circle.

If we want the consistency and certainty of our real number system to extend to include imaginary numbers, then we need to define precisely and unambiguously how imaginary numbers relate to real numbers. Imaginary numbers as they are currently defined are invalid - there is no mathematical bridge explicitly linking real and imaginary terms.

This paper asserts that just as real numbers appear on the real number line, imaginary numbers appear on the imaginary unit circle. It builds a foundational bridge linking these disparate number systems. Then spin is introduced based on this solid mathematical foundation.

The eloquence and symmetry of the new spin equations make it clear that the circle interpretation of imaginary numbers is correct. Another confirmation is that we can now mathematically deduce the "driving force" behind particle spin. This gives us a new level of understanding of the Schrodinger equation.

Introduction

There are two types of spin:

1. imaginary spin based on the constant i
2. real spin based on the functions: e^x, $\cos(x)$, and $\sin(x)$

These two types of spin are integrated by the equation.

$$e^{ix} = \cos(x) + i\sin(x) \tag{1}$$

Remarkably, this equation was first written three centuries ago by Euler. Presumably Euler had no knowledge of spin, however he also was famous for his work with the power series, and it turns out that the power series are closely related to spin. His equation has now evolved into a complex spin equation - that generates a real wave.

$$e^{ix} + e^{-ix} = 2\cos(x) \tag{2}$$

Our new understanding of spin allows us to derive an energy and a momentum formula from this spin equation. There is also a way to convert the momentum equation to a kinetic energy equation. When the two energy equations are combined they form the time dependent Schrodinger equation.

1 Background

1.1 Traditional View of Imaginary Numbers

The constant i (sometimes referred to as j) is currently defined solely as the square root of -1. Descartes was uncomfortable with this constant, because there was no way to conceptualize what it meant to be the square root of -1, and also because there were no practical applications (at that time) that used this constant. In his book *La Géométrie* in 1637, he disparagingly coined the term "imaginary numbers", suggesting they were not real numbers.

Ironically, the diagrams used today to represent imaginary numbers are based on Descartes' Cartesian coordinate system. Argand diagrams use the x axis to represent real numbers and the y axis to represent imaginary numbers. Since the coordinate points on these diagrams have both a real component and an imaginary component, these numbers are said to be complex. Similarly, the intersection of the real and imaginary axes are sometimes referred to as the complex plane.

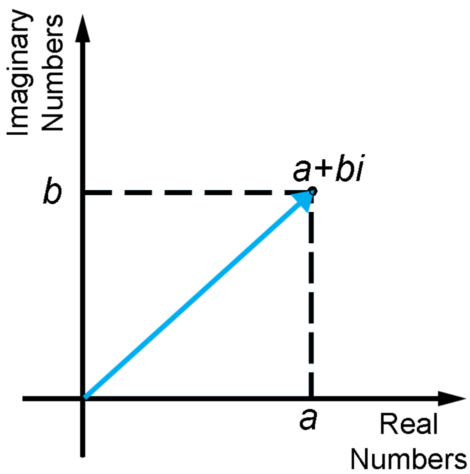


Figure 1: Argand Diagram

1.2 What Do We Really Know ?

Before imaginary numbers can enter the real number system, we need to know conceptually and mathematically how they relate. Yet the constant i is not decipherable and there are no mathematical operations or properties that allow us to relate imaginary numbers to real numbers. All we know is i squared gives -1, but we don't know how it arrived at -1, and we don't know how i relates to real numbers.

We are told that i is equal to the square root of -1. but we still need to know:
– Conceptually how do we envision i ?
– What does it mean to multiply two imaginary numbers ?
– How does i relate to real numbers ?

The current definition of i is based on arbitrary answers to those questions. However, without a foundation that relates imaginary and real numbers, complex terms such as $3i$ are mathematically meaningless - and not valid numbers.

1.3 Bridging Real and Imaginary Numbers

Conceptually i is a quarter of a turn on a circle. This competing concept is not new, but now it has evolved into the foundation for spin.

Spin equations are complex and they generate answers that contain a mixture of real and imaginary terms. When a spin equation is "paired" with its complex conjugate, the complex terms in the answer always appear with their conjugate and cancel out. This means "pairing" a complex spin equation with its conjugate yields an all real answer. This is the mathematical foundation linking imaginary and real numbers.

Note that we still have no way to relate i with a real number. So every point on the complex plane is still invalid. But we don't need to know how i directly relates to real numbers, because we know how to combine two imaginary numbers to get a real number.

2 Spin Based on Imaginary Numbers

2.1 Imaginary Unit Circle

Imaginary spin can be graphically represented on the imaginary unit circle using the expressions i^n and i^{-n}.

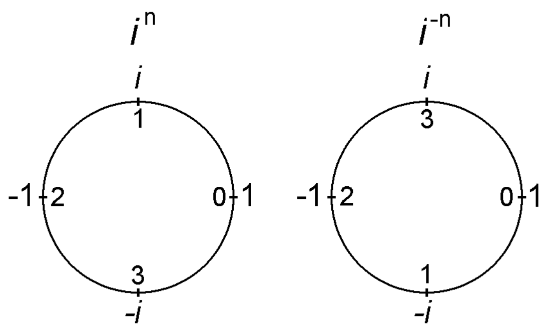


FIGURE 2: The imaginary unit circle

The expression i^n starts at the origin (with a step number of 0), and steps around the circle in a counterclockwise direction. This endless looping process is spin. Similarly, the expression i^{-n} starts at the origin, but steps around the circle in a clockwise direction.

The table below illustrates that spin loops endlessly (the values repeat every four numbers).

n	i^n	i^{-n}
0	1	1
1	i	$-i$
2	-1	-1
3	$-i$	i
4	1	1
5	i	$-i$
6	-1	-1
7	$-i$	i
8	1	1

TABLE 1: Generated spin terms

The equation i^n spins around taking multiplicative steps. For example two steps of i is equal to $i \times i$ (i^2 is equal to -1). Four steps of i is equal to $i \times i \times i \times i$. Those four steps bring you around and back to the origin with a value of 1 ($i^4 = 1$).

2.2 Complex Conjugates

The expressions i^n and i^{-n} generate both real and complex terms. It is therefore problematic to use either one of these expressions in formulas, because real world problems can not be solved with imaginary solutions (there is no way to interpret the result).

However i^n and i^{-n} are complex conjugates (the same complex equation spinning in the opposite direction). If they are "paired" together, then the real terms add together and the complex terms cancel out. We can see in the table below that no matter what value n has, $i^n + i^{-n}$ always yields a real value.

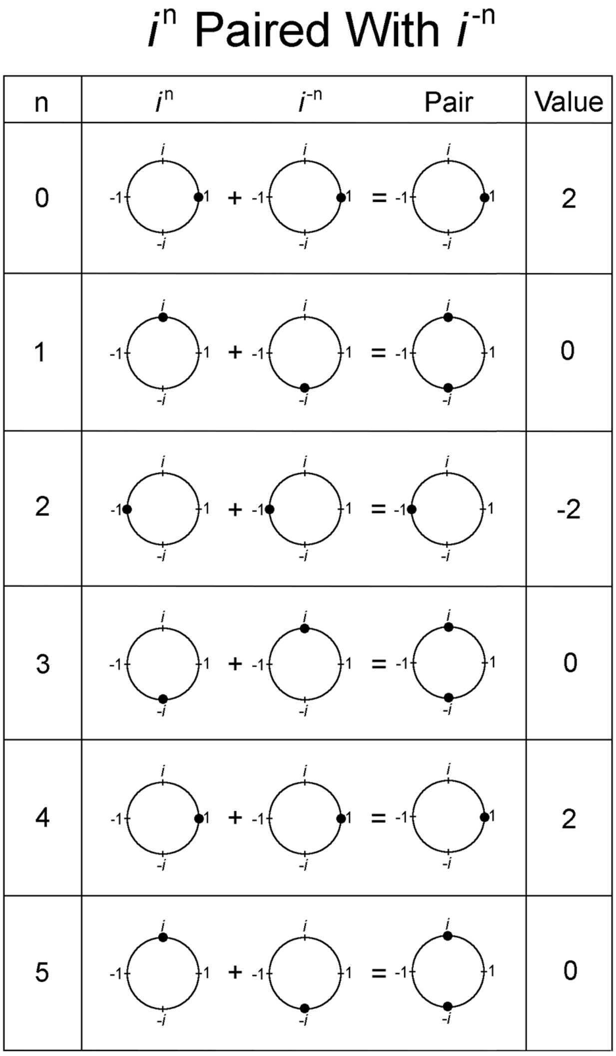
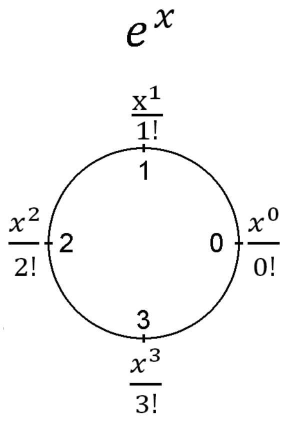


Figure 3: Complex conjugates paired together

3 Spin Based on Real Numbers

3.1 e^x Used in Spin Equations

The most fundamental real spin equation is e^x. e^x (or $\frac{x^n}{n!}$) can be graphically represented by starting at the origin ($n = 0$) and stepping counterclockwise around the circle (spinning). With each step, a new term in the answer is created (see formula 3).



n	$x^n/n!$
0	$x^0/0!$
1	$x^1/1!$
2	$x^2/2!$
3	$x^3/3!$
4	$x^4/4!$
5	$x^5/5!$

$$e^x = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots \tag{3}$$

The series generated by the spin equation e^x, is identical to the e^x power series.

3.2 Derivatives

At first glance it appears that e^x deals with exponents - x does grow exponentially. However there is also the n factorial term in the denominator that needs to be explained.

$$e^x = \frac{x^n}{n!} \tag{4}$$

The explanation involves derivatives. The derivative of any term in the e^x series, is equal to the value of the previous term. So, for example, the derivative of the third term in the e^x series is equal to the value of the second term.

$$\frac{d}{dx}\frac{x^3}{3!} = \frac{3x^2}{3!} = \frac{x^2}{2!} \tag{5}$$

It is interesting that for every term in any of the power series, the factorial value of the denominator is always equal to the exponential value of the numerator. Even when you take the derivative of a term, these two values change but still remain equal.

The derivative of the e^x function is calculated by taking the derivative of each of the terms in the series. Note that the derivative of e^x is simply e^x.

$$\begin{aligned}
e^x &= \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots && (6)\\
\frac{d}{dx}e^x &= 0 + \frac{1x^0}{1!} + \frac{2x^1}{2!} + \frac{3x^2}{3!} + \frac{4x^3}{4!} + \frac{5x^4}{5!} + \dots && (7)\\
\frac{d}{dx}e^x &= 0 + \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{4!} + \frac{x^4}{4!} + \dots = e^x && (8)\\
\frac{d}{dx}e^x &= e^x && (9)
\end{aligned}$$

Similarly, the derivatives of the cos(x) function are calculated by taking the derivative of each of the individual terms.

$$
\begin{aligned}
\cos x &= \frac{x^0}{0!} - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \dots && (10) \\
\frac{d}{dx}\cos x &= 0 - \frac{2x^1}{2!} + \frac{4x^3}{4!} - \frac{6x^5}{6!} + \frac{8x^7}{8!} - \frac{10x^9}{10!} + \dots && (11) \\
&= 0 - \frac{x^1}{1!} + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \frac{x^9}{9!} + \dots = -\sin x && (12) \\
\frac{d^2}{dx^2}\cos x &= 0 - \frac{1x^0}{1!} + \frac{3x^2}{3!} - \frac{5x^4}{5!} + \frac{7x^6}{7!} - \frac{9x^8}{9!} \dots && (13) \\
&= 0 - \frac{x^0}{0!} + \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \frac{x^8}{8!} + \dots = -\cos x && (14) \\
\frac{d^3}{dx^3}\cos x &= 0 - 0 + \frac{2x^1}{2!} - \frac{4x^3}{4!} + \frac{6x^5}{6!} - \frac{8x^7}{8!} \dots && (15) \\
&= 0 - 0 + \frac{x^1}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots = \sin x && (16) \\
\frac{d^4}{dx^4}\cos x &= 0 - 0 + \frac{1x^0}{1!} - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} \dots && (17) \\
&= 0 - 0 + \frac{x^0}{0!} - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \dots = \cos x && (18)
\end{aligned}
$$

The first derivative of the cos function is equal to the -sin. The second, third, and fourth derivatives are the -cos, sin, and cos respectively. This illustrates how these derivatives loop endlessly (spin).

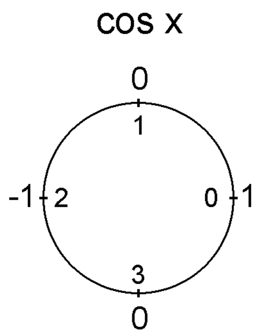
3.3 Cosine and Sine in Spin Equations

The cosine and sine functions are the other real functions used with spin. They are very similar to the function e^x. They both start at the origin and step counterclockwise around the circle. What is different is that they use a mask for interpreting each term in the e^x series. The mask value used to compute the $\cos x$ is:

$$\cos x = (i^n + i^{-n})/2 \tag{19}$$

So the value for any term a in the $\cos(x)$ series is equal to:

$$Term_a = \cos(x) \times \frac{x^a}{a!} \tag{20}$$

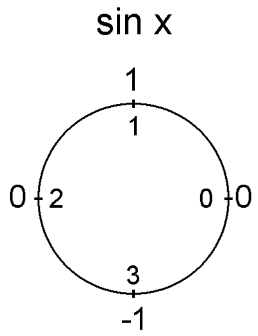


n	x	$\cos x$	$(i^n + i^{-n})/2$	$x^n/n!$
0	0	1	1	$x^0/0!$
1	$\pi/2$	0	0	$x^1/1!$
2	π	-1	-1	$x^2/2!$
3	$3\pi/2$	0	0	$x^3/3!$
4	2π	1	1	$x^4/4!$
5	$5\pi/2$	0	0	$x^5/5!$

$$\cos x = \frac{x^0}{0!} - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \dots \tag{21}$$

Similarly, determining each value of the $\sin x$ series also involves using a mask, but this time we use what is in the $\sin x$ column as the mask. So to determine the value of any term a in the $\sin x$ series:

$$Term_a = \sin(x) \times \frac{x^a}{a!} \tag{22}$$



n	x	$\sin x$	$x^n/n!$
0	0	0	$x^0/0!$
1	$\pi/2$	1	$x^1/1!$
2	π	0	$x^2/2!$
3	$3\pi/2$	-1	$x^3/3!$
4	2π	0	$x^4/4!$
5	$5\pi/2$	1	$x^5/5!$

$$\sin x = \frac{x^1}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \dots \tag{23}$$

All three power series (e^x, $\cos x$, and $\sin x$) are generated by spin equations.

4 Integrating Real and Imaginary Spin

It's not obvious how imaginary spin (based on i) relates to real spin (based on e^x, $\cos x$, and $\sin x$). However, Euler wrote an equation - now referred to as Euler's Formula - that integrates these two types of spin:

$$e^{ix} = \cos x + i \sin x \tag{24}$$

Below is his interesting derivation of this equation, which is based on the power series for e^x. (the explanation follows the derivation)

$$\begin{aligned}
e^x &= \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots && (25)\\
e^{ix} &= \frac{i^0x^0}{0!} + \frac{i^1x^1}{1!} + \frac{i^2x^2}{2!} + \frac{i^3x^3}{3!} + \frac{i^4x^4}{4!} + \frac{i^5x^5}{5!} + \ldots && (26)\\
e^{ix} &= \frac{x^0}{0!} + \frac{ix^1}{1!} - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} + \ldots && (27)\\
e^{ix} &= \left(\frac{x^0}{0!} - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right) + \left(\frac{ix^1}{1!} - \frac{ix^3}{3!} + \frac{ix^5}{5!} - \ldots\right) && (28)\\
e^{ix} &= \left(\frac{x^0}{0!} - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right) + i\left(\frac{x^1}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right) && (29)\\
e^{ix} &= \cos x + i \sin x && (30)
\end{aligned}$$

Formula 26 shows x replaced with ix. Formula 27 is a simplification of the imaginary terms. Formula 28 groups the terms with even and odd exponents. Formula 29 factors out an i from the the odd exponents group.

Remarkably, the terms grouped within parentheses in formula 29, are the power series for the $\cos x$ and $\sin x$ respectively. Formula 30 simply substitutes the terms "$\cos x$" and "$\sin x$" for the respective series.

Since e^{ix} is a spin equation, we can graph it and look at the table values to better understand its properties.

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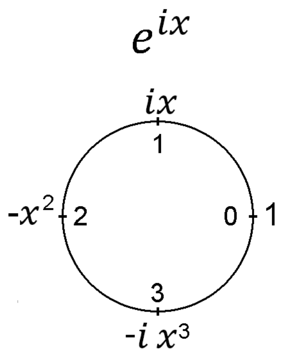

n	x	$\sin x$	$i \sin x$	$\cos x$	e^{ix}
0	0	0	0	1	1
1	$\frac{1}{2}\pi$	1	i	0	i
2	π	0	0	-1	-1
3	$\frac{3}{2}\pi$	-1	$-i$	0	$-i$
4	2π	0	0	1	1
5	$\frac{5}{2}\pi$	1	i	0	i

$$e^{ix} = \cos x + i \sin x \tag{31}$$

The table for e^{ix} looks complicated but it is very similar to previous tables. There is a new "$i \sin x$" column which is i times the "$\sin x$" column. The last column (e^{ix}), is simply "$i \sin x + \cos x$".

The equation e^{ix} starts at the origin ($n = 0$) and steps counterclockwise around the circle.

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4.1 Euler's' Identity

Starting with the complex Euler equation (formula 32), if you substitute π for x and simplify, you get an equation referred to as Euler's identity (shown in formula 35).

$$e^{ix} = \cos x + i \sin x \quad (32)$$
$$e^{i\pi} = \cos \pi + i \sin \pi \quad (33)$$
$$e^{i\pi} = -1 + 0 \quad (34)$$
$$e^{i\pi} + 1 = 0 \quad (35)$$

It is a famous identity because it integrates the three main mathematical constants: e, i, and π. The physicist Richard Feynman called Euler's identity "the most remarkable formula in mathematics". In 1988, a survey by the Mathematical Intelligencer reported that its readers voted this equation the "Most beautiful mathematical formula ever".

Mathematicians appreciated the elegance of this identity, even though its meaning was unclear. Now we know that this identity is really a spin equation that is easy to understand with spin concepts. If we look back at the e^{ix} table (used with formula 31) we see that e^{ix} is equal to -1 when $x = \pi$. So the identity basically says $1 + -1 = 0$.

4.2 Deriving the Complex Conjugate of e^{ix}

The odd powered terms generated by the equation e^{ix} contain imaginary numbers. This is a problem since imaginary terms in the answer can not be resolved. However, if we pair the e^{ix} equation with its complex conjugate e^{-ix}, then the complex terms cancel out, leaving only the real terms.

Below is a way to derive an equation that pairs e^{ix} (formula 36) with its complex conjugate. The first step is to determine the complex conjugate of e^{ix} which is e^{-ix}. We can get this by starting with the formula for e^{ix} and then substituting $-x$ for x (formula 37).

$$e^{ix} = \cos(x) + i\sin(x) \tag{36}$$
$$e^{i(-x)} = \cos(-x) + i\sin(-x) \tag{37}$$

Then with the help of the following two identities (formulas 38 and 39), $e^{i(-x)}$ can be simplified to formula 40.

$$\cos(-x) = \cos(x) \tag{38}$$
$$\sin(-x) = -\sin(x) \tag{39}$$

$$e^{-ix} = \cos(x) - i\sin(x) \tag{40}$$

At this point we have a formula for both e^{ix} (formula 36) and and its complex conjugate e^{-ix} (formula 40). All we have to do is add those two equations together and we get an equation (formula 41) that always yields real answers.

$$e^{ix} + e^{-ix} = 2\cos(x) \tag{41}$$

(For mathematical consistency, there needs to be a new form of brackets placed around $e^{ix} + e^{-ix}$ to indicate that the two paired expressions must be treated as a single mathematical entity.)

4.3 Elements of the Spin Equation

This paper views (Ψ) as a complex spin equation which takes the form:

$$\Psi = e^{ix} \tag{42}$$

The complex conjugate of Ψ is $\Psi^\star$ - the same equation spinning backwards:

$$\Psi^\star = e^{-ix} \tag{43}$$

When Ψ is added to $\Psi^\star$, it generates a real wave (ψ) with the characteristics of $2\cos x$

$$e^{ix} + e^{-ix} = 2\cos(x) \tag{44}$$

$$\psi = 2\cos(x) \tag{45}$$

5 Computing Energy

5.1 Single Variable Solution (t)

If you multiply Planck's reduced constant ($\hbar$) by the rate-of-spin you get the energy (E).

$$\hbar \times \text{Rate-of-spin} = E \tag{46}$$

Which means

$$\text{Rate-of-spin} = \frac{E}{\hbar} \tag{47}$$

Since Ψ is a spin equation with a constant velocity, the derivative of Ψ (with respect to time) is the change of spin over the change in time. So we can build an energy formula based on this Rate-of-spin.

$$\frac{\partial \Psi}{\partial t} = \frac{\Delta \Psi}{\Delta t} = \text{Rate-of-spin} \tag{48}$$

$$\frac{\partial \Psi}{\partial t} = \text{Rate-of-spin} = \frac{E}{\hbar} \tag{49}$$

$$\hbar \frac{\partial \Psi}{\partial t} = E \tag{50}$$

We can explicitly derive the same energy formula by taking the first derivative of Ψ (and then multiply both sides by $-i\hbar$.

$$\Psi(t) = e^{i\frac{E}{\hbar}t} \tag{51}$$

$$\frac{\partial \Psi}{\partial t} = i\frac{E}{\hbar}\Psi \tag{52}$$

$$-i\hbar \frac{\partial \Psi}{\partial t} = E\Psi \tag{53}$$

We will see later that when Ψ is paired with its complex conjugate the effect that i has on the equation disappears. Similarly the discrepancy in the minus sign will also disappear leaving the basic formula intact.

5.2 Two Variable Solution (x, t)

Just as we used the derivative with respect to time to build an energy formula, the derivative with respect to x can be used to create a momentum formula. We know the rate used for a time equation is $E/\hbar$, similarly the rate to use for computing momentum is $p/\hbar$. So a solution for Ψ that varies with both x and t could be:

$$\Psi = e^{i(\frac{p}{\hbar}x - \frac{E}{\hbar}t)} \tag{54}$$

With this spin equation, we can take derivatives to create the momentum and energy formulas. The energy formula comes by taking the first derivative of Ψ (t) and then multiplying both sides of the equation by $i\hbar$.

$$\frac{\partial \Psi}{\partial t} = -i\frac{E}{\hbar}\Psi \tag{55}$$

$$i\hbar\frac{\partial \Psi}{\partial t} = E\Psi \tag{56}$$

Formula 56 says that multiplying $\hbar$ by the derivative of Ψ (rate of spin) equals the energy of Ψ.

Similarly we can create a momentum formula by taking the derivative of Ψ (x), and then multiplying through by $-i\hbar$.

$$\frac{\partial \Psi}{\partial x} = i\frac{p}{\hbar}\Psi \tag{57}$$

$$-i\hbar\frac{\partial \Psi}{\partial x} = p\Psi \tag{58}$$

The momentum (sometimes represented as mv) can also be converted to kinetic energy ($\frac{1}{2}mv^2$). It is technically easier to convert energy from p^2 (than p), so below we determine the second derivative of Ψ(x). Then we isolate p^2, and replace it with m^2v^2 in formula 63.

$$\frac{\partial\Psi}{\partial x} = i\frac{p}{\hbar}\Psi \tag{59}$$

$$\frac{\partial^2\Psi}{\partial x^2} = i^2\frac{p^2}{\hbar^2}\Psi \tag{60}$$

$$\frac{\partial^2\Psi}{\partial x^2} = -\frac{p^2}{\hbar^2}\Psi \tag{61}$$

$$-\hbar^2\frac{\partial^2\Psi}{\partial x^2} = p^2\Psi \tag{62}$$

$$-\hbar^2\frac{\partial^2\Psi}{\partial x^2} = m^2v^2\Psi \tag{63}$$

Now if we divide both sides by $2m$ we get the kinetic energy.

$$\frac{-\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} = \frac{1}{2}mv^2\Psi = \text{Kinetic energy} \tag{64}$$

We also know that the total energy (E) is equal to the kinetic energy (KE) plus the potential energy (PE).

$$\text{E} = \text{KE} + \text{PE} \tag{65}$$

If we let υ equal the potential energy, then the total energy of Ψ is.

$$E\Psi = \frac{-\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2}\Psi + \upsilon\Psi \tag{66}$$

Finally, if we substitute in our energy equation (formula 56) for $E\Psi$ we get the time dependent Schrodinger equation.

$$i\hbar\frac{\partial\Psi}{\partial t} = \frac{-\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2}\Psi + \upsilon\Psi \tag{67}$$

6 New Definition of i

In order to introduce spin, a solid foundation has been built between imaginary and real numbers. However the definition of i still needs to be updated to reflect these new concepts. The new definition of i is:

The step unit in the imaginary unit circle;
equal in value to the fourth root of 1.

$$i = \sqrt[4]{1}$$

Summary

Imaginary numbers were introduced into our real number system without a mathematical foundation. With no explicit guide showing how to relate real and imaginary numbers, arbitrary assumptions were made. While those assumptions allowed us to successfully work with imaginary numbers, some distortions in our concepts occurred.

Spin is based on imaginary numbers. Spin equations look like waves when you plot them on a Cartesian (x, y) style graph. When you plot them on the imaginary unit circle, they spin.

There were hints of spin in our number system already. The power series (for e^x, $\cos x$, and $\sin x$) were all spin terms. The Euler formula (really a half of a formula) is a spin equation, as is the Schrodinger equation.

When Ψ is used to represent a complex solution it takes the form:

$$\Psi = e^{ix} \tag{68}$$

Ψ must be paired with its complex conjugate ($\Psi^\star$) in order to generate the real wave (ψ).

$$\begin{aligned} \Psi^\star &= e^{-ix} && (69) \\ e^{ix} + e^{-ix} &= 2\cos(x) && (70) \\ \psi &= 2\cos(x) && (71) \end{aligned}$$

In order to form energy and momentum equations, the coefficients for x and t are set to $(\frac{p}{\hbar})$ and $(\frac{E}{\hbar})$ respectively:

$$\Psi = e^{i(\frac{p}{\hbar}x - \frac{E}{\hbar}t)} \tag{72}$$

To create an energy formula, we took the derivative of Ψ and multiplied both sides by $i\hbar$.

$$i\hbar\frac{\partial\Psi}{\partial t} = E\Psi \tag{73}$$

To create a formula for momentum we took the second derivative of Ψ (with respect to x). The momentum formula is then converted into kinetic energy by dividing by $2m$.

$$\begin{aligned}
\frac{\partial^2\Psi}{\partial x^2} &= -\frac{p^2}{\hbar^2}\Psi & (74)\\
-\hbar^2\frac{\partial^2\Psi}{\partial x^2} &= p^2\Psi & (75)\\
-\hbar^2\frac{\partial^2\Psi}{\partial x^2} &= m^2v^2\Psi & (76)\\
\frac{-\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} &= \frac{1}{2}mv^2\Psi & (77)\\
\frac{-\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} &= \text{kinetic energy}\Psi & (78)
\end{aligned}$$

The total energy (E) of Ψ is the kinetic energy (KE) plus potential energy (PE). So we can substitute in the kinetic energy and let υ represent the potential energy, to get the time independent Schrodinger equation.

$$\begin{aligned}
E\Psi &= KE\Psi + PE\Psi & (79)\\
E\Psi &= \frac{-\hbar^2}{2m}\frac{\partial^2\Psi}{\partial t^2}\Psi + \upsilon\Psi & (80)
\end{aligned}$$

Finally we can replace $E\Psi$ with our energy equation (formula 73) to obtain the time dependent Schrodinger equation.

$$i\hbar\frac{\partial\Psi}{\partial t} = \frac{-\hbar^2}{2m}\frac{\partial^2\Psi}{\partial t^2}\Psi + \upsilon\Psi \tag{81}$$