Introducing Spin

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Abstract

The foundation is provided for bringing the property of spin into the mathematical domain. Two types of spin are introduced:
– Imaginary spin, based on the constant i.
– Real spin, based on the functions: e^x, $\cos x$, and $\sin x$.

These two types of spin are then integrated by an Euler equation that is over 200 years old.

$$e^{ix} = \cos x + i \sin x \tag{1}$$

This mysterious equation (with x set equal to π) was voted the "the Most Beautiful Mathematical Formula Ever" by readers of Mathematical Intelligencer in 1988. Now we know that it is a spin equation, and it is easily understood.

However, in order to use this equation, it must be paired with its complex conjugate to cancel out the imaginary terms. This is the process that allows the complex world of spin, to enter the real world of mathematics.

1 Background

The constant i (sometimes referred to as j) is currently defined as the square root of -1. Descartes was uncomfortable with this constant, because there was no way to conceptualize what it meant to be the square root of -1, and also because there were no practical applications that used this constant. In his book *La Géométrie* in 1637, he disparagingly coined the term "imaginary numbers", suggesting they were not real numbers.

Ironically, the diagrams used today to represent imaginary numbers are based on Descartes' Cartesian coordinate system. Argand diagrams use the x axis to represent real numbers and the y axis to represent imaginary numbers. Since the coordinate points on these diagrams have both a real component and an imaginary component, these numbers are said to be complex. Similarly, the intersection of the real and imaginary axes are sometimes referred to as the complex plane.

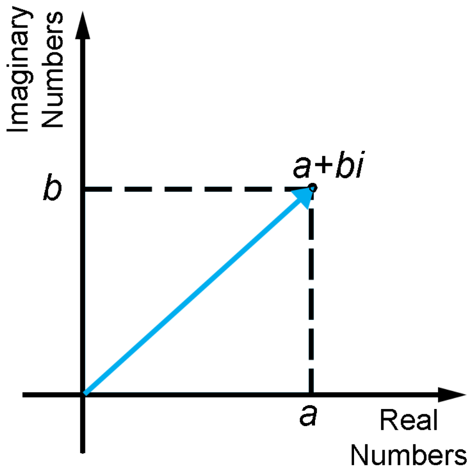


FIGURE 1: Argand Diagram

While there is consistency in viewing imaginary numbers with Argand diagrams, there still is no practical value to this approach. There are no real world problems that can be solved by working with imaginary numbers in this way. There are applications that use imaginary numbers, but they involve working with circles and the property of spin.

2 Spin Based on Imaginary Numbers

2.1 Unitary Circles

Unitary circles model the property of spin. They start at the origin and spin around in one, two, or four steps. These steps are multiplicative, which is why they each have a unitary formula with the step unit raised to the number of steps equaling one.

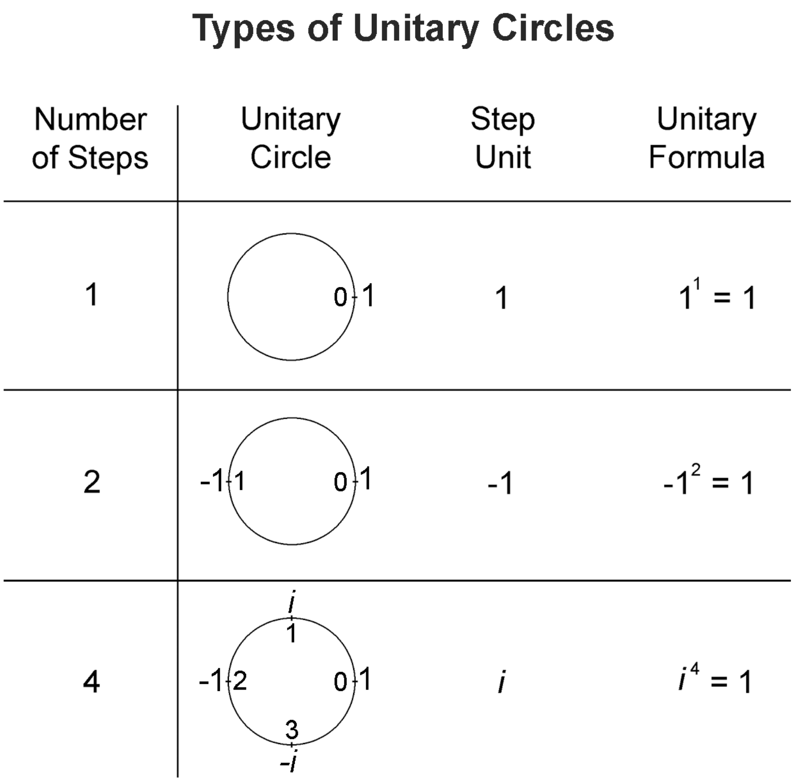


FIGURE 2: Unitary circles

The last circle, with a step unit of i, is called the imaginary unit circle.

2.2 Imaginary Unit Circle

The imaginary unit circle is a graphical representation of the expressions i^n and i^{-n}.

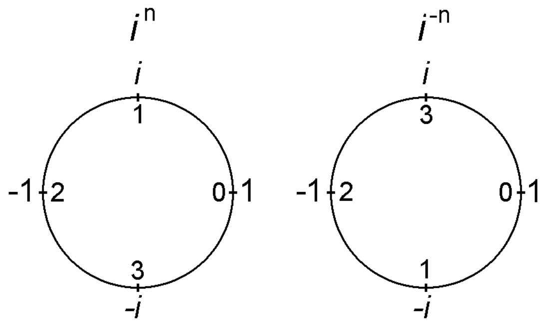


FIGURE 3: The imaginary unit circle

The expression i^n starts at the origin (with a step number of 0), and steps around the circle in a counterclockwise direction. Similarly, the expression i^{-n} starts at the origin, but steps around the circle in a clockwise direction. With each step, new terms are generated in an endless series.

n	i^n	i^{-n}
0	1	1
1	i	$-i$
2	-1	-1
3	$-i$	i
4	1	1
5	i	$-i$
6	-1	-1
7	$-i$	i

TABLE 1: Generated spin terms

2.3 Complex Conjugates

The expressions i^n and i^{-n} generate both real and complex terms. It is therefore problematic to use these expressions in formulas, because real world problems require only real answers.

However i^n can be "paired" with its complex conjugate i^{-n}, so that the real terms add together and the complex terms cancel out. Pairing binds i^n and i^{-n} into a single mathematical entity.

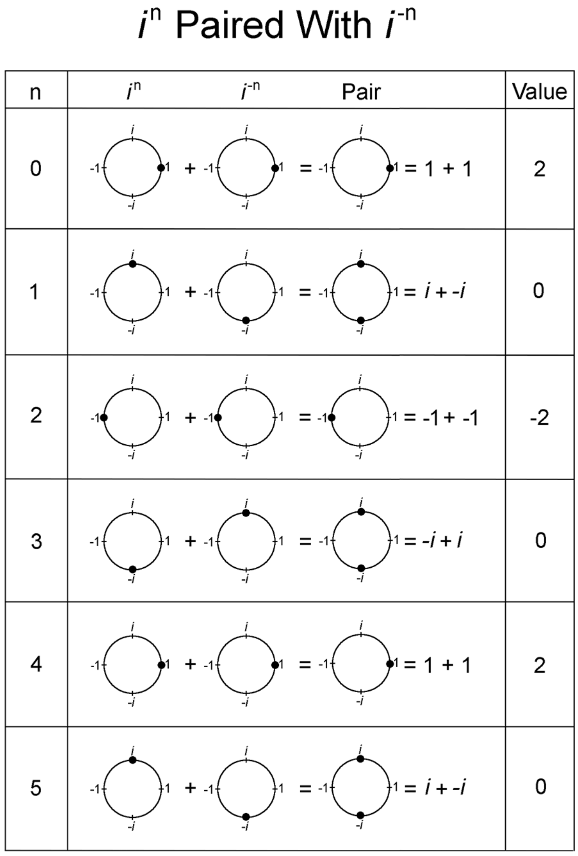
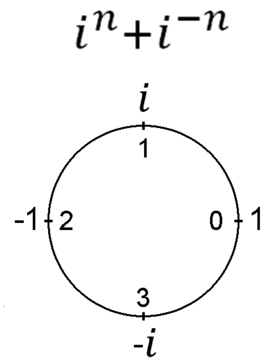


FIGURE 4: Complex conjugates bound together

Although spin equations generate an infinite number of terms, these terms can still add up to a constant. For example when the equation $i^n + i^{-n}$ is divided by $n+1$, it adds up to $\frac{\pi}{2}$.



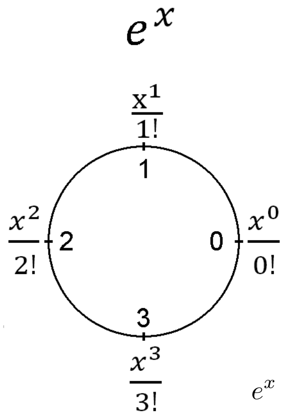
n	i^n	i^{-n}	$i^n + i^{-n}$
0	1	1	2
1	i	$-i$	0
2	-1	-1	-2
3	$-i$	i	0
4	1	1	2
5	i	$-i$	0

$$\sum_{n=0}^{\infty} \frac{i^n + i^{-n}}{n+1} = \frac{2}{1} - \frac{2}{3} + \frac{2}{5} - \frac{2}{7} + \frac{2}{9} \cdots = \frac{\Pi}{2} \tag{2}$$

3 Spin Based on Real Numbers

3.1 e^x Spin Equation

The basic equation for real spin is e^x - which seems reasonable since x grows exponentially with each step. It starts at the origin ($n = 0$) and steps counterclockwise around the circle. With each step a new term is created.



n	$x^n/n!$
0	$x^0/0!$
1	$x^1/1!$
2	$x^2/2!$
3	$x^3/3!$
4	$x^4/4!$
5	$x^5/5!$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots \tag{3}$$

This series generated by the spin equation e^x, is identical to the e^x power series. (Now we know the e^x power series was generated by a spin equation.)

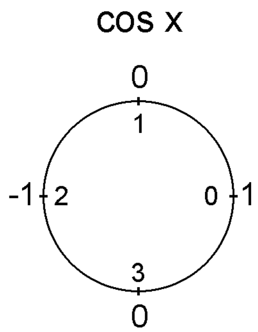
Each term in the e^x series carries a "running value". When the value of any term is multiplied by x/n, the running value for the next term in the series is generated. More specifically, if you let a represent any n, then:

$$\frac{x^a}{a!} \times \frac{x}{a+1} = \frac{x^{a+1}}{(a+1)!} \tag{4}$$

This shows us that a multiplication operation takes us from one term to the next.

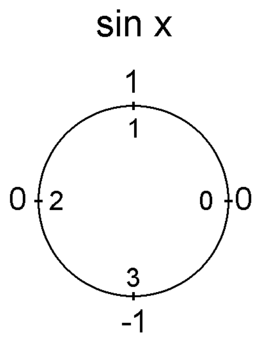
3.2 Cos and Sin Spin Equations

The cos and sin functions are very similar to the function e^x. They both start at the origin and step counterclockwise around the circle. What is different, is that these functions create a mask for interpreting each term in the e^x series. For both the cos and sin series, the mask is simply the $\cos x$ or $\sin x$ (respectively) for any given x.



n	x	$\cos x$	$x^n/n!$
0	0	1	$x^0/0!$
1	$\pi/2$	0	$x^1/1!$
2	π	-1	$x^2/2!$
3	$3\pi/2$	0	$x^3/3!$
4	2π	1	$x^4/4!$
5	$5\pi/2$	0	$x^5/5!$

$$\cos x = \frac{x^0}{0!} - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \dots \tag{5}$$



n	x	$\sin x$	$x^n/n!$
0	0	0	$x^0/0!$
1	$\pi/2$	1	$x^1/1!$
2	π	0	$x^2/2!$
3	$3\pi/2$	-1	$x^3/3!$
4	2π	0	$x^4/4!$
5	$5\pi/2$	1	$x^5/5!$

$$\sin x = \frac{x^1}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \dots \tag{6}$$

The $\cos x$ and $\sin x$ spin equations (formulas 5 and 6), generate the $\cos x$ and $\sin x$ power series.

4 Integrating Real and Imaginary Spin

It's not obvious how imaginary spin (which is based on the constant i) relates to real spin (which is based on e^x, $\cos x$, and $\sin x$). However, Euler wrote an equation more than 200 years ago that integrates these two types of spin:

$$e^{ix} = \cos x + i \sin x \tag{7}$$

Below is his very interesting derivation of this equation, based on the power series for e^x.

$$\begin{aligned}
e^x &= \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots && (8)\\
e^{ix} &= \frac{i^0x^0}{0!} + \frac{i^1x^1}{1!} + \frac{i^2x^2}{2!} + \frac{i^3x^3}{3!} + \frac{i^4x^4}{4!} + \frac{i^5x^5}{5!} + \dots && (9)\\
e^{ix} &= \frac{x^0}{0!} + \frac{ix^1}{1!} - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} + \dots && (10)\\
e^{ix} &= \left(\frac{x^0}{0!} - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right) + \left(\frac{ix^1}{1!} - \frac{ix^3}{3!} + \frac{ix^5}{5!} - \dots\right) && (11)\\
e^{ix} &= \left(\frac{x^0}{0!} - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right) + i\left(\frac{x^1}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right) && (12)\\
e^{ix} &= \cos x + i \sin x && (13)
\end{aligned}$$

Since the e^x series is true for any x, formula 9 shows x replaced with ix. Formula 10 is a simplification of the imaginary terms (remember $i^3 = -i$). Formula 11 groups the terms with even and odd exponents. Formula 12 factors out an i from the the odd exponents group.

Remarkably, the terms grouped within parentheses in formula 12, are the power series for the $\cos x$ and $\sin x$ respectively. Formula 13 simply substitutes the terms "$\cos x$" and "$\sin x$" for the respective series.

We can graph the e^{ix} equation to better understand its properties.

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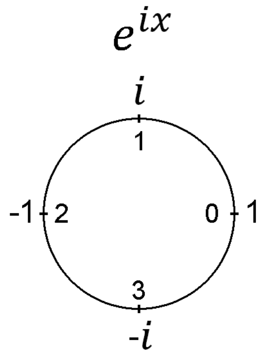

n	x	$\sin x$	$i\sin x$	$\cos x$	e^{ix}
0	0	0	0	1	1
1	$\frac{1}{2}\pi$	1	i	0	i
2	π	0	0	-1	-1
3	$\frac{3}{2}\pi$	-1	$-i$	0	$-i$
4	2π	0	0	1	1
5	$\frac{5}{2}\pi$	1	i	0	i

$$e^{ix} = \cos x + i \sin x \tag{14}$$

The table for e^{ix} looks complicated but it is very similar to previous tables. There is a new "$i\sin x$" column which is i times the "$\sin x$" column. The last column (e^{ix}), is simply "$i\sin x + \cos x$".

We can see that e^{ix} starts at the origin ($n = 0$) and steps counterclockwise around the circle. It appears to be a valid complex spin equation equal to "$\cos x + i\sin x$".

5 Euler's' Identity

Starting with the complex Euler equation (formula 15), if you substitute π for x and simplify, you get an equation referred to as Euler's identity (shown in formula 18).

$$\begin{aligned}
e^{ix} &= \cos x + i \sin x && (15)\\
e^{i\pi} &= \cos \pi + i \sin \pi && (16)\\
e^{i\pi} &= -1 + 0 && (17)\\
e^{i\pi} + 1 &= 0 && (18)
\end{aligned}$$

It is a famous identity because it integrates the three main mathematical constants: e, i, and π. The physicist Richard Feynman called Euler's identity "the most remarkable formula in mathematics". In 1988, a survey by the Mathematical Intelligencer reported that its readers voted this equation the "Most beautiful mathematical formula ever".

Mathematicians appreciated the elegance of this identity, even though its meaning was unclear. Now we know that this identity refers to the second step of a complex spin equation. Basically, e^{ix} is $\cos x$ for all even powered terms in the equation. On the second step, both $e^{i\pi}$ and $\cos \pi$ equal -1 (See the e^{ix} table and formula 14 for details.) Once we knew it was a spin equation, it was easy to understand.

6 Making the Imaginary Real

The odd powered terms in the infinite series e^{ix} contain imaginary numbers. This is a problem since real answers can not contain imaginary terms. However, if we pair the e^{ix} equation with its complex conjugate e^{-ix}, then the complex terms cancel out, leaving only the real terms.

Below is a way to derive an equation that pairs e^{ix} (formula 19) with its complex conjugate. The complex conjugate of a spin equation, is the same equation spinning backwards, so the derivation begins by substituting $-x$ for x (formula 20).

$$e^{ix} = \cos x + i\sin x \tag{19}$$
$$e^{i(-x)} = \cos(-x) + i\sin(-x) \tag{20}$$

Then with the help of the following two identitities (formulas 21 and 22), $e^{i(-x)}$ can be simplified to formula 23 .

$$\cos(-x) = \cos(x) \tag{21}$$
$$\sin(-x) = -\sin(x) \tag{22}$$

$$e^{-ix} = \cos(x) - i\sin(x) \tag{23}$$

Finally by adding formula 19 to formula 23, we get a formula that adds two complex spin equations and yields a real number (formula 24). This is the process that allows imaginary numbers to be used in conjunction with real numbers.

$$e^{ix} + e^{-ix} = 2\cos x \tag{24}$$

Below is an example of e^{ix} paired with its conjugate in a formula.

$$\sum_{n=0}^{\infty} \frac{e^{ix} + e^{-ix}}{n+1} = \frac{2}{1} - \frac{2}{3} + \frac{2}{5} - \frac{2}{7} + \frac{2}{9} - \cdots = \frac{\pi}{2} \tag{25}$$

7 Summary

Unitary circles model spin. The imaginary unit circle is a unitary circle, with the unitary formula

$$i^4 = 1 \tag{26}$$

The new definition of i is:

> The step unit in the imaginary unit circle; equal in value to the fourth root of 1.

$$i = \sqrt[4]{1} \tag{27}$$

i is used with the real functions: $e^x, \cos x$, and $\sin x$ to form the following complex spin equations.

$$e^{ix} = \cos x + i \sin x \tag{28}$$
$$e^{ix} + e^{-ix} = 2 \cos x \tag{29}$$

Our new knowledge of spin gives us a clear understanding of Euler's complex equation (formula 28). Formula 29 illustrates how to pair two complex spin equations to yield a real result ($2 \cos x$). This is how imaginary based spin can enter the real world of mathematics.