General Relativity: In Acknowledgement Of Professor Gerardus ‘t Hooft, Nobel Laureate

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ABSTRACT

I extend my thanks to Professor Gerardus ‘t Hooft, Nobel Laureate in Physics, for making more widely known my work on black hole theory, big bang cosmology, and Einstein’s General Theory of Relativity, by means of his personal website, and for providing me thereby with the opportunity to address the subject matter - supported by extensive references to primary sources for further information - in relation to his many comments, by means of this dedicated paper. The extensive mathematical appendices herein are not prerequisite to understanding the text.

Table of contents

I got my own way, to go,
And now I want,
To take your minds;
I, believe, if you could see,
The blood between the lines,
I, believe, that you could be,
A better kind;
Please lead the way so the unborn can play,
On some greener hill;
Laugh as the flames eat their burning remains,
Fools die laughing still.[1]

[1] ‘*Fools*’, Deep Purple, Fireball, 1971, (Gillan, I., Glover, R., Lord, J., Blackmore, R., Paice, I.)

I. Introduction

Gerardus ‘t Hooft is a Dutch professor of physics at the University of Utrecht in the Netherlands. He is a winner of the Nobel Prize for physics. He is currently, and for some years has been, the Editor in Chief of the journal *Foundations of Physics*. He has kindly brought attention to my writings on black holes, big bang cosmology, and General Relativity, on his personal website. I’m honoured that Professor ‘t Hooft has taken the time and trouble to inform people of my research proving the falsity of black hole theory, big bang cosmology, and Einstein's General Theory of Relativity. Although he comments on the works of five particular scientists, he has allocated perhaps the most of his comments to me.

Mr. 't Hooft [1] refers cryptically to the five scientists as Mr. L, Mr. C, Mr. DC, Mr. E, and Mr. AL, although it is a well known secret that Mr. L is Dr. Chung Lo of the Applied and Pure Research Institute, Mr. C is me, Mr. DC is Dimi Chakalov (independent researcher)[2], Mr. E is Professor Myron W. Evans of the Alpha Institute for Advanced Study, and Mr. AL is Professor Angelo Loinger of the Dipartimento di Fisica, Universitá di Milano, Italy; for those Readers who were not aware of the well known secret. Mr. 't Hooft provided a link on his webpage to an interesting paper by Professor Loinger, but none, unfortunately, to me or the other scientists. I therefore elaborate herein on the many comments Mr. 't Hooft has made on his webpage concerning me and my scientific work.

I shall begin by comparing the generic defining characteristics of all alleged black hole universes to all alleged big bang universes as they require no mathematics to fully understand.

II. Black holes and big bangs in contrast

There are four different types of black hole universes advanced by the astrophysical scientists; (a) non-rotating charge neutral, (b) non-rotating charged, (c) rotating charge neutral, (d) rotating charged. Black hole masses or 'sizes', are not types, just masses or sizes of the foregoing types. There are three purported types of big bang universes and they are characterised by their constant k-curvatures; (a) k = -1, negative spacetime curvature and spatially infinite, (b) k = 0, flat spacetime and spatially infinite, (c) k = 1, positive spacetime curvature and spatially finite. Compare now the generic defining characteristics of all black hole universes with those of all big bang universes [2, 3, 4, 5].

All black hole universes:

(1) are spatially infinite
(2) are eternal
(3) contain only one mass
(4) are not expanding (i.e. are static or stationary)
(5) are either asymptotically flat or asymptotically curved.

All big bang universes:

(1) are either spatially finite (1 case; k = 1) or spatially infinite (2 different cases; k = -1, k = 0)
(2) are of finite age (~13.8 billion years)
(3) contain radiation and many masses
(4) are expanding (i.e. are non-static)
(5) are not asymptotically anything.

Note also that no black hole universe even possesses a big bang universe k-curvature.

Comparison of the defining characteristics of all black hole universes with all big bang universes immediately reveals that they are contradictory and so they are mutually exclusive; they can't co-exist. No proposed black hole universe can be superposed with any other type of black hole universe, with any big bang universe, or with itself. Similarly, no proposed type of big bang universe can be superposed with any other type of big bang universe, with any black hole universe, or with itself. All proponents of black holes are blissfully unaware of these simple contradictions and so they combine (i.e. superpose) their black hole universes with black hole universes and with big bang universes to conjure up black hole big bang hybrid universes *ad arbitrium*, and without ever specifying what black hole universes in what big bang universes they intend.

[2] http://www.god-does-not-play-dice.net

Furthermore, General Relativity is a nonlinear theory and so the Principle of Superposition is invalid therein. Let ***X*** be some alleged black hole universe and ***Y*** be some alleged big bang universe. Then the linear combination (i.e. superposition) ***X*** + ***Y*** is not a universe. Indeed, ***X*** and ***Y*** pertain to completely different sets of Einstein field equations and so they have absolutely nothing to do with one another whatsoever.

Despite the contradictory nature of the defining characteristics of black hole universes and big bang universes, and despite the fact that the Principle of Superposition is invalid in General Relativity, Mr. 't Hooft [1, 6] superposes and says that multiple black holes exist, along with other matter such as stars and galaxies, and all together in some (unspecified) big bang universe [7].

"We not only accept the existence of black holes, we also understand how they can actually form under various circumstances. Theory allows us to calculate the behavior of material particles, fields or other substances near or inside a black hole. What is more, astronomers have now identified numerous objects in the heavens that completely match the detailed descriptions theoreticians have derived. These objects cannot be interpreted as anything else but black holes. The 'astronomical black holes' exhibit no clash whatsoever with other physical laws. Indeed, they have become rich sources of knowledge about physical phenomena under extreme conditions. General Relativity itself can also now be examined up to great accuracies." [6]

Mr. 't Hooft [7] begins his exposition of big bang creationism with the following words,

"General relativity plays an important role in cosmology. The simplest theory is that at a certain moment "t = 0", the universe started off from a singularity, after which it began to expand."

and he concludes from the Friedman-Robertson-Walker metrics that,

"All solutions start with a 'big bang' at t = 0." [7]

All so-called black hole solutions for various respective sets of Einstein field equations are also said to pertain to stars and other masses, including the Sun and the Earth. For instance, according to Mr. 't Hooft [7],

"*Einstein's equation, (7.26), should be exactly valid. Therefore it is interesting to search for exact solutions. The simplest and most important one is empty space surrounding a static star or planet. There, one has*

$$T_{\mu\nu} = 0."$$

Consequently, all the generic defining characteristics listed above for black hole universes apply equally to stars and planets and such, and they too are supposed to subsist in some unspecified big bang universe. Black hole universes differ however to those of stars and planets described by the very same equations on a secondary level. For instance, all black holes have a so-called 'event horizon' within which is located an 'infinitely dense singularity' at which spacetime is 'infinitely curved'; stars and planets have no event horizons or singularities. Mr. 't Hooft [1, 6, 7], as is usual for cosmologists, urges that singularities, which are actually just places in a mathematical expression where it is undefined, are physical entities. Mr. 't Hooft, along with the astrophysical scientists, reifies points in an equation where that equation is undefined.

Since Einstein's gravitational field is spacetime curvature, it follows that the cosmologists, including Mr. 't Hooft, necessarily maintain that Einstein's gravity is infinite at a black hole singularity. These infinities of density, spacetime curvature, and gravity are also said to be physically real. For instance, according to Hawking [8],

"*The work that Roger Penrose and I did between 1965 and 1970 showed that, according to general relativity, there must be a singularity of infinite density, within the black hole.*"

According to Carroll and Ostlie [9],

"A nonrotating black hole has a particularly simple structure. At the center is the singularity, a point of zero volume and infinite density where all of the black hole's mass is located. Spacetime is infinitely curved at the singularity. . . . The black hole's singularity is a real physical entity. It is not a mathematical artifact . . ."

According to Dodson and Poston [10],

"Once a body of matter, of any mass m, lies inside its Schwarzschild radius 2m it undergoes gravitational collapse . . . and the singularity becomes physical, not a limiting fiction."

According to Penrose [11],

"As r decreases, the space-time curvature mounts (in proportion to r^{-3}), becoming theoretically infinite at r = 0."

And according to Mr. 't Hooft [1],

"C is 'self taught', so he had no math courses and so does not know what almost means here, in terms of carefully chosen limiting procedures."

How does Mr. 't Hooft know if I have taken any mathematics courses or not? He doesn't! He certainly never asked me about it. What evidence does he adduce for his charge? None! Mr. 't Hooft just invented this charge for his own convenience. And for what it's worth, I have taken formal university courses in mathematics; not that it makes any difference to the scientific realities.

As for "*carefully chosen limiting procedures*", Dodson and Poston have already told us that a black hole singularity is "*not a limiting fiction*". Carroll and Ostlie have already told us that "*The black hole's singularity is a real physical entity. It is not a mathematical artifact*". Hawking and Penrose have already told us that "*there must be a singularity of infinite density, within the black hole.*" Penrose has already told us that spacetime curvature becomes "*theoretically infinite at r = 0.*"

It is not difficult to see when a limiting procedure is employed or not, and it is certainly not employed by the foregoing Authors, in their very own words. Such is the nature of the alleged black hole.

There are two types of black hole singularity reported by cosmologists and astronomers, according to whether or not their black hole is rotating. In the case of no rotation the singularity is a point; in the case of rotation the singularity is the circumference of a circle. Cosmologists and astronomers call them 'physical singularities'; and so does Mr. 't Hooft [6]. These and other mathematical singularities of black hole equations are reified so as to contain the masses of black holes and to locate their event horizons. Black holes are said to range in size (by means of their masses) from micro to mini to intermediate to supermassive to ultra-supermassive, up to billions of solar masses.

Since singularities are actually only places in an equation where the equation is undefined, owing for example, to a division by zero, singularities are not real physical entities, contrary to the claims of the cosmologists and astronomers.

Similarly, astrophysical scientists assert that there was a big bang singularity, also possessing various associated physically real infinities. According to Hawking [12],

"*At the big bang itself, the universe is thought to have had zero size, and to have been infinitely hot*."

That which has zero size has no volume and hence can't contain mass or have a temperature. What is temperature? According to the physicists and the chemists it is the motion of atoms and molecules. The more energy imparted to the atoms and molecules the faster they move about and so the higher the temperature. In the case of a solid the atoms or molecules vibrate about their equilibrium positions in a lattice structure and this vibration increases with increased temperature. According to Pauling [13],

"*As the temperature rises, the molecules become more and more agitated; each one bounds back and forth more and more vigorously in the little space left for it by its neighbours, and each one strikes its neighbours more and more strongly as it rebounds from them*."

Increased energy causes atoms or molecules of a solid to break down the long range order of its lattice structure to form a liquid or gas. Liquids have short range order, or long range disorder. Gases have a great molecular or atomic disorder. In the case of an ideal gas its temperature is proportional to the mean kinetic energy of its molecules [14, 15, 16],

$$\frac{3}{2}kT = \frac{1}{2}m\left\langle v^2 \right\rangle$$

wherein $<v^2>$ is the mean squared molecular speed, *m* the molecular mass, and *k* is Boltzman's constant[3].

Now that which has zero size has no space for atoms and molecules to exist in or for them to move about in. And just how fast must atoms and molecules be moving about to be infinitely hot? Zero size and infinitely hot - there is no such thing. Nonetheless, according to Misner, Thorne and Wheeler [17],

"One crucial assumption underlies the standard hot big-bang model: that the universe 'began' in a state of rapid expansion from a very nearly homogeneous, isotropic condition of infinite (or near infinite) density and pressure."

Just how close to infinite must one get to be "*near infinite*"? There are no such things as infinite or "*near infinite*" density and pressure either, just as nothing can have infinite gravity.

Near infinities of various sorts are routinely entertained by cosmologists and astronomers. Here is another example; this time it's Professor Lawrence Krauss [18] of Arizona State University, who says,

"*But is that, in fact, because of discovering that empty space has energy, it seems quite plausible that our universe may be just one universe in what could be almost an infinite number of universes and in every universe the laws of physics are different and they come into existence when the universe comes into existence*."

Just how close to infinite is "*almost an infinite number*"? There is no such thing as "*almost an infinite number*" at all.

[3] It has been shown that Boltzman's constant is not constant, since Kirchhoff's Law of Thermal Emission is not universal [100-104].

Krauss [18] reaffirms Hawking's zero size beginning of the big bang universes with the following,

"*There's no real particles but it actually has properties but the point is that you can go much further and say there's no space, no time, no universe and not even any fundamental laws and it could all spontaneously arise and it seems to me if you have no laws, no space, no time, no particles, no radiation, it is a pretty good approximation of nothing.*"

Thus, the Universe sprang into existence from absolutely nothing, by some big bang creationism, "*at time t = 0*" [7] and nothing, apparently, is "*a good approximation of nothing*" [18]. And not only is nothing a good approximation of nothing, Krauss [18] says,

"*But I would argue that nothing is a physical quantity. It's the absence of something*."

Krauss [19] reiterated the big bang universes *creation ex nihilo* dogma, thus,

"*There was nothing there. There was absolutely no space, no time, no matter, no radiation. Space and time themselves popped into existence which is one of the reasons why it is hard …"*

Yet despite the zero size, the infinities and near infinities possessed by nothing, and big bang creation *ex nihilo*, Hawking [12] still admits that,

"*energy cannot be created out of nothing*"

Thus stands yet another contradiction.

III. A black hole is a universe

Consider now a black hole universe; the type does not matter. Each and every black hole is indeed an independent universe by the very definition of a black hole, no less than the big bang universes are independent universes, although the proponents of black holes and big bangs, including Mr. 't Hooft, do not realise this.

The black hole universe is not contained within its so-called 'event horizon' because its spacetime supposedly extends indefinitely far from its so-called 'singularity'. Recall from the list of generic defining characteristics that all types of black hole universes are spatially infinite and eternal, and that they are either asymptotically flat or asymptotically curved. There is no bound on asymptotic, for otherwise it would not be asymptotic, and so every type of black hole constitutes an independent universe, bearing in mind also that each different type of black hole universe pertains to a different set of Einstein field equations as well, and therefore have nothing to do with one another whatsoever. Without the asymptotic condition one can write as many non-asymptotic non-equivalent solutions to the corresponding Einstein field equations for the supposed different types of black holes as one pleases, none of which contains a black hole.

According to the Dictionary of Geophysics, Astrophysics and Astronomy [20],

"*Black holes were first discovered as purely mathematical solutions of Einstein's field equations. This solution, the Schwarzschild black hole, is a nonlinear solution of the Einstein equations of General Relativity. It contains no matter, and exists forever in an asymptotically flat space-time."*

According to Penrose [11],

"The Kerr-Newman solutions … are explicit asymptotically flat stationary solutions of the Einstein-Maxwell equation (λ = 0) involving just three free parameters m, a and e. … the mass, as measured asymptotically, is the

parameter m (in gravitational units). The solution also possesses angular momentum, of magnitude am. Finally, the total charge is given by e. When a = e = 0 we get the Schwarzschild solution."

According to Wald [21],

"The charged Kerr metrics are all stationary and axisymmetric … They are asymptotically flat…"

I illustrate the black hole universe in figure 1.

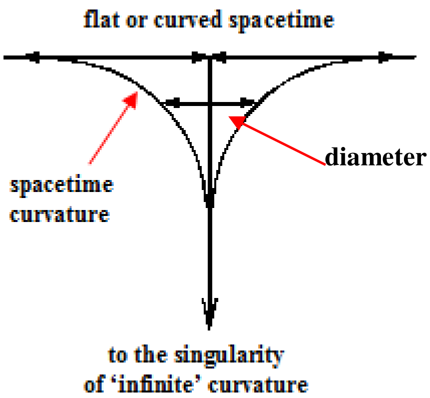

Figure 1

As the 'radial' distance from the black hole singularity increases indefinitely the spacetime curvature asymptotically approaches either flat or curved spacetime; thus, if R_p is the radial distance, $R_p \rightarrow \infty$. Note again that at the singularity gravity is infinite owing to infinite spacetime curvature there. This is what Mr. 't Hooft [6] calls a "*physical singularity*" or "*curvature singularity*". Furthermore, as the 'radial distance' increases it approaches and then grows larger than the radius of the event horizon of the black hole, the so-called 'Schwarzschild radius', also sometimes called the 'gravitational radius'. The 'Schwarzschild radius' is what Mr. 't Hooft and the astrophysical scientists call a 'coordinate singularity', which they say can be removed by some change of coordinate system.

Consider now a black hole 'binary system'. Such a binary system is also supposed to be in some (unspecified) big bang universe. I have already shown above that no black hole universe can be combined with any other universe or with itself, and so the notions of a black hole binary system and black hole collisions and mergers are inconsistent with the theory of black holes itself. To reaffirm this conclusion refer to figure 2 in which two supposed black holes are depicted.

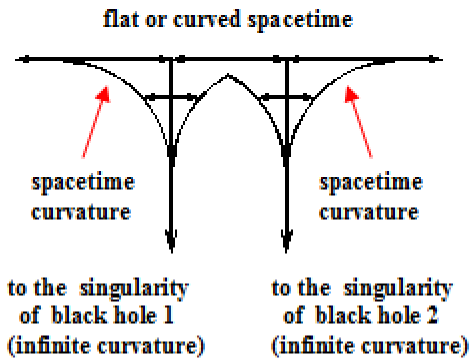

Figure 2

Recall again that the spacetimes of all black hole universes are either asymptotically flat or asymptotically curved, by definition. Note that in figure 2 it is immediately apparent that each black hole significantly disturbs the asymptotic nature of the spacetime of the other black hole and so neither of their spacetimes is asymptotically anything. Indeed, each black hole encounters an infinite spacetime curvature (infinite gravity) at the singularity of the other. This is true no matter how far from one another the black holes might be imagined, because there is no bound on asymptotic, for otherwise it would not be asymptotic. Thus the presence of another black hole violates the very definition of a black hole itself and so there can't be multiple black holes. Thus the black hole is necessarily a one-mass universe, on the assumption that the related equations even contain a mass in the first place. Such a model bears no relation to reality. Nonetheless it is routinely claimed by cosmologists and astronomers that not only are there billions of black holes (types unspecified), they are all present in some big bang universe (also unspecified), none of which can be superposed. NASA scientists, for example, have reported that they have found 2.5 million black holes (types unspecified) with their WISE survey [22]. But then none of their black holes are asymptotically anything since each and every one of them encounters 2, 499, 999 infinite spacetime curvatures around it, and so none of their black holes even satisfies the definition of a black hole. And all these black hole universes, despite being eternal, are inside some big bang expanding universe of finite age, ~13.8 billion years. Notwithstanding, Daniel Stern [22], a Principal Scientist for the NASA/JPL WISE Survey, reports,

"*We've got the black holes cornered*."

Astronomer Royal, Martin Rees [23], says,

"Black holes, the most remarkable consequences of Einstein's theory, are not just theoretical constructs. There are huge numbers of them in our Galaxy and in every other galaxy, each being the remnant of a star and weighing several times as much as the Sun. There are much larger ones, too, in the centers of galaxies."

All the different black hole 'solutions' are also applicable to stars and planets and such. Thus, these equations don't permit the presence of more than one star or planet in the universe. In the case of a body such as a star, the only significant difference in figures 1 and 2 is that the spacetime does not go to infinite curvature at the star, because there is no singularity and no event horizon in the case of a star (or planet).

IV. Gravitational collapse

Mr 't Hooft [1] adds his own invention to the notion of the mass of a black hole and its infinite gravity, in his discussion of the formation of a black hole.

"Matter travels onwards to the singularity at r = 0, and becomes invisible to the outside observer. All this is elementary exercise, and not in doubt by any serious researcher. However, one does see that the Schwarzschild solution (or its Kerr or Kerr-Newman generalization) emerges only partly: it is the solution in the forward time direction, but the part corresponding to a horizon in the past is actually modified by the contracting ball of matter. All this is well-known. An observer cannot look that far towards the past, so he will interpret the resulting metric as an accurate realization of the Schwarzschild metric. And its mass? The mass is dictated by energy conservation. What used to be the mass of a contracting star is turned into mass of a 'ball of pure gravity'. I actually don't care much about the particular language one should use here; for all practical purposes the best description is that a black hole has formed." [1]

Note that Mr 't Hooft urges that a mathematical point (and indeed the circumference of a circle too) can contain matter. But that is quite impossible - one might just as well claim that the centre of mass of a body (a mathematical artifice) is a real object, and has an infinite density. Also note that this mass, from a star, that forms his black hole, produces a "*ball of pure gravity*". However, the mathematical point he reifies, at his "*singularity at r = 0*", for infinite gravity, is not a ball, and neither is the universe that contained his star in the first place.

Recall that all the purported black hole solutions to Einstein's field equations each constitute an independent universe that contains only one mass, that of the black hole itself, on account of the asymptotic nature of their respective spacetimes. Mr. 't Hooft [1, 6, 7] refers only to asymptotically flat black hole universes, by virtue of his invoking of only Schwarzschild, Reissner-Nordström, Kerr, and Kerr-Newman black holes. Recall further, that all black hole equations, according to the proponents thereof, pertain to the 'outside' of a star without any change in their form; the only difference being that a star has no event horizon and no singularity, and so all the generic defining characteristics of all black hole universes also pertain to stars. Consequently, to result in any one of these solutions for the formation of a black hole it must begin with a universe that contains only one mass, such as a lone star. If Mr. 't Hooft, to form a black hole, begins, as he apparently does, with a universe full of stars, since he talks of clusters of stars [1], he does not begin with a relativistic universe, but a Newtonian universe. Indeed, according to Mr. 't Hooft [1],

"And now there is a thing that L and C fail to grasp: a black hole can be seen to be formed when matter implodes. Start with a regular, spherically symmetric (or approximately spherically symmetric) configuration of matter, such as a heavy star or a star cluster."

Since a black hole is actually, according to the cosmologists' actual definition of a black hole, a one mass universe, with the collapse of Mr 't Hooft's star into a black hole, the rest of the Universe must somehow completely disappear, but without falling into his newly formed black hole. Energy is therefore not conserved at all. And a Newtonian universe, which contains as many stars as one pleases to consider, can't magically transform itself into a one-mass black hole universe by means of the irresistible 'gravitational collapse' of a single star. Since the black hole equations (metrics) also apply to a star or planet, the star that 'collapses' to form a black hole must be the only mass in the Universe in the first place.

Furthermore, the gravity at the singularity of a black hole is infinite because spacetime is supposedly infinitely curved there – so a finite amount of mass 'collapses' to produce infinite gravity! This finite mass is converted into infinite "*pure gravity*" by Mr. 't Hooft [1]. Moreover, according to him, matter no longer even induces spacetime curvature by its presence: gravity can exist without matter to cause it. Indeed, according to Mr. 't Hooft [1],

*"But where does the black hole mass come from? Where is the source of this mass? $R_{\mu\nu}$ = 0 seems to imply that there is no matter at all, and yet the thing has mass! Here, both L and C suffer from the misconception that a gravitational field cannot have a mass of its own. But Einstein's equations are non-linear, and this is why gravitational fields **can** be the source of additional amount of gravity, so that a gravitational field can support itself. In particle theories, similar things can happen if fields obey non-linear equations, we call these solutions "solitons". A black hole looks like a soliton, but actually it is a bit more complicated than that."*

Mr. 't Hooft alters Einstein's theory *ad arbitrium* so that he can have gravitational fields not caused by the presence of material sources and that have a mass of their own. Contrast his notions with Einstein's actual theory. According to Einstein [24],

"We make a distinction hereafter between 'gravitational field' and 'matter' in this way, that we denote everything but the gravitational field as 'matter'. Our use of the word therefore includes not only matter in the ordinary sense, but the electromagnetic field as well."

Einstein [25] also asserts,

"*In the general theory of relativity the doctrine of space and time, or kinematics, no longer figures as a fundamental independent of the rest of physics. The geometrical behaviour of bodies and the motion of clocks rather depend on gravitational fields, which in their turn are produced by matter.*"

According to Pauli [26],

"*Since gravitation is determined by the matter present, the same must then be postulated for geometry, too. The geometry of space is not given a priori, but is only determined by matter.*"

According to Weyl [27],

"*Again, just as the electric field, for its part, depends upon the charges and is instrumental in producing mechanical interaction between the charges, so we must assume here that the metrical field (or, in mathematical language, the tensor with components g_{ik}) is related to the material filling the world*."

According to McMahon [28],

"*In general relativity, the stress-energy or energy-momentum tensor T^{ab} acts as the source of the gravitational field. It is related to the Einstein tensor and hence to the curvature of spacetime via the Einstein equation*."

According to Carroll and Ostlie [9],

"*Mass acts on spacetime, telling it how to curve. Spacetime in turn acts on mass, telling it how to move*."

According to Einstein [29],

"*space as opposed to 'what fills space', which is dependent on the coordinates, has no separate existence*"

According to Einstein [30],

"*I wish to show that space-time is not necessarily something to which one can ascribe a separate existence, independently of the actual objects of physical reality.*"

Thus, Einstein's gravitational field does not have a mass of its own at all, although it is fancied to possess energy and momentum [1, 24, 31, 32].

Although, on the one hand, Mr. 't Hooft [1] alleges, incorrectly, that Einstein's gravitational field does not require a material source, because it "*can have a mass of its own*", he also, on the other hand, says that Einstein's gravitational field must have a material source,

"*Clearly, the mass density, or equivalently, energy density $\rho(\vec{x};t)$ must play the role as a source. However, it is the 00 component of a tensor $T_{\mu\nu}(x)$, the mass-energy-momentum distribution of matter. So, this tensor must act as the source of the gravitational field*." [6]

Mr. 't Hooft [1] says he does not care about the language used in describing a black hole. Indeed; and so he foists his own language upon black hole mass and its related infinite gravity merely by means of linguistic licentiousness.

Now gravity is not a force in General Relativity because it is curvature of spacetime according to Einstein, but gravity is a force in Newton's theory. Nonetheless, Mr. 't Hooft invokes Newton's gravitational forces to enable black hole forming 'gravitational collapse'. Mr. 't Hooft [1] says of his collapsing star or star cluster,

"Assume that it obeys an equation of state. If, according to this equation of state, the pressure stays sufficiently low, one can

calculate that this ball of matter will contract under its own weight.”

Mr. ‘t Hooft [6] also says,

“One must ask what happens when larger quantities of mass are concentrated in a small enough volume. If no stable soution (sic) exists, this must mean that the system collapses under its own weight.”

However, weight is a force, Newton’s force of gravity, not a curvature of spacetime. Despite the methods of Mr. ‘t Hooft, although also routine for astronomers and cosmologists, Newtonian forces of gravity can’t be invoked for gravity in General Relativity. As de Sitter [33] remarked,

“In Einstein’s new theory, gravitation is of a much more fundamental nature: it becomes almost a property of space. … Gravitation is thus, properly speaking, not a ‘force’ in the new theory.”

V. Black hole escape velocity

They don’t realise it, but according to all proponents of black holes, of which Mr. ‘t Hooft is a typical example, their black holes all have both an escape velocity and no escape velocity simultaneously at the very same place; which is of course quite impossible, and therefore again completely invalidates the theory of black holes. However, since none of the proponents of black holes understands what escape velocity means, this additional contradiction has also escaped them.

On the one hand it is asserted by cosmologists and astronomers that their black holes have an escape velocity. According to the Dictionary of Geophysics, Astrophysics and Astronomy [20],

*“**black hole** A region of spacetime from which the escape velocity exceeds the velocity of light.*”

According to Hawking [8],

“*I had already discussed with Roger Penrose the idea of defining a black hole as a set of events from which it is not possible to escape to a large distance. It means that the boundary of the black hole, the event horizon, is formed by rays of light that just fail to get away from the black hole. Instead, they stay forever hovering on the edge of the black hole*.”

According to the Collins Encyclopædia of the Universe [34],

“***black hole** A massive object so dense that no light or any other radiation can escape from it; its escape velocity exceeds the speed of light*.”

According to O’Neill [35],

“*No particle, whether material or lightlike, can escape from the black hole*”

According to Mr. ‘t Hooft [6] the escape velocity of a black hole is at least the speed of light,

“A black hole is characterized by the presence of a region in space-time from which no trajectories can be found that escape to infinity while keeping a velocity smaller than that of light.”

According to Joss Bland-Hawthorn [36], Professor of Astrophysics at the Institute for Astronomy at the University of Sydney,

“A black hole is, ah, a massive object, and it’s something which is so massive that light can’t even escape. … some objects are so massive that the escape speed is basically the

speed of light and therefore not even light escapes. … so black holes themselves are, are basically inert, massive and nothing escapes."

So it is routinely claimed by proponents of black holes that they do have an escape velocity. Bland-Hawthorn's escape velocity is a particularly curious one: if the escape velocity of a black hole is the speed of light and light travels at the speed of light, then surely light must not only leave or emerge, but also escape. However, Bland-Hawthorn assures all and sundry, on national television, that because the escape speed of a black hole is that of light, light can't escape!

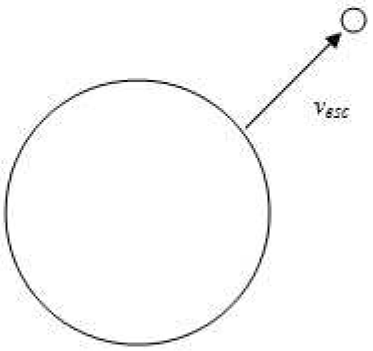

Figure 3
The small body escapes from the large body at speed v_{esc}.

Figure 3 simply depicts escape velocity. The small body escapes from the large body if it is initially propelled from the latter at the escape speed.

On the other hand the proponents of black holes also routinely claim that nothing can even leave or emerge from a black hole, let alone escape from it. Things can go into a black hole but nothing can come out of it. A journey into a black hole is a one way trip since anything that crosses its event horizon is inexorably destined, say the cosmologists, to be obliterated by crashing into and merging with the black hole's singularity. According to Chandrasekhar [37],

"*The problem we now consider is that of the gravitational collapse of a body to a volume so small that a trapped surface forms around it; as we have stated, from such a surface no light can emerge*."

According to d'Inverno [38],

"It is clear from this picture that the surface r = 2m is a one-way membrane, letting future-directed timelike and null curves cross only from the outside (region I) to the inside (region II)."

According to Hughes [39],

"*Things can go into the horizon (from r > 2M to r < 2M), but they cannot get out; once inside, all causal trajectories (timelike or null) take us inexorably into the classical singularity at r =0. … The defining property of black holes is their event horizon. Rather than a true surface, black holes have a 'one-way membrane' through which stuff can go in but cannot come out*."

According to Taylor and Wheeler [40],

"*Einstein predicts that nothing, not even light, can be successfully launched outward from the horizon ... and that light launched outward EXACTLY at the horizon will never increase its radial position by so much as a millimeter.*"

According to O'Neill [35],

"*In the exceptional case of a ∂_v photon parametrizing the positive v axis, r = 2M, though it is racing 'outward' at the speed of light the pull of the black hole holds it hovering at rest.*"

According to Dirac [41],

"*Thus we cannot have direct observational knowledge of the region r < 2m. Such a region is called a black hole, because things* *can fall into it (taking an infinite time, by our clocks, to do so) but nothing can come out*."

According to Hawking and Ellis [42],

"*The most obvious asymmetry is that the surface r = 2m acts as a one-way membrane, letting future-directed timelike and null curves cross only from the outside (r > 2m) to the inside (r < 2m).*"

And according to Mr. 't Hooft [6],

"It turned out that, at least in principle, a space traveller could go all the way in such a 'thing' but never return. Not even light could emerge out of the central region of these solutions. It was John Archibald Wheeler who dubbed these strange objects 'black holes'".

But escape velocity does not mean that things cannot leave or emerge, only that they cannot escape unless they are propelled at or greater than the escape velocity. Throw a ball into the air. Did it leave the Earth's surface? Of course! Did it escape from the Earth's gravity? No. This is simply depicted in figure 4.

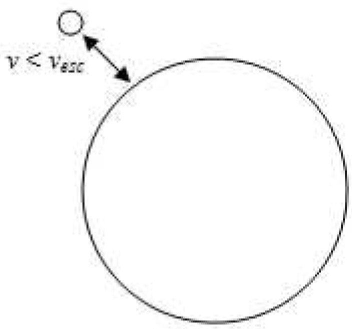

Figure 4
The small body leaves or emerges but cannot escape because $v < v_{esc}$. It falls back down after leaving or emerging.

If the initial speed of the small body in figure 4 is less than v_{esc} then it will not escape; it will rise to some maximum distance depending upon its initial speed and then fall back down. Hence, escape velocity means that things can either leave or escape from the surface of some other body, depending upon initial speed of propulsion. But this is not so in the case of the black holes, because nothing is able to even leave a black hole event horizon. Even light hovers "*forever*" at the event horizon. Things can only go into a black hole; nothing can even leave its event horizon or emerge from below its event-horizon. The black hole event horizon is therefore often referred to as a "*one-way membrane*" [38, 39, 42]. This is simply depicted in figure 5.

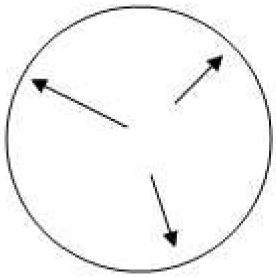

Figure 5
Nothing can even leave the black hole event horizon or emerge from beneath it. Light itself 'hovers forever' at the event horizon. The black hole event horizon has no escape velocity.

Thus, proponents of the black hole, including Mr. 't Hooft, do in fact claim that their black holes have and do not have an escape velocity simultaneously, and at the same place. Contra-hype! Proponents of black holes don't even understand escape velocity.

It's also important to note that escape velocity is an implicit two-body relation; one body escapes from another body (see figures 3 and 4). There's no meaning to escape velocity in a model of the Universe that contains only one mass, and such a model bears no relation to reality anyhow. But all black holes are independent universes which contain only one mass, on account of their asymptotic flatness or asymptotic curvedness. Despite this, proponents of black holes and big bangs, such as Mr. 't Hooft, talk about untold numbers of black holes present in some unspecified expanding big bang universe that also contains many masses other than black holes.

The escape velocity of a black hole is, as I have already revealed, claimed by the proponents thereof, to be ≥ *c*, the speed of light in vacuo. Recall that Mr. ‘t Hooft [6] has also alluded to this when he claims that black holes have an escape velocity. In order to see how Mr. ‘t Hooft and the astrophysical scientists obtain the value of their black hole escape velocity consider Hilbert’s solution, with a positive constant *m*, for static empty spacetime described by Einstein’s so-called ‘field equations’ $R_{\mu\nu} = 0$,

$$ds^2 = \left(1 - \frac{2m}{r}\right)dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2 d\Omega^2$$

$$0 \le r \qquad (1)$$

$$d\Omega^2 = \left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$

In this expression both *c* and *G* are set to unity. Note that the coefficients of the squared differential elements (i.e. the components of the metric tensor) of (1) do not depend on the time *t* and so the black hole obtained from (1) is eternal (or static). According to the astrophysical scientists the quantity *m* in expression (1) is the mass of the body producing the gravitational field. Mr. ‘t Hooft [6] also identifies *m* in expression (1) as the gravity inducing mass,

“*Newton's constant G has been absorbed in the definition of the mass parameter: M = Gm.*”

The astrophysical scientists say that Hilbert’s metric (1) describes the gravitational field ‘outside’ a body such as a star, and also a black hole. Expression (1) is almost always called ‘Schwarzschild’s solution’ by cosmologists. However, it is not Schwarzschild’s solution, which can be easily verified by reading Schwarzschild’s original paper [43]. Rewriting (1) with *c* and *G* explicitly, so that nothing is hidden, gives,

$$ds^2 = c^2\left(1 - \frac{2Gm}{c^2 r}\right)dt^2 - \left(1 - \frac{2Gm}{c^2 r}\right)^{-1} dr^2 - r^2 d\Omega^2$$

$$0 \le r \qquad (2)$$

$$d\Omega^2 = \left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$

According to Mr. ‘t Hooft [6] and all other proponents of black holes, there is a ‘coordinate’ or ‘apparent’ singularity’ at,

$$r = r_s = \frac{2Gm}{c^2} \qquad (3)$$

It is from equation (3) that they obtain the value of the ‘radius’ of the black hole event horizon, the so-called ‘Schwarzschild radius’. They mistakenly think that *r* in (1) and (2) is the radius therein.

Solving (3) for *c* yields,

$$c = \sqrt{\frac{2Gm}{r_s}} \qquad (4)$$

It is from equation (4) that the strange ‘escape velocity’ of a black hole is adduced as ≥ *c* by the proponents of black holes. However, equation (4) is nothing other than Newton’s expression for escape velocity. Since Newton’s expression, although containing only one mass term, *m*, is an implicit two-body relation, it cannot rightly appear in what is the solution to a one-body problem. It appears in (2) simply because the astrophysical scientists put it there, *post hoc*, in order to make a mass appear in it to satisfy the initial claim that $R_{\mu\nu} = 0$, where $T_{\mu\nu} = 0$, describes Einstein’s gravitational field *outside a body such as a star*, bearing in mind that Einstein’s gravitational field must have a material source. For example, according to Mr. ‘t Hooft [7], *"Einstein's equation, (7.26), should be exactly valid. Therefore it is interesting to search for exact solutions. The simplest and most important one is empty space surrounding a static star or planet. There, one has*

$$T_{\mu\nu} = 0"$$

Note that Mr. 't Hooft thus acknowledges that Hilbert's solution pertains to a static problem ("*a static star or planet*") and that the space surrounding this hypothetical static star or planet is "*empty space*". Indeed, according to Einstein [24], $T_{\mu\nu} = 0$ produces "*The field equations of gravitation in the absence of matter*".

Furthermore, since equation (3) is Newtonian it is the critical radius for the formation of the theoretical Michell-Laplace dark body, since *r* is the radius in Newton's expression, but the Michell-Laplace dark body is not a black hole because it does not possess any of the characteristics of a black hole, other than possessing mass.

VI. The radius of a black hole

As noted above, the 'Schwarzschild radius' is, according to the astrophysical scientists, and Mr. 't Hooft [6, 7], the radius of the event horizon of a black hole, which they in fact obtain from Newton's expression (4) for escape velocity. It is also claimed that bodies such as stars and planets have a Schwarzschild radius. One regularly finds in the literature, for example, that the Schwarzschild radius of the Sun is ~ 3km, and that of the Earth ~1cm. According to d'Inverno [38],

"The Schwarzschild radius for the Earth is about 1.0 cm and that of the Sun is 3.0 km."
According to Wald [21],

"*For example, a Schwarzschild black hole of mass equal to that of the Earth, $M_E = 6 \times 10^{27}$g, has $r_s = 2GM_E/c^2 \sim 1$ cm. … A black hole of one solar mass has a Schwarzschild radius of only 3km*."

According to McMahon [28],

"*For ordinary stars, the Schwarzschild radius lies deep in the stellar interior*."

In Hilbert's [44 - 46] equations (1) and (2), the quantity *r* therein has never been correctly identified by the astrophysical scientists. It has been variously and vaguely called the "*areal radius*" [11, 21, 37, 39, 47], the "*coordinate radius*" [13], the "*distance*" [27, 48], "*the radius*" [6, 10, 20, 28, 48-59], the "*radius of a 2-sphere*" [60], the "*radial coordinate*" [9, 17, 20, 28, 37, 40, 42], the "*reduced circumference*" [39], the "*radial space coordinate*" [61]. What does Mr 't Hooft call it? In his lecture notes on the theory of black holes, Mr. 't Hooft [6] says it's the "*radial coordinate*". In relation to the following metrical ground-form,

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2 d\Omega^2$$

$$d\Omega^2 = \left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

Mr. 't Hooft [6] says it's "*the radius r*". In his lecture notes on General Relativity Mr. 't Hooft again calls it the "*radius*", thus,

"*'ordinary' stars and planets contain matter ($T_{\mu\nu} \neq 0$) within a certain radius r > 2M, so that for them the validity of the Schwarzschild solution stops there*." [7]

In 2007 and 2008 I had some email exchange with Mr. 't Hooft about his *radial coordinate* come *radius* come *distance* come whatever else, amongst other things. In September 2007 he wrote to me that *r* in (2) is,

"*a gauge choice: it determines the coordinate r*" [62]

In May 2008 Mr. 't Hooft wrote this to me,

"*As for 'r' in Schwarzschild, any choice for the radial coordinate would do, but, in the spherically symmetric case, the choice that turns the angular distance into that of a sphere with radius r is the most convenient one. In physics, we call that a coordinate choice or gauge choice. Yes, if you keep this r constant, then the curvature in the angular directions indeed happens te* (sic) *be that of a sphere with radius r. It is that by choice*." [62]

From the above passage it is evident that Mr. 't Hooft says that his "*radial coordinate*" r in Hilbert's metric (since he calls Hilbert's solution 'Schwarzschild's solution') is also the "*radius r*". No matter what they call it the astrophysical scientists always treat r in (1) and (2) as the radius, and refer to $r = 0$ as *the origin*, where their black hole's mass is located, where spacetime is 'infinitely curved', and where the density is infinite!

Despite his various claims as to the identity of r, in the very same email exchange with me Mr. 't Hooft wrote,

"*Of course, no astronomer in his right mind would claim that r stands for a spatial distance"* [62]

Notwithstanding his hypothesised right mindedness of astronomers, Mr. 't Hooft [7] also says,

"*...where r_0 is the smallest distance of the light ray to the central source.*"

Here Mr. 't Hooft calls $r = r_0$ a distance and also the radius (implicitly) in the one sentence, bearing in mind that he is referring to a spherically symmetric configuration. Stefan Gillessen is an astronomer at the Max Planck Institute for Extraterrestrial Physics; he [63] also says that r in (2) is "*the radius*", and although also claiming in news reports and published papers in journals to have found a black hole, with his colleagues, at Sgt A*, he has admitted that not only did he and his colleagues not find a black hole at Sgt A*, nobody has ever found a black hole anywhere, amongst other admissions [63]. This has not stopped Gillessen from continued claims for a black hole at Sgt A* and from receiving research grants to study this nonexistent black hole [63].

Note that Mr. 't Hooft has given four different 'definitions' of r, but none of them are correct, and neither are any of the other 'definitions' proposed by the astrophysical scientists. Yet Mr. 't Hooft objects to my correct identification of what the radius is in Hilbert's metric, and my correct identification of r therein,

"*Mr. C. adds more claims to this: In our modern notation, a radial coordinate r is used to describe the Schwarzschild solution, the prototype of a black hole. 'That's not a radial distance!' he shouts. 'To get the radial distance you have to integrate the square root of the radial component g_{rr} of the metric!!' Now that happens to be right, but a non-issue; in practice we use r just because it is a more convenient coordinate, and every astrophysicist knows that an accurate calculation of the radial distance, if needed, would be obtained by doing exactly that integral*." [7]

So although Mr. 't Hooft admits that I am right again, he nonetheless clings to his "*radial coordinate r*", which he has already also said is the "*radius r*" [62], and other things besides. As for his claim that every astrophysicist knows what the radial distance in Hilbert's metric really is, that is patently false, as Gillessen [63], a typical example, attests, as do my many citations above. The 'Schwarzschild radius' and the 'gravitational radius' also attest to the routine identification of r by astrophysical scientists as the radius in

(1) and (2), by means of Newton's expression for escape velocity (4), and the claim that in (2) $0 \leq r$ with 'the origin' at $r = 0$. Contrary to Mr. 't Hooft's assertion, the correct identification of r in (2) is not a "*non-issue*", but a very important issue.

It was during my aforementioned email exchange with Mr. 't Hooft that I informed him of the true identity of r in Hilbert's metric; that r is in fact the inverse square root of the Gaussian curvature of the spherically symmetric geodesic surface in the spatial section of Hilbert's metric. He subsequently acknowledged that I am correct, as quoted above. But here again, for convenience is what he said on this issue,

"*Yes, if you keep this r constant, then the curvature in the angular directions indeed happens te (sic) be that of a sphere with radius r. It is that by choice*." [62]

Note that although Mr. 't Hooft admitted the truth of my argument about 'curvature' he still incorrectly says that r is the radius of a sphere, and that it is such by choice! Well, the fact that it is the inverse square root of the Gaussian curvature of a spherical surface means that it is not the radius of anything, and a sphere is not a surface because the former is three-dimensional but the latter is two-dimensional. As for there being any choice, that too is patently false because the metric determines what r is, not the arbitrary choice of astrophysical scientists and Mr. 't Hooft. This is a question of pure mathematics, as I will now prove, although I have expounded it in a number of my papers [64].

The squared differential element of arc-length of a curve in any surface is given by the First Fundamental Quadratic Form for a surface,

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2 \qquad (5)$$

wherein u and v are curvilinear coordinates and $E = E(u,v)$, $F = F(u,v)$, $G = G(u,v)$. The only independent variables are u and v and so this is a two-dimensional metric. If either u or v is constant the resulting line-elements describe parametric curves in the surface. The differential element of surface area is given by,

$$dA = \left|\sqrt{EG - F^2}\,du\,dv\right| \qquad (6)$$

Writing the coefficients in (5) in matrix form gives,

$$a_{ik} = \begin{bmatrix} E & F \\ F & G \end{bmatrix}$$

the determinate of which is,

$$a = EG - F^2$$

and so the differential element of area can be written as,

$$dA = \sqrt{a}\,du\,dv \qquad (6a)$$

Definition 1 (Bending Invariant): *In relation to the First Fundamental Quadratic Form for a surface,* an *expression which depends only on E, F, G and their first and second derivatives is called a bending invariant.*

Definition 2 (Spherical Surface): *A surface of constant positive Gaussian curvature is called a spherical surface.*

Definition 3 (Pseudospherical Surface): *A surface of constant negative Gaussian curvature is called a pseudo-spherical surface.*

Definition 4 (Plane Surface): *The surface of constant zero Gaussian curvature is the plane surface.*

Theorem 1 - 'Theorema Egregium' of Gauss: *The Gaussian curvature K at any point P of a surface depends only on the values at P of the coefficients in the First Fundamental Form and their first and second derivatives.*

It follows from **Definition 1** that the Gaussian curvature is a bending invariant. Interestingly, Gaussian curvature is the only second-order differential invariant of 2-dimensional Riemannian metrics.

It is of utmost importance to note that the intrinsic geometry of a surface is entirely independent of any embedding space;

"*And in any case, if the metric form of a surface is known for a certain system of intrinsic coordinates, then all the results concerning the intrinsic geometry of this surface can be obtained without appealing to the embedding space.*" [65]

Hilbert's metric (2) consists of a timelike part and a spacelike part. The timelike part is that which contains *dt*; all the rest is the spacelike part. The spacelike part is three-dimensional. Using the spacelike part one can calculate the length of curves in the space, the radial distance to any point therein, the volume of some part thereof, the area of a surface therein, etc. A 3-dimensional spherically symmetric metric manifold has the following metrical ground-form [66],

$$ds^2 = A^2(k)dk^2 + k^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right) \quad (7)$$

Note that expression (7) is a positive-definite quadratic form.

The spatial section of (2) is,

$$ds^2 = \left(1 - \frac{2Gm}{c^2 r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right) \quad (8)$$

This has the same metrical ground-form as (7), so it describes a 3-dimensional spherically symmetric space, provided the coefficient of dr^2 is not negative, because (7) is a positive-definite quadratic form. Thus, the coefficient of dr^2 in (8) can never be negative if (8) is to describe a 3-dimensional spherically symmetric space. Since the intrinsic geometry of a surface is entirely independent of any embedding space, the properties of the surface embedded in (8) can be ascertained from the metric for the surface itself. The surface in (8) is described by,

$$ds^2 = r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right) \quad (9)$$

Note that there are only two variables in this expression, ϑ and φ, since (9) is obtained from (8) by setting $r = constant \neq 0$. Since there are no cross terms in (9), i.e. no $d\vartheta d\varphi$, its metric tensor is diagonal.

Expression (9) has the form of (5), and so it is a particular First Fundamental Quadratic Form for a surface. This is easily seen by the following identifications,

$u = \vartheta,\ v = \varphi,\ E = r^2,\ F = 0,\ G = r^2\sin^2\vartheta$ (10)

Now calculate the Gaussian curvature *K* of this surface by using the relation,

$$K = \frac{R_{1212}}{g} \quad (11)$$

where R_{1212} is a component of the Riemann-Christoffel curvature tensor of the first kind and *g* is the determinant of the metric tensor for (9). To apply (11) to (9), utilize the following relations for a diagonal metric tensor,

$$R_{\mu\nu\rho\sigma} = g_{\mu\lambda} R^{\lambda}_{\ .\nu\rho\sigma}$$

$$R^{1}_{.212} = \frac{\partial \Gamma^{1}_{22}}{\partial x^1} - \frac{\partial \Gamma^{1}_{21}}{\partial x^2} + \Gamma^{k}_{22}\Gamma^{1}_{k1} - \Gamma^{k}_{21}\Gamma^{1}_{k2}$$

$$\Gamma^i_{ij} = \Gamma^i_{ji} = \frac{\partial\left(\frac{1}{2}\ln|g_{ii}|\right)}{\partial x^j}$$

$$\Gamma^i_{jj} = -\frac{1}{2g_{ii}}\frac{\partial g_{jj}}{\partial x^i} \quad (i \neq j) \qquad (12)$$

and all other Γ^i_{jk} vanish. In the above, *i, j, k* = 1, 2; $x^1 = \vartheta$, $x^2 = \varphi$. Applying expressions (11) and (12) to (9) yields,

$$K = \frac{1}{r^2} \qquad (13)$$

which is a positive constant Gaussian curvature, and hence, by **Definition 2**, (9) describes *a spherical surface*.

From (13),

$$r = \frac{1}{\sqrt{K}} \qquad (14)$$

and so *r* in (2) is the inverse square root of the Gaussian curvature of the spherically symmetric geodesic surface of the spatial section thereof. Thus *r* is neither a radius nor a distance in (9) and (2). It is defined by (13) via the expression (11), and therefore has a clear and definite intrinsic geometric identity. The result (13) obtains because the surface (9) is independent of any embedding space whatsoever and so does not change if it is embedded into some higher dimensional space.

Consequently, contrary to Mr. 't Hooft's [1] claim, there is no choice in the 'definition' of *r* in Hilbert's metric (2) because it is fully determined by the intrinsic geometry of the metric. Hence, *r* is not a 'radial coordinate', not a 'distance', not 'a gauge choice that determines *r*', and is not 'the radius', in (2). The 'Schwarzschild radius' is therefore not the radius of anything in (2), since it's not even a distance in (2).

Despite this irrefutable mathematical fact, Mr. 't Hooft [1] says,

"'r is defined by the inverse of the Gaussian curvature', C continues, but this happens to be true only for the spherically symmetric case. For the Kerr and Kerr-Newman metric, this is no longer true. Moreover, the Gaussian curvature is not locally measurable so a bad definition indeed for a radial coordinate. And why should one need such a definition? We have invariance under coordinate transformations. If so desired, we can use any coordinate we like. The Kruskal-Szekeres coordinates are an example. The Finkelstein coordinates another. Look at the many different ways one can map the surface of the Earth on a flat surface. Is one mapping more fundamental than another?"

It is trivially true that *r* in the Kerr and Kerr-Newman metrics is not simply the inverse square root of the Gaussian curvature of a spherically symmetric geodesic surface in the spatial section thereof, because the Kerr and Kerr-Newman metrics are not spherically symmetric! However, this does not change the fact that *r* in the Kerr and Kerr-Newman metrics is neither the radius nor even a distance therein, and is defined in terms of the associated Gaussian curvature, as calculation of the Gaussian curvature of the surface in the spatial section of the Kerr and Kerr-Newman metrics again attests (see Appendix A). Consider the Kerr-Newman metric in the so-called 'Boyer-Lindquist coordinates',

$$ds^2 = -\left(\frac{\Delta - a^2\sin^2\theta}{\rho^2}\right)dt^2 -$$

$$-\frac{2a\sin^2\theta\left(r^2 + a^2 - \Delta\right)}{\rho^2}dt\,d\varphi$$

$$+\frac{\left(r^2+a^2\right)^2-a^2\Delta\sin^2\theta}{\rho^2}\sin^2\theta\, d\varphi^2+$$
$$+\frac{\rho^2}{\Delta}dr^2+\rho^2 d\theta^2$$
(15)

$$\Delta = r^2-2mr+a^2+q^2$$
$$\rho^2=r^2+a^2\cos^2\theta$$

In this metric *a* supposedly accounts for angular momentum, *q* supposedly stands for electric charge, and *m* is to be the mass that carries the charge and angular momentum. Note that both *c* and *G* are again set to unity and so both disappear from the metric, just as they do in Hilbert's solution (1). Once again, the intrinsic geometry of the surface embedded in the metric is independent of the embedding space. The infinitely dense singularity of the Kerr-Newman metric for a black hole is the circumference of a circle!

Furthermore, Mr. 't Hooft's [1] claim that the correct identity of *r* in (2) is "*a bad definition indeed for a radial coordinate"* is quite nonsensical since the correct identification of *r* is not a 'definition' by the choice of astrophysical scientists or Mr. 't Hooft, and is not a 'radial coordinate' or 'the radius' (terms which Mr. 't Hooft uses equivalently). Contrary to Mr. 't Hooft's claims, there is no choice in the matter; it is fixed by the geometry of the metric itself.

Note that Mr. 't Hooft [1, 6, 7] appeals to the Kruskal-Szekeres 'coordinates' and the Eddington-Finkelstein 'coordinates', as if they change the foregoing geometrical facts. They don't! In both these sets of 'coordinates' *r* maintains its identity as in (2). This is easily determined by inspection. Consider the so-called 'Kruskal-Szekeres extension',

$$ds^2=\frac{32m^3}{r}e^{-r/2m}\left(dv^2-du^2\right)-r^2d\Omega^2$$
(16)

$$d\Omega^2=\left(d\theta^2+\sin^2\theta\, d\varphi^2\right)$$

$$0\le r$$

The surface embedded in (16) is exactly the same as in (2). Consider the 'Eddington-Finkelstein extension',

$$ds^2=\left(1-\frac{2m}{r}\right)dv^2-2dvdr-r^2d\Omega^2$$
$$d\Omega^2=\left(d\theta^2+\sin^2\theta\, d\varphi^2\right)$$
(17)

$$0\le r$$

The surface embedded in (17) is again precisely the same as in (2).

Since the surface in both (16) and (17) is exactly that in (2), *r* in (16) and (17) has the very same identity as in (2). Mr. 't Hooft's [1] analogy of a mapping of the surface of the Earth to a flat surface, in various ways, is misleading because such mappings change the spherical surface of Earth into the flat plane, which, *ipso facto*, is not a spherical surface. The Gaussian curvature of the plane is zero; that of a spherical surface is not zero.

Consider now the spatial section of the Kerr-Newman metric, which is obtained by setting *t = constant* in the metric (15),

$$ds^2=\frac{\rho^2}{\Delta}dr^2+\rho^2d\theta^2+$$
$$+\frac{\left(r^2+a^2\right)^2-a^2\Delta\sin^2\theta}{\rho^2}\sin^2\theta\, d\varphi^2$$

$$\Delta = r^2-2mr+a^2+q^2$$
$$\rho^2=r^2+a^2\cos^2\theta$$
(18)

Since expression (18) does not have the form of expression (7), it is not spherically symmetric. This is reaffirmed by the Gaussian curvature of the surface in the spatial section of (15), the latter obtained from (18) by setting r = *constant* ≠ 0,

$$ds^2 = \rho^2 d\theta^2 + \frac{\left(r^2 + a^2\right)^2 - a^2 \Delta \sin^2\theta}{\rho^2} \sin^2\theta \, d\varphi^2 \tag{19}$$

$$\Delta = r^2 - 2mr + a^2 + q^2$$
$$\rho^2 = r^2 + a^2 \cos^2\theta$$

Note that if $a = 0$ expression (19) reduces to expression (9) and (15) reduces to the Reissner-Nordström solution, which is spherically symmetric. If both $a = 0$ and $q = 0$ then expression (15) reduces to Hilbert's solution, in the form of expression (1).

Expression (19) has the form of expression (10) and is therefore a particular First Fundamental Quadratic Form for a surface, from which the Gaussian curvature can be calculated (see Appendix A). Once again the Gaussian curvature of (19) definitely identifies the quantity r in the Kerr-Newman metric (15), which does not change by (19) being embedded in (15), and so r is neither the radius nor even a distance in the Kerr-Newman metric. The Gaussian curvature of the surface (19) is not a constant positive quantity (see Appendix A) and so, by **Definition 2** it is not a spherical surface. Therefore, despite Mr. 't Hooft's assertions, r in the Kerr-Newman metric is neither the radius nor even a distance therein and so it is not a "*radial coordinate*" (whatever he really means by this vague term) because it is strictly identified in relation to the Gaussian curvature of the surface in the spatial section thereof, entirely independent of any embedding space.

What then is the actual radius in Hilbert's metric (2)? Recall that Mr. 't Hooft [1] also admitted that my identification of the radius in Hilbert's metric is actually correct. Let R_p denote the radius. Consider Hilbert's metric in the following form,

$$ds^2 = \left(1 - \frac{\alpha}{r}\right) dt^2 - \left(1 - \frac{\alpha}{r}\right)^{-1} dr^2 - r^2 d\Omega^2$$

$$0 \le r \tag{20}$$

$$d\Omega^2 = \left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$

wherein α is merely a positive constant. Then the radius is given by,

$$R_p = \int \frac{dr}{\sqrt{1 - \frac{\alpha}{r}}}$$

And so,

$$R_p = \sqrt{r(r-\alpha)} + \alpha \ln\left(\frac{\sqrt{r} + \sqrt{r-\alpha}}{\sqrt{\alpha}}\right) \tag{21}$$

If α is assigned the value $\alpha = 2Gm/c^2$ by means of Newton's equation (4), then Hilbert's metric (2) results in (20), but not Hilbert's solution, because according to (21) when the radius $R_p = 0$, $r = \alpha$. Values $0 \le r < \alpha$ are impossible because they would make the radius R_p take imaginary (i.e. complex) values.

VII. Metric 'extensions'

Since black hole universes have been proven fallacious in the previous sections herein, discussion of the so-called 'metric extensions' for them is merely a formal mathematical exercise, which I will limit here to the consideration of Schwarzschild spacetime because similar results obtain for the other equally phantasmagorial types of black hole universes (see Appendices A, B and C).

Mr. 't Hooft [1] complains that I insist on a metric signature (+, −, −, −) for Hilbert's metric (2). He says, "*'The horizon is a real singularity because at that spot the metric signature switches from (+, −, −, −) to (−, +, −, −)', C continues. This is wrong. The switch takes place when the usual Schwarzschild coordinates are used, but does not imply any singularity. The switch disappears in coordinates that are regular at the horizon, such as the Kruskal-Szekeres coordinates. That's why there is no physical singularity at the horizon*." [1]

First consider the signature switch of Hilbert's metric. The components of Hilbert's metric tensor are,

$$g_{00} = \left(1 - \frac{2Gm}{c^2 r}\right) \qquad g_{11} = -\left(1 - \frac{2Gm}{c^2 r}\right)^{-1}$$

$$g_{22} = -r^2 \qquad g_{33} = -r^2 \sin^2\theta \qquad (22)$$

When $r > 2Gm/c^2$, $g_{00} > 0$, $g_{11} < 0$, $g_{22} < 0$, and $g_{33} < 0$; consequently the signature is (+, −, −, −). If $0 < r < 2Gm/c^2$, then $g_{00} < 0$, $g_{11} > 0$, $g_{22} < 0$, and $g_{33} < 0$; consequently the signature changes to (−, +, −, −). Such a signature change is inconsistent with that of Minkowski spacetime in which Special Relativity is couched, because Minkowski spacetime has the fixed Lorentz signature of (+, −, −, −). It is also inconsistent with the metric ground-form (7) for a 3-dimensional spherically symmetric space because then the spatial section is no longer positive definite. Also, Hilbert's metric is actually undefined at $r = 2Gm/c^2$ and at the 'origin' $r = 0$, owing to divisions by zero in both cases. The Dirac Delta Function does not in fact circumvent this.

Furthermore, according to the astrophysical scientists, when $0 \le r < 2Gm/c^2$, the quantities t and r exchange their rôles, i.e. t becomes spacelike and r becomes timelike. Since time marches forwards they then maintain that anything that enters a black hole must collide and merge with its singularity because time drives it there inexorably; a time gradient becomes the driver. Some astrophysical scientists begin with the signature (−, +, +, +) for Hilbert's metric as opposed to the more usual (+, −, −, −), but all the alleged effects are still the same. According to Misner, Thorne and Wheeler [17], who use the spacetime signature (−, +, +, +) for Hilbert's solution (1),

"*The most obvious pathology at r = 2M is the reversal there of the roles of t and r as timelike and spacelike coordinates. In the region r > 2M, the t direction, ∂/∂t, is timelike (g_{tt} < 0) and the r direction, ∂/∂r, is spacelike (g_{rr} > 0); but in the region r < 2M, ∂/∂t, is spacelike (g_{tt} > 0) and ∂/∂r, is timelike (g_{rr} < 0).*

"*What does it mean for r to 'change in character from a spacelike coordinate to a timelike one'? The explorer in his jet-powered spaceship prior to arrival at r = 2M always has the option to turn on his jets and change his motion from decreasing r (infall) to increasing r (escape). Quite the contrary in the situation when he has once allowed himself to fall inside r = 2M. Then the further decrease of r represents the passage of time. No command that the traveler can give to his jet engine will turn back time. That unseen power of the world which drags everyone forward willy-nilly from age twenty to forty and from forty to eighty also drags the rocket in from time coordinate r = 2M to the later time coordinate r = 0. No human act of will, no engine, no rocket, no force (see exercise 31.3) can make time stand still. As surely as cells die, as surely as the traveler's watch ticks away 'the unforgiving minutes', with equal certainty, and with never one halt along the way, r drops from 2M to 0.*

According to Chandrasekhar [37],

"*There is no alternative to the matter collapsing to an infinite density at a* *singularity once a point of no-return is passed. The reason is that once the event horizon is passed, all time-like trajectories must necessarily get to the singularity: 'all the King's horses and all the King's men' cannot prevent it*."

According to Carroll [67],

"*This is worth stressing; not only can you not escape back to region I, you cannot even stop yourself from moving in the direction of decreasing r, since this is simply the timelike direction. (This could have been seen in our original coordinate system; for r < 2GM, t becomes spacelike and r becomes timelike.) Thus you can no more stop moving toward the singularity than you can stop getting older.*"

According to Vladmimirov, Mitskiévich and Horský [68],

"*For $r < 2GM/c^2$, however, the component g_{00} becomes negative, and g_{rr}, positive, so that in this domain, the role of time-like coordinate is played by r, whereas that of space-like coordinate by t. Thus in this domain, the gravitational field depends significantly on time (r) and does not depend on the coordinate t.*"

In other words, for $0 \le r < 2Gm/c^2$, Hilbert's static solution for a static problem becomes a non-static solution for a static problem (recall that $R_{\mu\nu} = 0$ is Einstein's [24, 32, 33] fanciful field equations for his static gravitational field in the absence of matter). To amplify this, set $t = r^*$ and $r = t^*$, and so for $0 \le r < 2m$, metric (1) becomes [69],

$$ds^2 = \left(1 - \frac{2m}{t^*}\right) dr^{*2} - \left(1 - \frac{2m}{t^*}\right)^{-1} dt^{*2} - t^{*2}\, d\Omega^2$$

$$0 \le t^* < 2m \qquad (23)$$

$$d\Omega^2 = \left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

which has the signature (−, +, −, −), which no longer has Lorentz character. It now becomes quite clear that this is a time-dependent metric since all the components of the metric tensor are functions of the timelike t^*, and so this metric bears no relationship to the original time-independent (i.e. static) problem to be solved [69].

Since it is claimed for Hilbert's metric (1) that $0 \le r$, this r passes right through the event horizon at the 'Schwarzschild radius' $r = 2m$ on its way down to $r = 0$. For instance, according to Misner, Thorne and Wheeler [17],
"*At r = 2M, where r and t exchange roles as space and time coordinates, g_{tt} vanishes while g_{rr} is infinite*."

In mathematical form this says,

$$g_{tt} = -\left(1 - \frac{2m}{2m}\right) = 0 \quad g_{rr} = \frac{1}{\left(1 - \frac{2m}{2m}\right)} = \frac{1}{0} = \infty$$

and according to Dirac [41], Hilbert's metric (1), in accordance with expressions (22),

"*becomes singular at r = 2m, because then $g_{00} = 0$ and $g_{11} = -\infty$.*"

This is however incorrect since division by zero is undefined. Despite this elementary mathematical fact the astrophysical scientists permit division by zero by a smooth passage of r down through $r = 2m$ in Hilbert's solution and claim that r and t exchange rôles [17, 37, 67, 68] for $0 \le r < 2m$ according to expression (23).

Recall that in Hilbert's solution (2) Mr. 't Hooft [1, 6, 7] and the astrophysical scientists claim that the 'Schwarzschild radius' $r = r_s = 2Gm/c^2$ is a removable 'coordinate singularity', and that $r = 0$ is the 'true' or 'physical singularity' or 'curvature singularity'. Mr. 't Hooft [1, 6, 7] employs the usual methods of the cosmologists by invoking the so-called 'Kruskal-Szekeres coordinates' and 'Eddington-Finkelstein coordinates'. It is by means of these 'coordinates' that Mr. 't Hooft [1] asserts that the switch in signature manifest in expression (23) is circumvented, despite (23) still being retained to argue for what happens after passing down through $r = 2Gm/c^2$ due to an exchange of the rôles of t and r. The very notion that such 'coordinates' are necessary is based on the false idea that r in Hilbert's metric (2) is the radius (distance) therein and hence must be able to take the values $0 \leq r$. However, by means of equation (21) it is clear that the radius does in fact take the values $0 \leq R_p$. It has already been proven in Section **VI** above that r in all the black hole solutions Mr. 't Hooft [1, 6, 7] utilises is neither the radius nor a distance and that this is also the case for the Kruskal-Szekeres and Eddington-Finkelstein 'coordinates'. Consequently, any *a priori* assertion as to the range of r in (2) has no valid basis [44]. Expression (21) determines, from the metric itself, the range on r in (2). To examine this issue further, consider Schwarzschild's [43] actual solution,

$$ds^2 = \left(1 - \frac{\alpha}{R}\right)dt^2 - \left(1 - \frac{\alpha}{R}\right)^{-1} dR^2 - R^2 d\Omega^2$$

$$d\Omega^2 = \left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$

$$R = \left(r^3 + \alpha^3\right)^{1/3}$$

$$0 \leq r \qquad (24)$$

Here α is a positive but otherwise indeterminable constant, and $r = \sqrt{x^2 + y^2 + z^2}$ where x, y, z are the usual Cartesian coordinates for 3-dimensional Euclidean space, the metric for which is,

$$ds^2 = dx^2 + dy^2 + dz^2 \qquad (25)$$

Converting (25) into spherical coordinates yields,

$$ds^2 = dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$

$$0 \leq r \qquad (26)$$

Note that when $r = 0$, Schwarzschild's metric is undefined, and the radius R_p is zero, consistent with equation (21). To see this just substitute r in (21) with $R(r)$ as defined in equations (24).

Metric (26) is the spatial section of Minkowski's spacetime metric, which is given by,

$$ds^2 = c^2dt^2 - dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$

$$0 \leq r \qquad (27)$$

Note that for every value of r in (27) there corresponds a unique value of the radius R_p for (24). The quantity r in (26) is not only the inverse square root of the Gaussian curvature of the spherically symmetric surface embedded therein, but is also the radius for the spherically symmetric 3-space (26), which is easily affirmed by a trivial calculation,

$$R_p = \int_0^r dr = r \qquad (28)$$

The spatial section of Schwarzschild's actual solution is given by,

$$ds^2 = \left(1 - \frac{\alpha}{R}\right)^{-1} dR^2 + R^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

$$R = \left(r^3 + \alpha^3\right)^{1/3} \qquad (29)$$

$$0 \leq r$$

Note that if $\alpha = 0$, Schwarzschild's metric (24) reduces to the flat spacetime of Minkowski (27) and the spatial section (29) of Schwarzschild's metric reduces to that for ordinary Euclidean 3-space (26).

The metric of (29) is undefined when $r = 0$, owing to a division by zero; otherwise it has the form of expression (7) and is thus a positive-definite quadratic form. Metric (29) can never be indefinite, i.e. its signature cannot change from (+, +, +) to (−, +, +), because (7) is always a positive-definite quadratic form. Similarly, metric (26) has the form of (7) and is a positive-definite quadratic form. It too can't change signature from (+, +, +) to (−, +, +). To be consistent with (7), (26) and (29), the spatial section (8) of Hilbert's metric must also be a positive-definite quadratic form. This means that Hilbert's metric (2) can't change its signature from (+, −, −, −) to (−, +, −, −), as expression (21) reaffirms. But it is by means of a change of signature of (2), producing (23), on the false assumption that r in (2) is the radius, that the Kruskal-Szekeres and Eddington-Finkelstein 'coordinates' are based, and employed by Mr. 't Hooft and the cosmologists.

To amplify that there can be no change of signature in (2), consider Brillouin's [69] solution,

$$ds^2 = \left(1 - \frac{2m}{R_c}\right)dt^2 - \left(1 - \frac{2m}{R_c}\right)^{-1} dR^2 - R_c^2 d\Omega^2$$

$$d\Omega^2 = \left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$

$$R_c = r + 2m$$

$$0 \le r \qquad (30)$$

Here m is a positive constant and $c = 1$.

Now consider Droste's [70] solution (again $c = 1$),

$$ds^2 = \left(1 - \frac{\alpha}{R}\right)dt^2 - \left(1 - \frac{\alpha}{R}\right)^{-1} dR^2 - R^2 d\Omega^2$$

$$d\Omega^2 = \left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$

$$R = r$$

$$\alpha \le r \qquad (31)$$

Solutions (24), (30), and (31) are equivalent. However, Hilbert's solution is not equivalent to them on account of $0 \le r$ in (2).

I have shown elsewhere [71 - 79] that all solutions equivalent to Schwarzschild's are generated by (using $c = 1$),

$$ds^2 = \left(1 - \frac{\alpha}{R_c}\right)dt^2 - \left(1 - \frac{\alpha}{R_c}\right)^{-1} dR_c^2 - R_c^2 d\Omega^2$$

$$d\Omega^2 = \left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$

$$R_c = \left(|r - r_o|^n + \alpha^n\right)^{1/n}$$

$$r, r_0 \in \Re \quad n \in \Re^+ \qquad (32)$$

where r_0 is an entirely arbitrary real number and n an entirely arbitrary positive real number. For instance, setting $r_0 = 0$, $n = 1$, $r \ge r_0$, Brillouin's solution (30) is obtained. Setting $r_0 = 0$, $n = 3$, $r \ge r_0$, Schwarzschild's actual solution (24) is obtained. Setting $r_0 = \alpha$, $n = 1$, $r \ge r_0$, Droste's solution (31) is obtained. Expressions (32) generate an infinite set of equivalent solutions. Expressions (32) are easily rendered in isotropic form as well (see [79] and Appendix D). The signature is always (+, −, −, −) in accordance with (27), except at $r = r_0$ where the metric is undefined.

Expressions (32) are not, from a purely mathematical perspective, restricted to values of $r \ge r_0$. The only value for which the metric (32) is undefined is $r = r_0$, and so there is only one singularity in (32). However, the radius for (32) is defined for all r and the radius $R_p(r_0) = 0$ for all arbitrary r_0 for all arbitrary n.

Hilbert's solution has no representation by (32) because it is not equivalent to any solution generated by it. Only Hilbert's *metrical form*, which is the same as Droste's, obtains from (32). Values $R_c < \alpha$ are not possible. R_c can only take the value of 0 if α = 0, in which case only Minkowski spacetime is produced [71 - 79].

Since (32) generates all the possible equivalent solutions in Schwarzschild form, if any one of them is extendible then all of them must be extendible. In other words, if any one of (32) can't be 'extended' then none can be extended. Thus, if Hilbert's solution is valid it must require that in Schwarzschild's actual solution $-\alpha \leq r$. Similarly this must require that $-\alpha \leq r$ in Brillouin's solution, and $0 \leq r$ in Droste's solution. It is evident from (32) that this is impossible. To amplify, consider the specific case $r_0 = 0$, $n = 2$, for which (32) yields,

$$ds^2 = \left(1 - \frac{\alpha}{R_c}\right)dt^2 - \left(1 - \frac{\alpha}{R_c}\right)^{-1} dR_c^2 - R_c^2 d\Omega^2$$

$$d\Omega^2 = \left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$

$$R_c = \left(r^2 + \alpha^2\right)^{1/2} \qquad (33)$$

According to Hilbert's solution this would require the range $-\alpha^2 \leq r^2$ in (33). However, although r can now take any real value whatsoever, r^2 cannot take values < 0. Thus, (33) cannot be 'extended' by any means. Since (33) is equivalent to (24), (30), and (31), none of the latter can be made equivalent to Hilbert's solution (2) either. Consequently, the supposed extension of Hilbert's metric to values $0 \leq r < 2m$ by means of the Kruskal-Szekeres 'coordinates, the Eddington-Finkelstein 'coordinates', and also the Lemaître 'coordinates', are all fallacious. Thus, in Hilbert's metric $0 \leq r < 2m$ is not valid [44 - 46, 71 - 79]. Mr. 't Hooft's [1, 6, 7] claims for the Kruskal-Szekeres and Eddington-Finkelstein 'coordinates' are both standard and patently false.

Putting R_c from (32) into the Kruskal-Szekeres form yields,

$$ds^2 = \frac{4\alpha^3}{R_c} e^{-R_c/\alpha}\left(dv^2 - du^2\right) - R_c^2 d\Omega^2$$

$$d\Omega^2 = \left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

$$\left(\frac{R_c}{\alpha} - 1\right)e^{R_c/\alpha} = u^2 - v^2$$

$$R_c = \left(\left|r - r_o\right|^n + \alpha^n\right)^{1/n} \qquad (34)$$

This does not extend Hilbert's metric to $0 \leq r$ since the minimum value of R_c is $R_c(r_0) = \alpha$ for all r_0 for all n. Metric (34) is not singular at $R_c(r_0)$ but it is degenerate there since then $u^2 = v^2$.

Putting R_c from (32) into the Eddington-Finkelstein form yields,

$$ds^2 = \left(1 - \frac{\alpha}{R_c}\right)dv^2 - 2dvdR_c - R_c^2 d\Omega^2$$

$$d\Omega^2 = \left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

$$v = t + R_c + \alpha \ln\left(\frac{R_c}{\alpha} - 1\right)$$

$$R_c = \left(\left|r - r_o\right|^n + \alpha^n\right)^{1/n} \qquad (35)$$

This too does not extend Hilbert's metric to $0 \leq r$ since the minimum value of R_c is $R_c(r_0) = \alpha$ for all r_0 for all n. Metric (35) is not singular anywhere, but it is degenerate at $R_c(r_0)$.

The Lemaître 'extension' has the form,

$$ds^2 = d\tau^2 - \frac{\alpha}{r}d\rho^2 - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

$$r = \left[\frac{3}{2}(\rho - \tau)\right]^{2/3} \alpha^{1/3} \qquad (36)$$

Putting R_c from (32) into the Lemaître form yields,

$$ds^2 = d\tau^2 - \frac{\alpha}{R_c} d\rho^2 - R_c^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

$$R_c = \left[\frac{3}{2}(\rho - \tau)\right]^{2/3} \alpha^{1/3} = \left(|r - r_0|^n + \alpha^n\right)^{1/n} \qquad (37)$$

Once again this does not extend Hilbert's metric to $0 \le r$ because the minimum value of R_c is $R_c(r_0) = \alpha$ for all r_0 for all n, at which the minimum value of $3(\rho - \tau)/2$ is α. Once again, metric (37) is not singular at $R_c(r_0)$, or anywhere for that matter.

Mr. 't Hooft and the astrophysical scientists claim that the Riemann tensor scalar curvature invariant (also called the Kretschmann scalar) must be unbounded at their 'physical' or 'curvature' singularity. They then claim that this justifies their 'extension' of Hilbert's solution to $0 \le r < 2m$. However, there is nothing in General Relativity or in pure mathematics that requires this condition to be met. In fact, it is not required at all because such curvature invariants are fully determined by the metric, not by any *a priori* assumed condition foisted upon it.

The Kretschmann scalar f is defined in terms of the Riemann-Christoffel curvature tensor of the first kind, as follows, $f = R^{\alpha\beta\mu\nu}R_{\alpha\beta\mu\nu}$.

In the case of the Schwarzschild form it is given by,

$$f = \frac{12\alpha^2}{R_c^6}$$

where from (32),

$$R_c = \left(|r - r_o|^n + \alpha^n\right)^{1/n}$$

Thus,

$$f = \frac{12\alpha^2}{\left(|r - r_0|^n + \alpha^n\right)^{6/n}}$$

Since (32) and all its equivalent solutions are inextendible the maximum value of f occurs at $r = r_0$, irrespective of the value of r_0 and irrespective of the value of n. Now $R_c(r_0) = \alpha$ and so the maximum value of the Kretschmann scalar is,

$$f(r_0) = \frac{12}{\alpha^4}$$

This is a finite curvature invariant for the Schwarzschild form.

Similarly, when $r = r_0$ the Gaussian curvature K of the spherically symmetric geodesic surface in the spatial section of the Schwarzschild form takes the value,

$$K = \frac{1}{\alpha^2}$$

which is also a finite curvature invariant, and is independent of the values of r_0 and n. Owing to (32) the curvature invariants f and K are always finite.

For the Schwarzschild form both f and K are curvatures that depend only upon position. There is another curvature that is of importance, which depends upon both position and a pair of directions determined by two vectors; it is called the Riemannian (or sectional) curvature K_s, and is given by,

$$K_s = \frac{R_{ijkl}U^iV^jU^kV^l}{G_{pqrs}U^pV^qU^rV^s}$$

$$G_{pqrs} \equiv g_{pr}g_{qs} - g_{ps}g_{qr}$$

where ***U*** = $\langle U^i \rangle$ and **V** = $\langle V^i \rangle$ are two linearly independent contravariant vectors of appropriate dimension. The Riemannian curvature of a metric space is a generalisation of the Gaussian curvature for a surface to spaces of dimension higher than 2. It is therefore not surprising that the Riemannian curvature reduces to Gaussian curvature in the case of dimension 2 (see equation (11) above), which is entirely independent of direction vectors – it is dependent only upon position.

In the case of a diagonal metric tensor the expression for the Riemannian curvature is simplified somewhat. The metric tensor of the spatial section of the Schwarzschild form is diagonal, and the Riemannian curvature for it is found to be given by (see Appendix B),

$$K_S = \frac{-\frac{\alpha}{2}\left(W_{1212} + W_{1313}\sin^2\theta\right) + \alpha R_c\left(R_c - \alpha\right)W_{2323}}{R_c^3\left(W_{1212} + W_{1313}\sin^2\theta\right) + R_c^4\sin^2\theta\left(R_c - \alpha\right)W_{2323}}$$

where in turn R_c is given by expression (32) (and expression (A17) in Appendix A) and the W_{ijkl} by the determinant product,

$$W_{ijkl} = \begin{vmatrix} U^i & U^j \\ V^i & V^j \end{vmatrix}\begin{vmatrix} U^k & U^l \\ V^k & V^l \end{vmatrix}$$

The most important result of all this is that when $r = r_0$ in (32), the Riemannian curvature of the spatial section of the Schwarzschild form is,

$$K_s = -\frac{1}{2\alpha^2}$$

which is entirely independent of any direction vectors **U** and **V**. This is another finite valued curvature invariant for the Schwarzschild form, and reaffirms that the Schwarzschild form cannot be extended.

Thus, there are no curvature singularities, no 'infinite curvatures', in the Schwarzschild form, contrary to the standard claims. All curvature invariants take finite values everywhere in the Schwarzschild form.

Similar results obtain for the other alleged black hole forms. For instance, the Kretschmann scalar for the Reissner-Nordström form is [72],

$$f = \frac{8\left\{6\left(\frac{\alpha}{2}R_c - q^2\right)^2 + q^4\right\}}{R_c^8}$$

$$R_c = \left(\left|r - r_0\right|^n + \xi^n\right)^{1/n}$$

$$\xi = \frac{\alpha}{2} + \sqrt{\frac{\alpha^2}{4} - q^2}$$

In this case when $r = r_0$ the Kretschmann scalar takes the value [72, 80],

$$f = \frac{8\left\{6\left(\frac{\alpha}{2}\xi - q^2\right)^2 + q^4\right\}}{\xi^8}$$

which is finite irrespective of the values of r_0 and n.

At $r = r_0$ the Gaussian curvature K for the spherically symmetric geodesic surface in the spatial section of the Reissner-Nordström form has the finite value (see Appendix A),

$$K = \frac{1}{\xi^2}$$

Note that if $q = 0$ all these curvature invariants reduce to that for the Schwarzschild form.

The Riemannian curvature for the spatial section of the Reissner-Nordström form is given by (see appendix B),

$$K_S = \frac{A+B}{C+D}$$

wherein,

$$A = -\frac{\left(\alpha R_c - 2q^2\right)}{2}\left(W_{1212} + W_{1313}\sin^2\theta\right)$$

$$B = \left(\alpha R_c - q^2\right)\left(R_c^2 - \alpha R_c + q^2\right)W_{2323}\sin^2\theta$$

$$C = R_c^4\left(W_{1212} + W_{1313}\sin^2\theta\right)$$

$$D = R_c^4\left(R_c^2 - \alpha R_c + q^2\right)W_{2323}\sin^2\theta$$

where again, in turn, R_c is given by expression (A17) and the W_{ijkl} by the determinant product,

$$W_{ijkl} = \begin{vmatrix} U^i & U^j \\ V^i & V^j \end{vmatrix}\begin{vmatrix} U^k & U^l \\ V^k & V^l \end{vmatrix}$$

Once again, when $r = r_0$ in (A17), the Riemannian curvature of the spatial section of the Reissner-Nordström form is,

$$K_s = \frac{-\left(\alpha\xi - 2q^2\right)}{2\xi^4}$$

or explicitly,

$$K_s = \frac{2q^2 - \dfrac{\alpha^2}{2} - \alpha\sqrt{\dfrac{\alpha^2}{4} - q^2}}{2\left(\dfrac{\alpha}{2} + \sqrt{\dfrac{\alpha^2}{4} - q^2}\right)^4}$$

since in this case $R_c(r_0) = \xi$, where, according to expression (A17),

$$\xi = \frac{\alpha}{2} + \sqrt{\frac{\alpha^2}{4} - q^2}$$

Note that if $q = 0$, the Riemannian curvature reduces to that for the Schwarzschild form.

Thus, there are no curvature singularities in the Reissner-Nordström form either [72, 73, 80], once again contrary to the standard claims.

Similar curvature invariants can be deduced for the Kerr and Kerr-Newman forms by means of equations (A17) (see Appendices A and B).

That none of the 'black hole' metrics can be extended to produce a black hole is reaffirmed yet again by considering the acceleration of a point in the Schwarzschild form. Doughty [81] has shown that the acceleration β of a point along a radial geodesic in the Schwarzschild manifold is given by the following form (see Appendix C),

$$\beta = \frac{\sqrt{-g_{11}}\left(-g^{11}\right)\left|\dfrac{\partial g_{00}}{\partial r}\right|}{2g_{00}}$$

where

$$g^{11} = \frac{1}{g_{11}} = -\left(1 - \frac{\alpha}{r}\right)$$

$$\frac{\partial g_{00}}{\partial r} = \frac{\partial}{\partial r}\left(1 - \frac{\alpha}{r}\right) = \frac{\alpha}{r^2}$$

From expressions (32), the radial acceleration is given explicitly,

$$\beta = \frac{\alpha}{2R_c^2\sqrt{1 - \dfrac{\alpha}{R_c}}}$$

$$R_c = \left(\left|r - r_o\right|^n + \alpha^n\right)^{1/n}$$

$$r, r_0 \in \Re \quad n \in \Re^+$$

Then $r \to r_0 \Rightarrow \beta \to \infty$, for all r_0 for all n.

In the case of the Reissner-Nordström form (see Appendix C) the acceleration of a point along a radial geodesic is given by,

$$\beta = \frac{\alpha R_c - 2q^2}{2R_c^2 \sqrt{R_c^2 - \alpha R_c + q^2}}$$

$$\xi = \frac{\alpha}{2} + \sqrt{\frac{\alpha^2}{4} - q^2} \qquad q^2 < \frac{\alpha^2}{4}$$

$$R_c = \left(\left|r - r_o\right|^n + \xi^n\right)^{1/n}$$

$$r, r_0 \in \Re \quad n \in \Re^+$$

which naturally reduces to that for the Schwarzschild form when $q = 0$; once again, $r \to r_0 \Rightarrow \beta \to \infty$, for all r_0 for all n.

Consequently,

$$r \to r_0 \Rightarrow \beta \to \infty$$

constitutes an invariant and hence reaffirms that the Schwarzschild and the Reissner-Nordström forms cannot be extended.

Nevertheless, unbeknown to the cosmologists, and Mr. ‘t Hooft, the acceleration approaches ∞ where, according to them, there is no matter! [44]. The mass of their black holes is located, they say, at their ‘curvature’ singularity, at their ‘origin’ $r = 0$, where their spacetime is ‘infinitely curved’.

VIII. Black hole universes contain no mass

Mr. ‘t Hooft [1] mocks me because I argue that black holes don’t exist on the grounds that no mass is present in the relevant field equations in the first place. The first section that he devotes to me on his webpage is titled, *"**Black holes do not exist,** they are solutions of the equation for the Ricci tensor $R_{\mu\nu} = 0$, so they cannot carry any mass. And what is usually called a "horizon" is actually a physical singularity."* [1]

Another section of his webpage, dedicated to me, is titled, *"**You can’t have massive objects near a black hole;** and you can’t have multiple black holes orbiting one another"* [1].

As I have already shown above, multiple black holes are inconsistent with the very definition of a black hole, and can’t exist in any of the alleged big bang universes either. Nonetheless Mr. ‘t Hooft superposes his many unspecified black holes upon some unspecified big bang universe. Also recall that the very same black hole equations also describe a star of the corresponding type. Since the alleged black hole is a one-mass universe according to its definition, it is physically meaningless. Stars exist, but they are not one-mass universes, so they too can’t be modelled by black hole equations. But Mr. ‘t Hooft vilifies me for arguing that the Einstein field equations $R_{\mu\nu} = 0$ contain no matter because in this case the energy-momentum tensor is zero (i.e. $T_{\mu\nu} = 0$). He goes on and on about test particles, and complains that,

“Mr. C attacks some generally accepted notions about black holes. It appears that the introduction of test particles is inadmissible to him. A test particle, freely falling in a gravitational field, feels no change in energy and momentum, and mathematically, we describe this situation in terms of comoving coordinate frames. This does not fit in C’s analysis, so, test particles are forbidden. A test particle is an object with almost no mass and almost no size, such as the space ship Cassini orbiting Saturn.” [1]

Mr. 't Hooft [1] also complains about me,

"*He has a problem with the notion of test particles, which are objects whose mass (and/or charge) is negligible for all practical purposes, so that they can be used as probes to investigate the properties of field configurations. Again, this is a question of making valid approximations in physics. A space ship such as the Cassini probe near Saturn, has mass, but it is far too light to have any effect on the planets and moons that it observes, so, its orbit is a geodesic as long as its engines are switched off. No physicist is surprised by these facts, but for C, approximations are inexcusable. For him, the Cassini probe cannot exist. Astrophysicists studying black holes routinely make the same assumptions. A valid question is: could the tiny effects of probes such as Cassini have explosive consequences for black holes or other solutions to Einstein's equations? You don't have to be a superb physicist - but you must have better intuitions than C - to conclude that such things do not happen.*"

Although Mr. 't Hooft harps on his test particles, they are located in some big bang universe that also allegedly contains many large masses, such as stars, galaxies, and untold numbers of black holes, despite the fact that the equations (metrics) for stars and black holes, being one and the same, don't contain any other masses whatsoever by their very definitions, and neither do any of the big bang universes.

It is not difficult to prove mathematically that $R_{\mu\nu} = 0$ actually contains no matter whatsoever and is therefore physically meaningless, and hence the black hole a figment of irrational imagination. First, according to Einstein [24, 32], his gravitational field equations are,

$$R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} = -\kappa T_{\mu\nu} \qquad (38)$$

If in (38) the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu}$ is used, these equations are compactly written as,

$$G_{\mu\nu} = -\kappa T_{\mu\nu} \qquad (39)$$

The Einstein tensor describes spacetime geometry (i.e. Einstein's gravitational field) and the material sources of his gravitational field are denoted by the energy-momentum tensor $T_{\mu\nu}$. Matter is the cause of Einstein's gravitational field as it induces by its presence curvature in his spacetime. Thus Einstein's field equations couple his gravitational field to its material sources. In words Einstein's field equations are just,

spacetime geometry = -κ(*material sources*)

Recall that according to Einstein [24], everything except his gravitational field is matter.

Einstein [24, 32] says that the field equations for his static gravitational field in the absence of matter are,

$$R_{\mu\nu} = 0 \qquad (40)$$

In words these equations are simply,

spacetime geometry = 0

Although equations (40) are not coupled to any material sources, since all matter is removed by setting $T_{\mu\nu} = 0$ (in which case $R = 0$ in (38)), Einstein nonetheless claims that equations (40) contain a massive source because they allegedly describe the gravitational field outside a body such as the Sun. Thus Einstein on the one hand removes all material sources by setting $T_{\mu\nu} = 0$ and on the other hand immediately reinstates the presence of a massive source with words (linguistic legerdemain) by alluding to a body outside of which equations (40) apply. After all, his gravitational field must be caused by matter: his gravitational field does not conjure itself up from nothing, and everything but the gravitational field is matter. Indeed, Einstein [31] refers to the 'Schwarzschild solution' for equations (40) as follows,

$$\text{“}\, ds^2 = \left(1 - \frac{A}{r}\right)dl^2 - \left[\frac{dr^2}{1 - \frac{A}{r}} + r^2\left(\sin^2\theta\, d\varphi^2 + d\theta^2\right)\right] \quad (109a)$$

$$A = \frac{\kappa M}{4\pi}$$

M denotes the sun's mass centrally symmetrically placed about the origin of co-ordinates; the solution (109a) is valid only outside this mass, where all the $T_{\mu\nu}$ vanish."

According to Einstein his equation (109a) contains a massive source, at "*the origin*", yet also according to Einstein his equations (40), from which (109a) is obtained, contain no matter. Thus Einstein's argument is a contradiction and therefore false. This contradiction is readily amplified by comparison to the 'field equations',

$$R_{\mu\nu} = \lambda g_{\mu\nu} \quad (41)$$

In words these equations are,

spacetime geometry = λ(metric tensor)

Here λ is the so-called 'cosmological constant', which is said to be tiny in magnitude. The solution for equations (41) is de Sitter's empty universe. It's empty because it contains no matter:

"*This is not a model of relativistic cosmology because it is devoid of matter.*" [38]

"*the de Sitter line element corresponds to a model which must strictly be taken as completely empty.*" [82]

"*the solution for an entirely empty world.*" [83]

"*there is no matter at all*!" [84]

Now note that in both equations (40) and (41) the energy-momentum tensor is zero ($T_{\mu\nu} = 0$). Thus, according to Einstein and his followers when the energy-momentum tensor is zero material sources are both present and absent. However, matter cannot be both present and absent by the very same mathematical constraint.

Since de Sitter's universe is devoid of matter by virtue of $T_{\mu\nu} = 0$, the 'Schwarzschild solution' must also be devoid of matter by the very same condition. Thus, equations (40) contain no matter. But it is upon equations (40) that all black holes rely. Thus, once again, no black hole solution has any physical meaning and so black holes are not predicted by General Relativity at all – they don't have any basis in any theory or observation, since Newton's theory does not predict black holes either, contrary to the claims of the astrophysical scientists [85]; and nobody has ever found a black hole [63]. Nonetheless, according to Hawking and Ellis [42],

"*Laplace essentially predicted the black hole…*"

According to the Cambridge Illustrated History of Astronomy [86],

"*Eighteenth-century speculators had discussed the characteristics of stars so dense that light would be prevented from leaving them by the strength of their gravitational attraction; and according to Einstein's General Relativity, such bizarre objects (today's 'black holes') were theoretically* *possible as end-products of stellar evolution, provided the stars were massive enough for their inward gravitational attraction to overwhelm the repulsive forces at work*."

In part C of Box 24.1 in their book 'Gravitation', Misner, Thorne and Wheeler [17] include the Michell-Laplace dark body under the heading of 'BLACK HOLES'. In section 24.2 they include a copy of the cover of Laplace's paper '*Exposition du Syetème du Monde*', and a page from his paper, in French, beside two papers, one by Oppenheimer and Volkov, the other by Oppenheimer and Snyder, on neutron stars and gravitational contraction respectively, and a paper by Baade and Zwicky on neutron stars. All these papers are denoted as 'Figure 24.1', with this caption:

"*Two important arrivals on the scene: the neutron star (1933) and the black hole (1795, 1939). No proper account of either can forego general relativity.*"

According to Chandrasekhar [37],

"*That such a contingency can arise was surmised already by Laplace in 1798. Laplace argued as follows. For a particle to escape from the surface of a spherical body of mass M and radius R, it must be projected with a velocity v such that* $\frac{1}{2}v^2 > GM/R$; *and it cannot escape if* $v^2 < 2GM/R$. *On the basis of this last inequality, Laplace concluded that if* $R < 2GM/c^2 = R_s$ *(say) where c denotes the velocity of light, then light will not be able to escape from such a body and we will not be able to see it!*

"*By a curious coincidence, the limit* R_s *discovered by Laplace is exactly the same that general relativity gives for the occurrence of the trapped surface around a spherical mass.*"

But it is not "*a curious coincidence*" that General Relativity gives the same R_s "*discovered by Laplace*" because the Newtonian expression for escape velocity (4) is deliberately inserted *post hoc* into Hilbert's solution (2) by the proponents of the black hole in order to make a mass appear in equations that contain no material source.

The Michell-Laplace dark body is not a black hole [87 - 90]. It possesses an escape velocity at its surface, but the black hole has both an escape velocity and no escape velocity simultaneously at its 'surface' (i.e. event horizon); masses and light can leave the Michell-Laplace dark body, but nothing can leave the black hole; it does not require irresistible gravitational collapse to form, whereas the black hole does; it has no (infinitely dense) singularity, but the black hole does; it has no event horizon, but the black hole does; it has 'infinite gravity' nowhere, but the black hole has infinite gravity at its singularity; there is always a class of observers that can see the Michell-Laplace dark body, but there is no class of observers that can see the black hole; the Michell-Laplace dark body persists in a space which by consistent theory contains other Michell-Laplace dark bodies and other matter and they can interact with themselves and other matter, but the spacetime of all types of black hole pertains to a universe that contains only one mass (but actually contains no mass by mathematical construction) and so cannot interact with any other masses; the space of the Michell-Laplace dark body is 3-dimensional and Euclidean, but the black hole is in a 4-dimensional non-Euclidean (pseudo-Riemannian) spacetime; the space of the Michell-Laplace dark body is not asymptotically anything whereas the spacetime of the black hole is asymptotically flat or asymptotically curved; the Michell-Laplace dark body does not 'curve' a spacetime, but the black hole does. Therefore, the Michell-Laplace dark body does not possess the characteristics of the black hole and so it is not a black hole.

Mr. ‘t Hooft’s [1] test particle in the spacetime of $R_{\mu\nu} = 0$ has no meaning either since $R_{\mu\nu} = 0$ is physically meaningless. Not only does $R_{\mu\nu} = 0$ contain no matter it also violates other physical principles of General Relativity. According to Einstein [32] his Principle of Equivalence and his Special Theory of Relativity must hold in his gravitational field,

“*Let now K be an inertial system. Masses which are sufficiently far from each other and from other bodies are then, with respect to K, free from acceleration. We shall also refer these masses to a system of co-ordinates K', uniformly accelerated with respect to K. Relatively to K' all the masses have equal and parallel accelerations; with respect to K' they behave just as if a gravitational field were present and K' were unaccelerated. Overlooking for the present the question as to the 'cause' of such a gravitational field, which will occupy us later, there is nothing to prevent our conceiving this gravitational field as real, that is, the conception that K' is 'at rest' and a gravitational field is present we may consider as equivalent to the conception that only K is an 'allowable' system of co-ordinates and no gravitational field is present. The assumption of the complete physical equivalence of the systems of coordinates, K and K', we call the 'principle of equivalence'; this principle is evidently intimately connected with the law of the equality between the inert and the gravitational mass, and signifies an extension of the principle of relativity to co-ordinate systems which are in non-uniform motion relatively to each other. In fact, through this conception we arrive at the unity of the nature of inertia and gravitation.*” [32]

“*Stated more exactly, there are finite regions, where, with respect to a suitably chosen space of reference, material particles move freely without acceleration, and in which the laws of the special theory of relativity, which have been developed above, hold with remarkable accuracy.*” [32]

Note that both the Principle of Equivalence and Special Relativity are defined in terms of the *a priori* presence of multiple arbitrarily large finite masses and photons. There can be no multiple arbitrarily large finite masses and photons in a spacetime that contains no matter by mathematical construction, and so neither the Principle of Equivalence nor Special Relativity can manifest therein. But $R_{\mu\nu} = 0$ is a spacetime that contains no matter by mathematical construction. Furthermore, Mr. ‘t Hooft’s test particle, be it the “*space ship Cassini orbiting Saturn*” or otherwise, must surely constitute a finite region in which Special Relativity must hold in accordance with Einstein’s tenets, assuming that Special Relativity is valid in the first place, and if so, multiple arbitrarily large finite masses and photons must be able to be present anywhere. This is impossible for $R_{\mu\nu} = 0$.

It follows from this that Einstein’s field equations do not in fact reduce to $R_{\mu\nu} = 0$ when $T_{\mu\nu} = 0$.

Notwithstanding the facts, the astrophysical scientists see black holes in multitudes, throughout the galaxies, at the centres of galaxies, in binary systems, and colliding and merging. According to Chandrasekhar [37],

“*From what I have said, collapse of the kind I have described must be of frequent occurrence in the Galaxy; and black-holes must be present in numbers comparable to, if not exceeding, those of the pulsars. While the black-holes will not be visible to external observers, they can nevertheless interact with one another and with the outside world through their external fields.*

"*In considering the energy that could be released by interactions with black holes, a theorem of Hawking is useful. Hawking's theorem states that in the interactions involving black holes, the total surface area of the boundaries of the black holes can never decrease; it can at best remain unchanged (if the conditions are stationary).*

"*Another example illustrating Hawking's theorem (and considered by him) is the following. Imagine two spherical (Schwarzschild) black holes, each of mass ½M, coalescing to form a single black hole; and let the black hole that is eventually left be, again, spherical and have a mass M*."

According to Hawking [8],

"*Also, suppose two black holes collided and merged together to form a single black hole. Then the area of the event horizon of the final black hole would be greater than the sum of the areas of the event horizons of the original black holes*."

And according to Mr. 't Hooft [6],

"*We not only accept the existence of black holes, we also understand how they can actually form under various circumstances. Theory allows us to calculate the behavior of material particles, fields or other substances near or inside a black hole. What is more, astronomers have now identified numerous objects in the heavens that completely match the detailed descriptions theoreticians have derived.*"

IX. Big bang universes are one-mass universes

All big bang models treat the universe, after the initial bang from nothing (or, semantically, a reified mathematical 'singularity'), as being entirely filled by a single continuous indivisible homogeneous distribution of matter of uniform macroscopic density and pressure. This continuous distribution of matter is given the form of an idealised fluid that completely fills the universe. For instance, according to Tolman [82],

"… *it must be remembered that these quantities apply to the idealized fluid in the model, which we have substituted in place of the matter and radiation actually present in the real universe*."

"*We may, however, introduce a more specific hypothesis by assuming that the material filling the model can be treated as a perfect fluid*."

The multiple black holes merging or colliding or capturing other matter or forming binary systems, the many stars and galaxies, and the radiation too that appear in big bang models is therefore inconsistent with the very basis of the models, and are obtained by invalid application of the Principle of Superposition. Tolman [82] reveals this explicitly,

"*We can then treat the universe as filled with a continuous distribution of fluid of proper macroscopic density ρ_{oo} and pressure p_o, and shall feel justified in making this simplification since our interest lies in obtaining a general framework for the behaviour of the universe as a whole, on which the details of local occurrences could be later superposed*."

However, the Principle of Superposition is not valid in General Relativity. Nonetheless, superposition is inadmissibly applied to obtain multiple masses, radiation and multiple black holes in big bang creation models.

Mr. 't Hooft [1, 6, 7] talks of multiple black holes, and other matter such as stars and planets, presumably in some big bang universe. In 2008 Mr. 't Hooft [62] wrote in an email to me,

"*Black holes can be in the vicinity of other black holes*."

Hence, his big bang universes are riddled with infinite 'gravitational fields' at the singularities of all his black holes, where spacetime curvature is infinite, and where the density is also infinite thereby violating into the bargain the uniform macroscopic density of all the one-mass big bang models. None of Mr. 't Hooft's black holes are asymptotically anything in his multiple black hole universe, and this violates the very definition of the black hole as well.

X. Einstein's gravitational waves and the usual conservation of energy and momentum

Mr. 't Hooft [1] first mentions me in the section of his webpage titled "***Einstein's equations for gravity are incorrect, they have no dynamical solutions, and do not imply gravitational waves as described in numerous text books***." In this section he derides Dr. Lo, but includes me, as follows,

"*Apparently, he fails to understand where the energy in a gravitational wave packet comes from, thinking that it is not given by Einstein's equations, a misconception that he shares with Mr. C. Due to the energy that should exist in a gravitational wave, gravity should interact with itself. Einstein's equation should have a term describing gravity's own energy. In fact, it does. This interaction is automatically included in Einstein's equations, because, indeed, the equations are non-linear, but neither L nor C appear to comprehend this*." [1]

Mr. 't Hooft has offered no evidence to support his claim that I think that Einstein's gravitational energy "*is not given by Einstein's equations*"; and for good reason – there is none. This is another false allegation that he has conveniently conjured up by means of his imagination. None of my papers [64] even remotely suggests Mr. 't Hooft's claim, and neither does our email communications [62].

Mr. 't Hooft goes on to explain his division of a metric $g_{\mu\nu}$ into two parts; a flat spacetime background $g^o_{\mu\nu}$ and a dynamical part $g^1_{\mu\nu}$, in order to account for Einstein's alleged gravitational waves; thus $g_{\mu\nu} = g^o_{\mu\nu} + g^1_{\mu\nu}$. He then makes the following remarks [1],

"*The stress-energy-momentum tensor can then be obtained routinely by considering infinitesimal variations of the background part, just like one does for any other type of matter field; the infinitesimal change of the total action (the space-time integral of the Lagrange density) then yields the stress-energy-momentum tensor. Of course, one finds that the dynamical part of the metric indeed carries energy and momentum, just as one expects in a gravitational field. As hydro-electric plants and the daily tides show, there's lots of energy in gravity, and this agrees perfectly with Einstein's original equations. In spite of DC calling it 'utter madness, this procedure works just perfectly. L and C shout that this stress-energy-momentum tensor is a 'pseudotensor'*."

Let's now investigate how Einstein fed the conservation of energy and momentum of his gravitational field and its material sources into his field equations.

It must first be noted that when Einstein talks of the conservation of energy and momentum he means that the sum of the energy and momentum of his gravitational field and its material sources is conserved in the usual way for a closed system, as experiment attests, for otherwise his theory would be in conflict with a vast array of experiments and therefore invalid. Einstein [31] emphasises that,

"It must be remembered that besides the energy density of the matter there must also be given an energy density of the gravitational field, so that there can be no talk of principles of conservation of energy and momentum of matter alone."

Mr. 't Hooft [1] acknowledges Einstein,

"*The truth is that gravitational energy plus material energy together obey the energy conservation law. We can understand this just as we have explained it for gravitational waves*."

Consider Einstein's field equations in the following form,

$$R_{uv} = -\kappa\left(T_{uv} - \frac{1}{2}Tg_{uv}\right) \tag{42}$$

According to Einstein when $T_{uv} = 0$, and hence $T = 0$, this reduces to,

$$R_{uv} = 0 \tag{43}$$

The solution to (43) is Schwarzschild's solution. It is routine amongst astrophysical scientists to consider a 'weak' gravitational field and a very slow moving 'particle' in relation to the 'Schwarzschild solution' to finally obtain an expression for the component of the metric tensor g_{00} in terms of the Newtonian potential function ϕ. The inclusion of ϕ in $g_{00,}$ although standard, is *ad hoc*, by means of a false analogy with Newton's theory, as explained above in relation to equation (4). Equations (43) are Einstein's analogue of the Laplace equation.

Eventually the divergence of the Newtonian potential function is often equated to R_{00} to obtain the Poisson equation by assuming a particular form for T_{00}. One can't use the 'Schwarzschild solution' to effect this analogue of the Poisson equation since (43) is allegedly an analogue of the Laplace equation. When Einstein developed his analogue of the Poisson equation he had no 'Schwarzschild solution' to work with. Instead he began with his analogue of the Laplace equation and attributed energy and momentum to his gravitational field, the latter he described by the following form of (43), with a constraint [24, 32],

$$\frac{\partial \Gamma^a_{uv}}{\partial x^a} + \Gamma^a_{ub}\Gamma^b_{va} = 0 \tag{44}$$

$$\sqrt{-g} = 1$$

Einstein writes the Christoffel symbol of the second kind as,

$$\Gamma^a_{bc} = -\frac{1}{2}g^{ad}\left(\frac{\partial g_{dc}}{\partial x^b} - \frac{\partial g_{cb}}{\partial x^d} + \frac{\partial g_{bd}}{\partial x^c}\right)$$

Einstein [24] proceeded from his analogue of the Laplace equation, equations (44), to his analogue of the Poisson equation. Using equations (44) he first alleged the conservation of the energy-momentum of his gravitational field by introducing his so-called 'pseudotensor', t^α_σ, via a Hamiltonian form of equations (44). According to Einstein [24] the components of his pseudotensor are,

"*the 'energy components' of the gravitational field*".

His conservation law for his gravitational field alone is by means of an ordinary divergence of t^α_σ, not a tensor divergence, since t^α_σ is not a tensor, and therefore in conflict with his tenet that all the equations of physics be covariant tensor expressions. He and his followers to this day attempt to justify this procedure on the basis that t^α_σ acts 'like a tensor' under linear transformations of coordinates. Nevertheless, this does not make t^{α}_{σ} a tensor. After a long-winded set of calculations Einstein [24] produces the ordinary divergence,

$$\frac{\partial t_{\sigma}^{\alpha}}{\partial x_{\alpha}} = 0 \qquad (45)$$

and proclaims a conservation law, but only for the energy and momentum of his gravitational field,

"*This equation expresses the law of conservation of momentum and energy for the gravitational field.*" [24]

Einstein then replaces equations (44) with the following,

$$\frac{\partial}{\partial x_{\alpha}}\left(g^{\sigma\beta}\Gamma^{\alpha}_{\mu\beta}\right) = -\kappa\left(t_{\mu}^{\sigma} - \frac{1}{2}\delta_{\mu}^{\sigma} t\right) \qquad (46)$$

$$\sqrt{-g} = 1$$

Equations (46) are still Einstein's proposed analogue of the Laplace equation. To get his analogue of the Poisson equation he simply adds a term for the material sources of his gravitational field, namely, his energy-momentum tensor T^{σ}_{μ}, thus[4],

"*The system of equation (51) shows how this energy-tensor (corresponding to the density ρ in Poisson's equation) is to be introduced into the field equations of gravitation. For if we consider a complete system (e.g. the solar system), the total mass of the system, and therefore its total gravitating action as well, will depend on the total energy of the system, and therefore on the ponderable energy together with the gravitational energy. This will allow itself to be expressed by introducing into (51), in place of the energy-components of the gravitational field alone, the sums $t^{\sigma}_{\mu} + T^{\sigma}_{\mu}$ of the energy-components of matter and of gravitational field. Thus instead of (51) we obtain the tensor equation*

$$\frac{\partial}{\partial x_{\alpha}}\left(g^{\sigma\beta}\Gamma^{\alpha}_{\mu\beta}\right) = -\kappa\left[\left(t_{\mu}^{\sigma} + T_{\mu}^{\sigma}\right) - \frac{1}{2}\delta_{\mu}^{\sigma}\left(t + T\right)\right] \qquad (52)$$

$$\sqrt{-g} = 1$$

where we have set $T = T^{\mu}_{\mu}$ (Laue'salar). These are the required general equations of gravitation in mixed form." [24]

Recall that Mr. 't Hooft [1] invoked "*a 'pseudotensor'*" in relation to the conservation of the energy and momentum of Einstein's shadowy gravitational waves, and mocks me for my rejection of it. The overt problem is that Einstein's pseudotensor is not a tensor and is therefore coordinate dependent. This is not in keeping with Einstein's requirement that all the equations of physics must be coordinate independent by means of tensor relations.

"*It is to be noted that t^{α}_{σ} is not a tensor*" [24]

"*Let us consider the energy of these waves. Owing to the pseudo-tensor not being a real tensor, we do not get, in general, a clear result independent of the coordinate system.*" [41]

"*It is not possible to obtain an expression for the energy of the gravitational field satisfying both the conditions: (i) when added to other forms of energy the total energy is conserved, and (ii) the energy within a definite (three dimensional) region at a certain time is independent of the coordinate system. Thus, in general, gravitational energy cannot be localized. The best we can do is to use the pseudo-tensor, which satisfies condition (i) but not (ii). It gives us approximate information about gravitational energy, which in some special cases can be accurate.*" [41]

[4] Einstein's equation (51) is equation (46) herein.

However, besides coordinate dependence there is an even more compelling reason to reject Einstein's pseudotensor; it is a meaningless concoction of mathematical symbols and therefore can't be used to represent any entity, to model any phenomena, or to make any calculations!

Definition 5 (Class of a Riemannian Metric): *Let φ be a Riemannian metric in the n variables $x^1, \ldots, x^n$. If σ is sufficiently large then n + σ functions $y^1, \ldots, y^{n+\sigma}$ of the x^i can be chosen such that,*

$$\varphi = (dy^1)^2 + \ldots + (dy^{n+\sigma})^2.$$

Let m be the smallest possible value for σ such that,

$$0 \le m \le \frac{n(n-1)}{2}.$$

Then m is called the class of the Riemannian metric φ. [91]

Theorem 2: *Metrics of zero class (of any number of variables n) are characterised by the necessary and sufficient condition that their Riemann-Christoffel curvature tensor vanishes identically.* [91]

In General Relativity the Riemann-Christoffel curvature tensor does not vanish identically [24].

Theorem 3: *Metrics φ of class zero have no non-zero differential invariants. Metrics of non-zero class have no first order differential invariants. The invariants greater than one are the invariants of φ, the Riemann-Christoffel curvature tensor, and its covariant derivatives.* [91]

Now Einstein's pseudotensor t^{α}_{σ} is defined as [24, 32],

$$\kappa t^{\alpha}_{\sigma} = \frac{1}{2}\delta^{\alpha}_{\sigma} g^{\mu\nu}\Gamma^{\lambda}_{\mu\beta}\Gamma^{\beta}_{\nu\lambda} - g^{\mu\nu}\Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\sigma}$$

wherein κ is a constant and δ^{α}_{σ} is the Kronecker-delta. Contract Einstein's pseudotensor by setting σ = α to yield the invariant $t = t^{\alpha}_{\alpha}$, thus,

$$\kappa t^{\alpha}_{\alpha} = \kappa t = \frac{1}{2}\delta^{\alpha}_{\alpha} g^{\mu\nu}\Gamma^{\lambda}_{\mu\beta}\Gamma^{\beta}_{\nu\lambda} - g^{\mu\nu}\Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha}$$

Since the $\Gamma^{\alpha}_{\beta\sigma}$ are functions only of the components of the metric tensor and their first derivatives, *t* is seen to be a first-order intrinsic differential invariant [91, 92], i.e. it is an invariant that depends solely upon the components of the metric tensor and their first derivatives. However, by **Theorem 3** this is impossible. Hence, by *reductio ad absurdum*, Einstein's pseudotensor is a meaningless concoction of mathematical symbols, and therefore, contrary to Einstein, the astrophysical scientists, and Mr. 't Hooft, it can't be used to make any calculations, to represent any physical quantity, or to model any physical phenomena, such as Einstein's ghostly gravitational waves.

The Landau-Lifshitz [93] pseudotensor is often used in place of Einstein's; however, it suffers from precisely the same defects as Einstein's and it is therefore also a meaningless concoction of mathematical symbols. All the so-called gravitational 'pseudotensors' share these fatal defects.

Einstein and the astrophysical scientists nonetheless permit his pseudotensor, and do calculations with it, as does Mr. 't Hooft [1] who says,

"*…and there's nothing wrong with a definition of energy, stress and momentum that's frame dependent, as long as energy and momentum are conserved.*"

The conservation of energy and momentum Mr. ‘t Hooft refers to is that usual for a closed system, as determined by experiments.

From Einstein’s equation (*52*) the total energy-momentum E, of his gravitational field and its material sources, is,

$$E = (t^{\sigma}{}_{\mu} + T^{\sigma}{}_{\mu}) \qquad (47)$$

This is still not a tensor expression, so Einstein can’t take a tensor divergence. He then takes the ordinary divergence to get [24],

$$\frac{\partial\left(t_{\mu}^{\sigma} + T_{\mu}^{\sigma}\right)}{\partial x_{\alpha}} = 0 \qquad (48)$$

and proclaims the usual conservation laws of energy and momentum for a closed system,

“*Thus it results from our field equations of gravitation that the laws of conservation of momentum and energy are satisfied*.” [24]

Compare now equation (42) with the equivalent forms,

$$R_{\nu}^{\mu} = -\kappa\left(T_{\nu}^{\mu} - \frac{1}{2}Tg_{\nu}^{\mu}\right) \qquad (49)$$

$$T_{\nu}^{\mu} = -\frac{1}{\kappa}\left(R_{\nu}^{\mu} - \frac{1}{2}Rg_{\nu}^{\mu}\right) \qquad (50)$$

Thus by (49), according to Einstein, if $T^{\mu}{}_{\nu} = 0$ then $R^{\mu}{}_{\nu} = 0$. But by (50), if $R^{\mu}{}_{\nu} = 0$ then $T^{\mu}{}_{\nu} = 0$. In other words, $R^{\mu}{}_{\nu}$ and $T^{\mu}{}_{\nu}$ must vanish identically – if there are no material sources then there is no gravitational field, and no universe. Bearing this in mind, and in view of (40) and (41), consideration of the conservation of energy and momentum, and tensor relations, Einstein’s field equations must take the following form [92, 94],

$$\frac{G_{\nu}^{\mu}}{\kappa} + T_{\nu}^{\mu} = 0 \qquad (51)$$

Comparing this to expression (47) it is clear that the $G^{\mu}{}_{\nu}/\kappa$ actually constitute the energy-momentum components of Einstein’s gravitational field, which is rather natural since the Einstein tensor $G^{\mu}{}_{\nu}$ describes the curvature of Einstein’s spacetime (i.e. his gravitational field), and that (51) also constitutes the total energy-momentum of Einstein’s gravitational field and its material sources. Unlike (47), expression (51) is a tensor expression. The tensor (covariant derivative) divergence of the left side of (51) is zero and therefore constitutes a conservation law for Einstein’s gravitational field and its material sources $T^{\mu}{}_{\nu}$.

However, the total energy-momentum of (51) is always zero, the G_{uv}/κ and the T_{uv} must vanish identically (i.e. when $T_{uv} = 0$, $G_{uv} = 0$, and vice-versa, producing the identity 0 = 0), and gravitational energy can’t be localised [92]. Moreover, since the total energy-momentum is always zero the usual conservation laws for energy and momentum for a closed system can’t be satisfied. General Relativity is therefore in conflict with a vast array of experiments on a fundamental level.

The so-called ‘cosmological constant’ can be easily included as follows,

$$\frac{\left(G_{\nu}^{\mu} + \lambda g_{\nu}^{\mu}\right)}{\kappa} + T_{\nu}^{\mu} = 0 \qquad (52)$$

In this case the energy-momentum components of Einstein’s gravitational field are given by $(G^{\mu}{}_{\nu} + \lambda g^{\mu}{}_{\nu})/\kappa$. The $G^{\mu}{}_{\nu}$, $g^{\mu}{}_{\nu}$, and $T^{\mu}{}_{\nu}$ must all vanish identically, and all the same consequences ensue just as for equation (51). Thus, if there is no material source, not only is there no gravitational field, there is no universe, and Einstein’s field equations violate the usual conservation of energy and momentum for a closed system and are thereby in conflict with a vast array of experiments.

Recall that Mr. ‘t Hooft [1] splits the metric tensor into two parts, a flat ‘background’ spacetime and a dynamical spacetime, as follows,

$$g_{\mu\nu} = g^{o}_{\mu\nu} + g^{1}_{\mu\nu} \tag{53}$$

This procedure is the so-called ‘linearisation’ of Einstein’s field equations. With this procedure Mr. ‘t Hooft [1] says,

“*The dynamical part, $g^{1}_{\mu\nu}$, is defined to include all the ripples of whatever gravitational wave one wishes to describe*.”

The linearisation procedure leads to the following alleged gravitational wave equation in empty spacetime,

$$\Box^2 g^{1}_{\mu\nu} = 0 \tag{54}$$

where the d’Alembertian operator is defined,

$$\Box^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2 - c^{-2}\partial^2/\partial t^2$$

and where c is the speed of light in vacuo. Quite often, as in the case of Hilbert’s solution (1), c is set to unity, in which case the d’Alembertian operator is,

$$\Box^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2 - \partial^2/\partial t^2$$

From expression (54) it is claimed that the speed of propagation of Einstein’s gravitational waves is the speed of light in vacuo. For instance, according to Foster and Nightingale [95],

“*… we see that gravitational radiation propagates through empty spacetime with the speed of light*.”

However, the speed of propagation of these alleged gravitational waves is coordinate dependent and therefore not unique. For instance, concerning equation (54), Eddington [83] noted that,

“*… the deviations of the gravitational potentials are propagated as waves with unit velocity, i.e. the velocity of light. But it must be remembered that this representation of the propagation, though always permissible, is not unique. ... All the coordinate-systems differ from Galilean coordinates by small quantities of the first order. The potentials $g_{\mu\nu}$ pertain not only to the gravitational influence which is objective reality, but also to the coordinate-system which we select arbitrarily. We can ‘propagate’ coordinate-changes with the speed of thought, and these may be mixed up at will with the more dilatory propagation discussed above. There does not seem to be any way of distinguishing a physical and a conventional part in the changes of the $g_{\mu\nu}$.*

“*The statement that in the relativity theory gravitational waves are propagated with the speed of light has, I believe, been based entirely upon the foregoing investigation; but it will be seen that it is only true in a very conventional sense. If coordinates are chosen so as to satisfy a certain condition which has no very clear geometrical importance, the speed is that of light; if the coordinates are slightly different the speed is altogether different from that of light. The result stands or falls by the choice of coordinates and, so far as can be judged, the coordinates here used were purposely introduced in order to obtain the simplification which results from representing the propagation as occurring with the speed of light. The argument thus follows a vicious circle*.”

Recall that Einstein’s pseudotensor represents the energy-momentum of his gravitational field alone. Mr. ‘t Hooft [1] says, "*Actually, one can define the energy density in different ways, since one has the freedom to add pure gradients to the energy density, without affecting the total integral, which represents the total energy, which is conserved. Allowing this, one might consider the Einstein tensor $G_{\mu\nu}$ itself to serve as the gravitational part of the stress-energy-momentum tensor, but there would be problems with such a choice.*

"*The definition using a background metric (which produces only terms that are quadratic in the first derivatives) is much better, and there's nothing wrong with a definition of energy, stress and momentum that's frame dependent, as long as energy and momentum are conserved. In short, if one wants only first derivatives, either frame dependence or background metric dependence are inevitable*.

"*…In spite of DC calling it 'utter madness', this procedure works just perfectly. L and C shout that this stress-energy-momentum tensor is a 'pseudotensor'.*"

However, all attempts to account for the energy-momentum of Einstein's gravitational field, and hence his 'gravitational waves', by means of a pseudotensor are futile. Consequently, General Relativity violates the usual conservation of energy and momentum for a closed system as determined by experiments. Equation (51) is the form that Einstein's field equations must take. Consequently, the search for Einstein's gravitational waves has from the outset been a search for that which does not exist. It is no wonder that no such waves have ever been detected.

XI. Functional analysis

Mr. 't Hooft [1] says of the five scientists he vilifies,

"*These self proclaimed scientists in turn blame me of 'not understanding functional analysis*'."

Mr. 't Hooft has offered no evidence for this allegation either. All we have is his word for it. I don't know whether or not any of the other four scientists Mr. 't Hooft vilifies on his webpage has made this accusation against him, but certainly I have never done so. In our email exchange I have accused him of other things, but strangely he has not cared to mention them, whereas I hide nothing [62].

In his final email, copied to me in 2008 but addressed to another, Mr. 't Hooft wrote,

"*O, yes, excerpts from my mail will probably emerge on some weblogs,* (sic) *drawn out of context and ornamented with comments*."

I refer readers again to [62] for confirmation of Mr. 't Hooft's context and to the contextualization of my 'ornaments'.

DEDICATION

In memory of my brother,

Paul Raymond Crothers
12th May 1960 – 25th December 2008

and my Uncle,

Gary Christopher Crothers
3rd June 1935 – 10th November 2013

APPENDIX A – GAUSSIAN CURVATURE

Gaussian curvature is an intrinsic geometric property of a surface. As such it is independent of any embedding space. All black hole spacetime metrics contain a surface from which various invariants and geometric identities can be deduced by purely mathematical means. The Kerr-Newman form subsumes the Kerr, Reissner-Nordström, and Schwarzschild forms. The Gaussian curvature of the surface in the Kerr-Newman metric therefore subsumes the Gaussian curvatures of the surfaces in the subordinate forms too. The Gaussian curvature reveals the type of surface and uniquely identifies the terms that appear in its general form. The Gaussian curvature demonstrates that no so-called black hole metric can in fact be extended to produce a black hole. The Gaussian curvature of the surface in the Kerr-Newman metrical ground-form and its subordinate metrics is determined as follows. The Kerr-Newman metric in Boyer-Lindquist coordinates is,

$$ds^2 = -\frac{\Delta - a^2\sin^2\theta}{\rho^2}dt^2 - \frac{2a\sin^2\theta\left(r^2+a^2-\Delta\right)}{\rho^2}dt\,d\varphi + \\ + \frac{\left(r^2+a^2\right)^2 - a^2\Delta\sin^2\theta}{\rho^2}\sin^2\theta\,d\varphi^2 + \\ + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2$$

$$\Delta = r^2 - 2mr + a^2 + q^2 \qquad \rho^2 = r^2 + a^2\cos^2\theta$$

$$0 \le r \tag{A1}$$

If r = *constant* ≠ 0 and t = *constant*, (A1) reduces to,

$$ds^2 = \rho^2 d\theta^2 + \frac{\left(r^2+a^2\right)^2 - a^2\Delta\sin^2\theta}{\rho^2}\sin^2\theta\,d\varphi^2$$

$$\Delta = r^2 - 2mr + a^2 + q^2 \qquad \rho^2 = r^2 + a^2\cos^2\theta \tag{A2}$$

Metric (A2) is a particular form of equation (5) of the First Fundamental Quadratic Form for a surface. The components of the metric tensor of (A2) are,

$$g_{11} = \rho^2 \qquad g_{22} = \frac{\left(r^2+a^2\right)^2 - a^2\Delta\sin^2\theta}{\rho^2}\sin^2\theta \tag{A3}$$

To facilitate the calculation of the Gaussian curvature of the surface described by (A2), make the following substitutions,

$$x^1 = \theta \qquad x^2 = \varphi \qquad f = \left(r^2 + a^2\right)^2 - a^2 \Delta \sin^2\theta$$
$$h = \rho^2 = r^2 + a^2\cos^2\theta \qquad \beta = \frac{f}{h} \tag{A4}$$
$$g_{11} = h \qquad g_{22} = \beta \sin^2\theta$$

Accordingly,

$$\frac{\partial f}{\partial \theta} = -2a^2 \Delta \sin\theta\cos\theta \qquad \frac{\partial h}{\partial \theta} = -2a^2 \sin\theta\cos\theta$$

$$\frac{\partial^2 f}{\partial \theta^2} = 2a^2 \Delta\left(\sin^2\theta - \cos^2\theta\right) \qquad \frac{\partial^2 h}{\partial \theta^2} = 2a^2\left(\sin^2\theta - \cos^2\theta\right)$$

$$\frac{\partial \beta}{\partial \theta} = \frac{2a^2 \sin\theta\cos\theta\left(f - \Delta h\right)}{h^2}$$

$$\frac{\partial^2 \beta}{\partial \theta^2} = \frac{\left\{2a^2 h\left(\sin^2\theta - \cos^2\theta\right)\left(\Delta h - f\right) - 8a^4 \sin^2\theta\cos^2\theta\left(\Delta h - f\right)\right\}}{h^3}$$

$$\frac{\partial g_{22}}{\partial \theta} = \frac{\partial \beta}{\partial \theta}\sin^2\theta + 2\beta\sin\theta\cos\theta \tag{A5}$$

From equations (12),

$$R^1_{.212} = \frac{\partial \Gamma^1_{22}}{\partial \theta} - \frac{\partial \Gamma^1_{12}}{\partial \varphi} + \Gamma^1_{22}\Gamma^1_{11} + \Gamma^2_{22}\Gamma^1_{12} - \Gamma^1_{12}\Gamma^1_{12} - \Gamma^2_{12}\Gamma^1_{22} \tag{A6}$$

bearing in mind the symmetry $\Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\gamma\beta}$.

According to (A3) the metric tensor is diagonal and so from equations (12),

$$\Gamma^i_{ij} = \Gamma^i_{ji} = \frac{\partial\left(\frac{1}{2}\ln\left|g_{ii}\right|\right)}{\partial x^j} \qquad \Gamma^i_{jj} = \frac{-1}{2g_{ii}}\frac{\partial g_{jj}}{\partial x^i} \quad \left(i \neq j\right)$$

from which it follows that

$$\Gamma^1_{12} = \frac{1}{2}\frac{\partial \ln\left|g_{11}\right|}{\partial x^2} = \frac{1}{2}\frac{\partial \ln\left|h\right|}{\partial \varphi} = 0$$

$$\Gamma^1_{22} = \frac{-1}{2g_{11}}\frac{\partial g_{22}}{\partial x^1} = \frac{-1}{2h}\frac{\partial g_{22}}{\partial \theta} = \frac{-\left(\frac{\partial \beta}{\partial \theta}\sin^2\theta + 2\beta\sin\theta\cos\theta\right)}{2h}$$

$$\Gamma^2_{21} = \frac{1}{2}\frac{\partial \ln|g_{22}|}{\partial x^1} = \frac{1}{2}\frac{\partial \ln|\beta\sin^2\theta|}{\partial \theta} = \frac{1}{2\beta\sin^2\theta}\left(\frac{\partial \beta}{\partial \theta}\sin^2\theta + 2\beta\sin\theta\cos\theta\right)$$

$$\Gamma^1_{11} = \frac{1}{2}\frac{\partial \ln|g_{11}|}{\partial x^1} = \frac{1}{2}\frac{\partial \ln|h|}{\partial \theta} = \frac{-a^2\sin\theta\cos\theta}{h} \qquad \text{(A7)}$$

Using expressions (A7), expression (A6) reduces to,

$$R^1_{.212} = \frac{\partial \Gamma^1_{22}}{\partial \theta} + \Gamma^1_{22}\Gamma^1_{11} - \Gamma^2_{12}\Gamma^1_{22} \qquad \text{(A8)}$$

From (A7),

$$\frac{\partial \Gamma^1_{22}}{\partial \theta} = \frac{\left[\left(\frac{\partial \beta}{\partial \theta}\sin^2\theta + \beta\sin 2\theta\right)\frac{\partial h}{\partial \theta} - h\left(\frac{\partial^2 \beta}{\partial \theta^2}\sin^2\theta + 2\frac{\partial \beta}{\partial \theta}\sin 2\theta + 2\beta\cos 2\theta\right)\right]}{2h^2}$$

$$\Gamma^1_{22}\Gamma^1_{11} = \frac{a^2\sin\theta\cos\theta}{2h^2}\left(\frac{\partial \beta}{\partial \theta}\sin^2\theta + 2\beta\sin\theta\cos\theta\right)$$

$$\Gamma^2_{12}\Gamma^1_{22} = \Gamma^2_{21}\Gamma^1_{22} = \frac{-\left(\frac{\partial \beta}{\partial \theta}\sin^2\theta + 2\beta\sin\theta\cos\theta\right)^2}{4h\beta\sin^2\theta} \qquad \text{(A9)}$$

Putting (A9) into (A8) gives,

$$R^1_{.212} = \frac{\left[\left(\frac{\partial \beta}{\partial \theta}\sin^2\theta + \beta\sin 2\theta\right)\frac{\partial h}{\partial \theta} - h\left(\frac{\partial^2 \beta}{\partial \theta^2}\sin^2\theta + 2\frac{\partial \beta}{\partial \theta}\sin 2\theta + 2\beta\cos 2\theta\right)\right]}{2h^2} +$$

$$+\frac{a^2\sin\theta\cos\theta}{2h^2}\left(\frac{\partial \beta}{\partial \theta}\sin^2\theta + 2\beta\sin\theta\cos\theta\right) + \frac{\left(\frac{\partial \beta}{\partial \theta}\sin^2\theta + 2\beta\sin\theta\cos\theta\right)^2}{4h\beta\sin^2\theta}$$

(A10)

Now,

$$R_{1212} = g_{11}R^1_{.212} = hR^1_{.212}$$

and from (A4), $g_{11} = h$. Hence,

$$R_{1212} = \frac{\left[\left(\frac{\partial\beta}{\partial\theta}\sin^2\theta + \beta\sin 2\theta\right)\frac{\partial h}{\partial\theta} - h\left(\frac{\partial^2\beta}{\partial\theta^2}\sin^2\theta + 2\frac{\partial\beta}{\partial\theta}\sin 2\theta + 2\beta\cos 2\theta\right)\right]}{2h} +$$

$$+\frac{a^2\sin\theta\cos\theta}{2h}\left(\frac{\partial\beta}{\partial\theta}\sin^2\theta + 2\beta\sin\theta\cos\theta\right) + \frac{\left(\frac{\partial\beta}{\partial\theta}\sin^2\theta + 2\beta\sin\theta\cos\theta\right)^2}{4\beta\sin^2\theta}$$

(A11)

From (A3) and (A4) the determinant g of the metric tensor is,

$$g = g_{11}g_{22} = h\beta\sin^2\theta = f\sin^2\theta \qquad \text{(A12)}$$

The Gaussian curvature K is,

$$K = \frac{R_{1212}}{g} = \frac{R_{1212}}{f\sin^2\theta} \qquad \text{(A13)}$$

Putting (A11) into (A13) yields,

$$K = \frac{\left[\left(\frac{\partial\beta}{\partial\theta}\sin^2\theta + \beta\sin 2\theta\right)\frac{\partial h}{\partial\theta} - h\left(\frac{\partial^2\beta}{\partial\theta^2}\sin^2\theta + 2\frac{\partial\beta}{\partial\theta}\sin 2\theta + 2\beta\cos 2\theta\right)\right]}{2hf\sin^2\theta} +$$

$$+\frac{a^2\sin\theta\cos\theta}{2hf\sin^2\theta}\left(\frac{\partial\beta}{\partial\theta}\sin^2\theta + 2\beta\sin\theta\cos\theta\right) + \frac{\left(\frac{\partial\beta}{\partial\theta}\sin^2\theta + 2\beta\sin\theta\cos\theta\right)^2}{4\beta f\sin^4\theta}$$

(A14)

After simplifying terms, (A14) becomes,

$$K = \frac{1}{2hf}\frac{\partial\beta}{\partial\theta}\frac{\partial h}{\partial\theta} - \frac{a^2\cos^2\theta}{h^2} - \frac{1}{2f}\frac{\partial^2\beta}{\partial\theta^2} + \frac{1}{h} + \frac{a^2\sin\theta\cos\theta}{2hf}\frac{\partial\beta}{\partial\theta} + \frac{h}{4f^2}\left(\frac{\partial\beta}{\partial\theta}\right)^2 +$$

$$+\frac{2a^2\cos^2\theta(f - \Delta h)}{h^2 f}$$

(A15)

It is clearly evident from (A15) that the Gaussian curvature is not a positive constant and so the surface (A2) is not a spherical surface. Thus, the Kerr-Newman metric (A1) is not spherically symmetric.

By virtue of (A15) the quantity *r* in the Kerr-Newman metric is neither the radius nor a distance therein, as it is defined by (A15) owing to the intrinsic geometry of the metric (A2). Since the intrinsic geometry of a surface is independent of any embedding space the quantity *r* in (A2) retains its identity when (A2) is embedded in the Kerr-Newman spacetime (A1).

Note that if the alleged angular momentum is zero, i.e. $a = 0$, then by (A4) and (A5),

$$h = r^2 \qquad f = r^4 \qquad \frac{\partial h}{\partial \theta} = 0 \qquad \frac{\partial \beta}{\partial \theta} = 0 \qquad \frac{\partial^2 \beta}{\partial \theta^2} = 0$$

and so (A15) reduces to,

$$K = \frac{1}{r^2} \tag{A16}$$

The Kerr-Newman metric (A1) then reduces to the Reissner-Nordström metric for an alleged charged non-rotating body, including the corresponding 'black hole', since the charge is not zero (i.e. $q \neq 0$). By (A16) the Reissner-Nordström metric is spherically symmetric, and the quantity *r* therein is neither the radius nor a distance.

If both *a* and *q* are zero, the Kerr-Newman metric (A1) reduces to Hilbert's metric and the Gaussian curvature of the surface therein is again given by (A16), and so *r* therein is neither the radius nor a distance in Hilbert's metric.

Since the metric of (A1) is a generalisation of Schwarzschild's metric, it is in turn a certain element of an infinite set of equivalent metrics, but for an incorrect range on *r*. It has been shown [71 - 79] that the correct form of the Kerr-Newman solution, although also physically meaningless, is obtained from,

$$\begin{aligned} ds^2 = -\frac{\Delta - a^2\sin^2\theta}{\rho^2}dt^2 - \frac{2a\sin^2\theta\left(R_c^2 + a^2 - \Delta\right)}{\rho^2}dt\,d\varphi + \\ + \frac{\left(R_c^2 + a^2\right)^2 - a^2\Delta\sin^2\theta}{\rho^2}\sin^2\theta\,d\varphi^2 + \\ + \frac{\rho^2}{\Delta}dR_c^2 + \rho^2 d\theta^2 \end{aligned}$$

$$\Delta = R_c^2 - \alpha R_c + a^2 + q^2 \qquad \rho^2 = R_c^2 + a^2\cos^2\theta$$

$$R_c = \left(\left|r - r_0\right|^n + \xi^n\right)^{1/n} \qquad r, r_0 \in \Re \quad n \in \Re^+$$

$$\xi = \frac{\alpha}{2} + \sqrt{\frac{\alpha^2}{4} - q^2 - a^2\cos^2\theta} \qquad a^2 + q^2 < \frac{\alpha^2}{4}$$

(A17)

Here r_0 and n are entirely arbitrary. Since $R_c(r_0) = \xi$ for all r_0 for all n, none can be extended. If $a = 0$ and $q = 0$, then (A17) reduces to the Schwarzschild form, equations (32), none of which can be extended. The expressions (A17) generate an infinite set of equivalent metrics which cannot be extended.

When $\vartheta = 0$ and $\vartheta = \pi$

From (A4), (A5), when $\vartheta = 0$ and $\vartheta = \pi$,

$$h = r^2 + a^2 \qquad f = \left(r^2 + a^2\right)^2 \qquad \frac{\partial\beta}{\partial\theta} = 0 \qquad \frac{\partial h}{\partial\theta} = 0$$

$$\Delta = r^2 - 2mr + a^2 + q^2$$

$$\frac{\partial^2\beta}{\partial\theta^2} = \frac{2a^2\left(f - \Delta h\right)}{h^2} \tag{A18}$$

Then (A15) reduces to,

$$K = -\frac{a^2}{h^2} - \frac{1}{2f}\frac{\partial^2\beta}{\partial\theta^2} + \frac{1}{h} \tag{A19}$$

Putting expressions (A18) into (A19) yields,

$$K = \frac{r^2}{\left(r^2 + a^2\right)^2} + \frac{a^2\left(2mr - q^2\right)}{\left(r^2 + a^2\right)^3} \tag{A20}$$

Then from (A17),

$$R_c = \left(\left|r - r_0\right|^n + \xi^n\right)^{1/n} \qquad r, r_0 \in \Re \quad n \in \Re^+$$

$$\xi = \frac{\alpha}{2} + \sqrt{\frac{\alpha^2}{4} - q^2 - a^2} \qquad a^2 + q^2 < \frac{\alpha^2}{4}$$

(A21)

and so with (A20)

$$K = \frac{R_c^2}{\left(R_c^2 + a^2\right)^2} + \frac{a^2\left(\alpha R_c - q^2\right)}{\left(R_c^2 + a^2\right)^3} \tag{A22}$$

By (A21) the minimum value of R_c is $R_c(r_0) = \xi$. Thus at $\vartheta = 0$ and at $\vartheta = \pi$ the maximum of K is the invariant,

$$K = \frac{\xi^2}{\left(\xi^2 + a^2\right)^2} + \frac{a^2\left(\alpha\xi - q^2\right)}{\left(\xi^2 + a^2\right)^3} \tag{A23}$$

When $\vartheta = \pi/2$

From (A4), (A5), when $\vartheta = \pi/2$,

$$h = r^2 \qquad f = \left(r^2 + a^2\right)^2 - a^2\Delta \qquad \frac{\partial\beta}{\partial\theta} = 0 \qquad \frac{\partial h}{\partial\theta} = 0$$

$$\frac{\partial^2\beta}{\partial\theta^2} = \frac{2a^2\left(\Delta h - f\right)}{h^2} = \frac{2a^2\left(r^2 + a^2\right)\left(q^2 - 2mr\right)}{r^4}$$

$$\Delta = r^2 - 2mr + a^2 + q^2$$

(A24)

Then (A15) reduces to,

$$K = -\frac{1}{2f}\frac{\partial^2\beta}{\partial\theta^2} + \frac{1}{h} \tag{A25}$$

or,

$$K = \frac{1}{h} - \frac{a^2\left(\Delta h - f\right)}{h^2 f} \tag{A26}$$

and so,

$$K = \frac{1}{r^2} + \frac{a^2\left(r^2 + a^2\right)\left(2mr - q^2\right)}{r^4\left[r^2\left(r^2 + a^2\right) + a^2\left(2mr - q^2\right)\right]} \tag{A27}$$

Then from (A17),

$$R_c = \left(\left|r - r_0\right|^n + \xi^n\right)^{\frac{1}{n}} \qquad r, r_0 \in \Re \quad n \in \Re^+$$

$$\xi = \frac{\alpha}{2} + \sqrt{\frac{\alpha^2}{4} - q^2} \qquad q^2 < \frac{\alpha^2}{4} \tag{A28}$$

and so with (A27),

$$K = \frac{1}{R_c^2} + \frac{a^2\left(R_c^2 + a^2\right)\left(\alpha R_c - q^2\right)}{R_c^4\left[R_c^2\left(R_c^2 + a^2\right) + a^2\left(\alpha R_c - q^2\right)\right]} \tag{A29}$$

By (A28) the minimum value of R_c is $R_c(r_0) = \xi$. Thus at $\vartheta = \pi/2$ the maximum of K is the invariant,

$$K = \frac{1}{\xi^2} + \frac{a^2\left(\xi^2 + a^2\right)\left(\alpha\xi - q^2\right)}{\xi^4\left[\xi^2\left(\xi^2 + a^2\right) + a^2\left(\alpha\xi - q^2\right)\right]} \tag{A30}$$

Note that if $a = 0$ then (A22) and (A29) reduce to,

$$K = \frac{1}{R_c^2} \tag{A31}$$

for a spherical surface, and the associated invariant at $R_c(r_0)$ is,

$$K = \frac{1}{\xi^2} \tag{A32}$$

where ξ is given by (A17). For $a = 0$, $q \neq 0$ then (A32) is,

$$K = \frac{1}{\left[\frac{\alpha}{2} + \sqrt{\frac{\alpha^2}{4} - q^2}\right]^2} \tag{A33}$$

which is an invariant for the Reissner-Nordström form. If both $a = 0$ and $q = 0$ then this reduces to,

$$K = \frac{1}{\alpha^2} \tag{A34}$$

which is an invariant for the Schwarzschild form.

Note that in all cases the Gaussian curvature of the surface in the spatial section is finite everywhere.

There is no black hole

By (A17) the minimum value for Δ is,

$$\Delta_{\min} = a^2 \sin^2 \theta \tag{A35}$$

which occurs when $r = r_0$, irrespective of the values of r_0 and n. $\Delta_{\min} = 0$ only when $\vartheta = 0$ and when $\vartheta = \pi$, in which cases (A17), and hence (A1), are undefined.

Similarly, the minimum value of ρ^2 is,

$$\rho^2_{\min} = \xi^2 + a^2 \cos^2 \theta \tag{A36}$$

which occurs when $r = r_0$, irrespective of the values of r_0 and n. Since ξ is always greater than zero, ρ^2 can never be zero.

Since (A1) is generated from (A17) in the case of $r_0 = \xi$, $n = 1$, $r > r_0$, it cannot be extended, and ρ^2 can never be zero. This is amplified by the case of $r_0 = 0$, $n = 2$ in (A17). Then,

$$R_c = \left(r^2 + \xi^2\right)^{\frac{1}{2}} \tag{A37}$$

which is defined for all real values of r and can never be zero. If (A1) can be extended then so must (A17), and hence the case of (A37). But the case (A37) cannot be extended because the square of a real number is never less than zero. Thus (A1) cannot be extended either.

There is only ever one singularity in every equivalent metric generated by (A17), and this can only occur at $r = r_0$, whether or not $a = 0$ or $q = 0$ or both are zero.

Thus, there is no event horizon and no static limit, and hence no black hole, associated with (A1), or any other of the metrics generated by (A17), all of which are equivalent.

There is no event horizon associated with any output from (A17), whether or not a and q are zero or not. Thus, there is no black hole in any case.

According to (A17), $R_c = R_c(r, \vartheta)$, since ξ depends on the value of ϑ. Hence h, f, β and Δ are also functions of (r, ϑ). To incorporate all permissible values of ϑ, the connexion coefficient Γ^1_{11} must be generalised to,

$$\Gamma^1_{11} = \frac{1}{2}\frac{\partial \ln|g_{11}|}{\partial x^1} = \frac{1}{2}\frac{\partial \ln|h|}{\partial \theta} = \frac{1}{2h}\frac{\partial h}{\partial \theta} = \frac{R_c \dfrac{\partial R_c}{\partial \theta} - a^2 \sin\theta\cos\theta}{h} \tag{A38}$$

Equation (A14) then becomes,

$$K = \frac{\left[\left(\frac{\partial \beta}{\partial \theta}\sin^2\theta + \beta \sin 2\theta\right)\frac{\partial h}{\partial \theta} - h\left(\frac{\partial^2 \beta}{\partial \theta^2}\sin^2\theta + 2\frac{\partial \beta}{\partial \theta}\sin 2\theta + 2\beta\cos 2\theta\right)\right]}{2hf\sin^2\theta}$$

$$-\frac{\left(\frac{\partial \beta}{\partial \theta}\sin^2\theta + \beta\sin 2\theta\right)\frac{\partial h}{\partial \theta}}{4hf\sin^2\theta} + \frac{h\left(\frac{\partial \beta}{\partial \theta}\sin^2\theta + \beta\sin 2\theta\right)^2}{4f^2\sin^4\theta}$$

(A39)

APPENDIX B – RIEMANNIAN CURVATURE

Riemannian (or sectional) curvature generalises to dimensions higher than 2 the Gaussian curvature of a surface. Consequently, in the case of a surface the Riemannian curvature reduces to Gaussian curvature. The Riemannian curvature of the Kerr-Newman form subsumes that for the Kerr, Reissner-Nordström, and Schwarzschild forms, and can be determined for the spatial sections thereof and for the whole '4-dimensional' metrics respectively. In this Appendix consideration will only be given to the Schwarzschild and Reissner-Nordström forms. Calculations for the Kerr and Kerr-Newman forms follow similar lines. Once again, the Riemannian curvature demonstrates yet again that none of the so-called black hole metrics can be extended to produce a black hole.

The Riemannian curvature K_S at any point in a metric space of dimensions $n > 2$ depends upon the Riemann-Christoffel curvature tensor of the first kind R_{ijkl}, the components of the metric tensor g_{ik}, and two arbitrary n-dimensional linearly independent contravariant direction vectors U^i and V^i, as follows:

$$K_S = \frac{R_{ijkl}U^iV^jU^kV^l}{G_{pqrs}U^pV^qU^rV^s} \qquad G_{pqrs} = g_{pr}g_{qs} - g_{ps}g_{qr}$$

Definition 6: *If the Riemannian curvature at any point is independent of direction vectors at that point then the point is called an isotropic point.*

It follows from (A13) and **Definition 6** that all points of a surface are isotropic.

Riemannian curvature of the spatial section of the Schwarzschild form

The spatial section of the Schwarzschild form is, from expressions (32),

$$ds^2 = \left(1 - \frac{\alpha}{R_c}\right)^{-1} dR_c^2 + R_c^2 d\theta^2 + R_c^2 \sin^2\theta \, d\varphi^2$$

$$R_c = \left(|r - r_o|^n + \alpha^n\right)^{1/n}$$

$$r, r_0 \in \Re \quad n \in \Re^+ \tag{B1}$$

The metric tensor is diagonal,

$$g_{ij} = \begin{bmatrix} \left(1 - \frac{\alpha}{R_c}\right)^{-1} & 0 & 0 \\ 0 & R_c^2 & 0 \\ 0 & 0 & R_c^2 \sin^2\theta \end{bmatrix} \tag{B2}$$

The components of the metric tensor are,

$$g_{11} = \left(1 - \frac{\alpha}{R_c}\right)^{-1} \qquad g_{22} = R_c^2 \qquad g_{33} = R_c^2 \sin^2\theta \tag{B3}$$

The components of the Riemann-Christoffel curvature tensor of the second kind are determined by,

$$R^i_{.jkl} = \frac{\partial \Gamma^i_{jl}}{\partial x^k} - \frac{\partial \Gamma^i_{jk}}{\partial x^l} + \Gamma^r_{jl}\Gamma^i_{rk} - \Gamma^r_{jk}\Gamma^i_{rl} \tag{B4}$$

Since (B2) is diagonal, the Christoffel symbols of the second kind can be calculated using the following relations,

$$\Gamma^i_{ij} = \Gamma^i_{ji} = \frac{\partial\left(\frac{1}{2}\ln|g_{ii}|\right)}{\partial x^j} \qquad \Gamma^i_{jj} = -\frac{1}{2g_{ii}}\frac{\partial g_{jj}}{\partial x^i} \quad (i \neq j) \tag{B5}$$

Make the following assignments,

$$x^1 = R_c \qquad x^2 = \theta \qquad x^3 = \varphi$$

There are 15 Christoffel symbols of the second kind to be considered. Calculation determines that there are only 7 non-zero such terms,

$$\Gamma^1_{11} = \frac{\alpha}{2R_c(\alpha - R_c)} \qquad \Gamma^2_{21} = \frac{1}{R_c} \qquad \Gamma^3_{31} = \frac{1}{R_c} \qquad \Gamma^3_{32} = \cot\theta$$

$$\Gamma^1_{22} = (\alpha - R_c) \qquad \Gamma^1_{33} = (\alpha - R_c)\sin^2\theta \qquad \Gamma^2_{33} = -\sin\theta\cos\theta \tag{B6}$$

The number of components of the Riemann-Christoffel curvature tensor that are not identically zero is $n^2(n^2 - 1)/12$, where n is the number of dimensions of the metric space, which in this case is 3. Thus there are 9(9 – 1)/12 = 6 components to consider. Calculation determines that there are only 3 non-zero components of the Riemann-Christoffel curvature tensor of the second kind,

$$R^1_{212} = \frac{-\alpha}{2R_c} \qquad R^1_{313} = \frac{-\alpha\sin^2\theta}{2R_c} \qquad R^2_{323} = \frac{\alpha\sin^2\theta}{R_c} \tag{B7}$$

The components of the Riemann-Christoffel curvature tensor of the first kind, R_{ijkl}, are determined by,

$$R_{ijkl} = g_{ir}R^r_{\ jkl} \tag{B8}$$

Putting expressions (B3) and (B7) into (B8) yields,

$$R_{1212} = \frac{-\alpha}{2(R_c - \alpha)} \qquad R_{1313} = \frac{-\alpha\sin^2\theta}{2(R_c - \alpha)} \qquad R_{2323} = \alpha R_c \sin^2\theta \tag{B9}$$

Let U^i and V^i be two arbitrary linearly independent contravariant direction vectors. Then for the problem at hand the Riemannian curvature K_s is given by,

$$K_S = \frac{R_{1212}W_{1212} + R_{1313}W_{1313} + R_{2323}W_{2323}}{G_{1212}W_{1212} + G_{1313}W_{1313} + G_{2323}W_{2323}}$$

$$W_{ijkl} = \begin{vmatrix} U^i & U^j \\ V^i & V^j \end{vmatrix} \begin{vmatrix} U^k & U^l \\ V^k & V^l \end{vmatrix}$$

$$R_c = \left(|r - r_o|^n + \alpha^n\right)^{1/n}$$

$$r, r_0 \in \Re \quad n \in \Re^+ \tag{B10}$$

Since the metric tensor is diagonal the non-zero G_{ijkl} are calculated by,

$$G_{ijij} = g_{ii} g_{jj} \qquad i < j \qquad (no\ summation) \tag{B11}$$

The non-zero G_{ijkl} are calculated,

$$G_{1212} = \frac{-R_c^3}{(\alpha - R_c)} \qquad G_{1313} = \frac{-R_c^3 \sin\ \theta}{(\alpha - R_c)} \qquad G_{2323} = R_c^4 \sin^2\theta \tag{B12}$$

Putting (B9) and (B12) into (B10) yields,

$$K_S = \frac{-\alpha\left(W_{1212} + W_{1313}\sin^2\theta\right) + 2\alpha R_c(R_c - \alpha)W_{2323}\sin^2\theta}{2R_c^3\left(W_{1212} + W_{1313}\sin^2\theta\right) + 2R_c^4(R_c - \alpha)W_{2323}\sin^2\theta}$$

$$W_{ijkl} = \begin{vmatrix} U^i & U^j \\ V^i & V^j \end{vmatrix}\begin{vmatrix} U^k & U^l \\ V^k & V^l \end{vmatrix}$$

$$R_c = \left(\left|r - r_o\right|^n + \alpha^n\right)^{1/n}$$

$$r, r_0 \in \Re \quad n \in \Re^+ \tag{B13}$$

Now $R_c(r_0) = \alpha\ \forall r_0\ \forall n$, in which case (B13) is,

$$K_s = -\frac{1}{2\alpha^2} \tag{B14}$$

which is *entirely independent of the direction vectors* U^i and V^i, and independent of ϑ. Thus r_0 produces an isotropic point. This reaffirms that the Schwarzschild form cannot be extended. Comparing (B14) with (A34) gives,

$$K_S = -\frac{K}{2} \tag{B15}$$

Thus, at $r = r_0$ the Riemannian curvature of the spatial section of the Schwarzschild form is the negative of half the Gaussian curvature of the spherical surface in the spatial section of the Schwarzschild form. (B15) is another curvature invariant for the Schwarzschild form.

(B13) depends on ϑ. When $\vartheta = 0$ and $\vartheta = \pi$, (B13) becomes (B14). When $\vartheta = \pi/2$, (B13) becomes,

$$K_S = \frac{-\alpha(W_{1212} + W_{1313}) + 2\alpha R_c (R_c - \alpha) W_{2323}}{2R_c^3 (W_{1212} + W_{1313}) + 2R_c^4 (R_c - \alpha) W_{2323}}$$

$$W_{ijkl} = \begin{vmatrix} U^i & U^j \\ V^i & V^j \end{vmatrix} \begin{vmatrix} U^k & U^l \\ V^k & V^l \end{vmatrix}$$

$$R_c = \left(|r - r_o|^n + \alpha^n \right)^{1/n}$$

$$r, r_0 \in \Re \quad n \in \Re^+ \tag{B13b}$$

This reaffirms that the Schwarzschild form cannot be extended.

Riemannian curvature of the spatial section of the Reissner-Nordström form

If *a* = 0 in expressions (A17), the Reissner-Nordström form is obtained, thus,

$$ds^2 = \left(1 - \frac{\alpha}{R_c} + \frac{q^2}{R_c^2}\right) dt^2 - \left(1 - \frac{\alpha}{R_c} + \frac{q^2}{R_c^2}\right)^{-1} dR_c^2 - R_c^2 d\theta^2 - R_c^2 \sin^2\theta \, d\varphi^2$$

$$R_c = \left(|r - r_o|^n + \xi^n \right)^{1/n}$$

$$r, r_0 \in \Re \quad n \in \Re^+$$

$$\xi = \frac{\alpha}{2} + \sqrt{\frac{\alpha^2}{4} - q^2}$$

(B16)

The spatial section is,

$$ds^2 = \left(1 - \frac{\alpha}{R_c} + \frac{q^2}{R_c^2}\right)^{-1} dR_c^2 + R_c^2 d\theta^2 + R_c^2 \sin^2\theta \, d\varphi^2 \tag{B17}$$

The metric tensor is diagonal,

$$g_{ij} = \begin{bmatrix} \left(1 - \frac{\alpha}{R_c} + \frac{q^2}{R_c^2}\right)^{-1} & 0 & 0 \\ 0 & R_c^2 & 0 \\ 0 & 0 & R_c^2 \sin^2\theta \end{bmatrix} \tag{B18}$$

The components of the metric tensor are,

$$g_{11} = \left(1 - \frac{\alpha}{R_c} + \frac{q^2}{R_c^2}\right)^{-1} \qquad g_{22} = R_c^2 \qquad g_{33} = R_c^2 \sin^2\theta \tag{B19}$$

Make the following assignments,

$$x^1 = R_c \qquad x^2 = \theta \qquad x^3 = \varphi$$

There are 15 Riemann-Christoffel symbols of the second kind to consider. Calculation determines that there are only 7 non-zero such terms,

$$\Gamma^1_{11} = \frac{-\left(\alpha R_c - 2q^2\right)}{2R_c\left(R_c^2 - \alpha R_c + q^2\right)} \qquad \Gamma^2_{21} = \frac{1}{R_c} \qquad \Gamma^3_{31} = \frac{1}{R_c} \qquad \Gamma^3_{32} = \cot\theta$$

$$\Gamma^1_{22} = \frac{-\left(R_c^2 - \alpha R_c + q^2\right)}{R_c} \qquad \Gamma^1_{33} = \frac{-\left(R_c^2 - \alpha R_c + q^2\right)\sin^2\theta}{R_c} \qquad \Gamma^2_{33} = -\sin\theta\cos\theta$$

There are 6 components of the Riemann-Christoffel curvature tensor to consider. Calculation determines that there are only 3 non-zero such terms,

$$R_{1212} = \frac{-\left(\alpha R_c - 2q^2\right)}{2\left(R_c^2 - \alpha R_2 + q^2\right)} \qquad R_{1313} = \frac{-\left(\alpha R_c - 2q^2\right)\sin^2\theta}{2\left(R_c^2 - \alpha R_c + q^2\right)} \qquad R_{2323} = \left(\alpha R_c - q^2\right)\sin^2\theta$$

There are only 3 non-zero G_{ijkl},

$$G_{1212} = \frac{R_c^4}{\left(R_c^2 - \alpha R_c + q^2\right)} \qquad G_{1313} = \frac{R_c^4\sin^2\theta}{\left(R_c^2 - \alpha R_c + q^2\right)} \qquad G_{2323} = R_c^4\sin^2\theta$$

The Riemannian curvature K_S is,

$$K_S = \frac{-\left(\alpha R_c - 2q^2\right)\left(W_{1212} + W_{1313}\sin^2\theta\right) + 2\left(R_c^2 - \alpha R_c + q^2\right)\left(\alpha R_c - q^2\right)W_{2323}\sin^2\theta}{2R_c^4\left(W_{1212} + W_{1313}\sin^2\theta\right) + 2R_c^4\left(R_c^2 - \alpha R_c + q^2\right)W_{2323}\sin^2\theta}$$

$$W_{ijkl} = \begin{vmatrix} U^i & U^j \\ V^i & V^j \end{vmatrix}\begin{vmatrix} U^k & U^l \\ V^k & V^l \end{vmatrix}$$

$$R_c = \left(\left|r - r_o\right|^n + \xi^n\right)^{1/n} \qquad r, r_0 \in \Re \quad n \in \Re^+$$

$$\xi = \frac{\alpha}{2} + \sqrt{\frac{\alpha^2}{4} - q^2} \qquad \text{(B20)}$$

Note that if q = 0, (B20) reduces to (B13) for the spatial section of the Schwarzschild form. Also note that for (B20) $R_c(r_0) = \xi$, where ξ is given by (B16), in which case the Riemannian curvature is,

$$K_s = -\frac{\left(\alpha\xi - 2q^2\right)}{2\xi^4} \qquad \text{(B21)}$$

which is *entirely independent of the direction vectors* U^i and V^i, and of ϑ. Thus, $r = r_0$ produces an isotropic point. This reaffirms that the Reissner-Nordström form cannot be extended.

Taking ξ from (B16) the Riemannian curvature is,

$$K_S = -\frac{\left(\alpha^2 + \alpha\sqrt{\alpha^2 - 4q^2} - 4q^2\right)}{4\left(\frac{\alpha}{2} + \frac{\sqrt{\alpha^2 - 4q^2}}{2}\right)^4} \qquad \text{(B21b)}$$

Once again, if q = 0, then (B21) reduces to (B14) for the spatial section of the Schwarzschild form, as easily seen from (B21b). (B21) is a curvature invariant for the Reissner-Nordström form.
(B20) depends on ϑ. When $\vartheta = 0$ and $\vartheta = \pi$, (B20) becomes,

$$K_s = -\frac{\left(\alpha R_c - 2q^2\right)}{2R_c^4} \qquad \text{(B21c)}$$

(B21c) produces isotropic points. When $\vartheta = \pi/2$, (B20) becomes,

$$K_S = \frac{-\left(\alpha R_c - 2q^2\right)\left(W_{1212} + W_{1313}\right) + 2\left(R_c^2 - \alpha R_c + q^2\right)\left(\alpha R_c - q^2\right)W_{2323}}{2R_c^4\left(W_{1212} + W_{1313}\right) + 2R_c^4\left(R_c^2 - \alpha R_c + q^2\right)W_{2323}}$$

$$W_{ijkl} = \begin{vmatrix} U^i & U^j \\ V^i & V^j \end{vmatrix}\begin{vmatrix} U^k & U^l \\ V^k & V^l \end{vmatrix}$$

$$R_c = \left(\left|r - r_o\right|^n + \xi^n\right)^{1/n} \qquad r, r_0 \in \Re \quad n \in \Re^+$$

$$\xi = \frac{\alpha}{2} + \sqrt{\frac{\alpha^2}{4} - q^2} \qquad \text{(B21d)}$$

Riemannian curvature of the Schwarzschild form

The Schwarzschild form is,

$$ds^2 = \left(1 - \frac{\alpha}{R_c}\right)dt^2 - \left(1 - \frac{\alpha}{R_c}\right)^{-1} dR_c^2 - R_c^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

$$R_c = \left(\left|r - r_0\right|^n + \alpha^n\right)^{1/n} \quad 0 \le r \quad r, r_0 \in \Re, \quad n \in \Re^+ \qquad \text{(B22)}$$

The metric tensor is diagonal,

$$g_{ik} = \begin{bmatrix} \left(1 - \frac{\alpha}{R_c}\right) & 0 & 0 & 0 \\ 0 & -\left(1 - \frac{\alpha}{R_c}\right)^{-1} & 0 & 0 \\ 0 & 0 & -R_c^2 & 0 \\ 0 & 0 & 0 & -R_c^2 \sin^2\theta \end{bmatrix} \qquad \text{(B23)}$$

The components of the metric tensor are,

$$g_{00} = \left(1 - \frac{\alpha}{R_c}\right) \qquad g_{11} = -\left(1 - \frac{\alpha}{R_c}\right)^{-1} \qquad g_{22} = -R_c^2 \qquad g_{33} = -R_c^2 \sin^2\theta \qquad \text{(B24)}$$

Make the following assignments,

$$x^0 = t \qquad x^1 = R_c \qquad x^2 = \theta \qquad x^3 = \varphi$$

There are 28 Christoffel symbols of the second kind to consider. Calculation shows that there are only 9 non-zero such terms,

$$\Gamma^0_{01} = \frac{\alpha}{2R_c(R_c - \alpha)} \qquad \Gamma^1_{00} = \frac{\alpha(R_c - \alpha)}{2R_c^3}$$

$$\Gamma^1_{11} = \frac{-\alpha}{2R_c(R_c - \alpha)} \qquad \Gamma^2_{21} = \frac{1}{R_c} \qquad \Gamma^3_{31} = \frac{1}{R_c} \qquad \Gamma^3_{32} = \cot\theta$$

$$\Gamma^1_{22} = -(R_c - \alpha) \qquad \Gamma^1_{33} = -(R_c - \alpha)\sin^2\theta \qquad \Gamma^2_{33} = -\sin\theta\cos\theta$$

Since the dimension of the space is 4 there are 16(16-1)/12 = 20 components of the Riemann-Christoffel curvature tensor to consider. Calculation determines that are only 6 non-zero such terms,

$$R_{0101} = \frac{\alpha}{R_c^3} \qquad R_{0202} = \frac{-\alpha(R_c-\alpha)}{2R_c^2} \qquad R_{0303} = \frac{-\alpha(R_c-\alpha)\sin^2\theta}{2R_c^2}$$

$$R_{1212} = \frac{\alpha}{2(R_c-\alpha)} \qquad R_{1313} = \frac{\alpha\sin^2\theta}{2(R_c-\alpha)} \qquad R_{2323} = -\alpha R_c\sin^2\theta$$

Since the metric tensor is diagonal there are only 6 non-zero components of the G_{ijkl},

$$G_{0101} = -1 \qquad G_{0202} = -R_c(R_c-\alpha) \qquad G_{0303} = -R_c(R_c-\alpha)\sin^2\theta$$

$$G_{1212} = \frac{R_c^3}{(R_c-\alpha)} \qquad G_{1313} = \frac{R_c^3\sin^2\theta}{(R_c-\alpha)} \qquad G_{2323} = R_c^4\sin^2\theta$$

The Riemannian curvature for the Schwarzschild form is therefore,

$$K_S = \frac{2\alpha(R_c-\alpha)W_{0101} - \alpha R_c(R_c-\alpha)^2W_{0202} - \alpha R_c(R_c-\alpha)^2W_{0303}\sin^2\theta + \alpha R_c^3W_{1212} + \alpha R_c^3W_{1313}\sin^2\theta - 2\alpha R_c^4(R_c-\alpha)W_{2323}\sin^2\theta}{-2R_c^3(R_c-\alpha)W_{0101} - 2R_c^4(R_c-\alpha)^2W_{0202} - 2R_c^4(R_c-\alpha)^2W_{0303}\sin^2\theta + 2R_c^6W_{1212} + 2R_c^6W_{1313}\sin^2\theta + 2R_c^7(R_c-\alpha)W_{2323}\sin^2\theta}$$

$$W_{ijkl} = \begin{vmatrix} U^i & U^j \\ V^i & V^j \end{vmatrix}\begin{vmatrix} U^k & U^l \\ V^k & V^l \end{vmatrix}$$

$$R_c = \left(|r-r_o|^n + \alpha^n\right)^{1/n}$$

$$r, r_0 \in \Re, \quad n \in \Re^+ \tag{B25}$$

By (B22), $R_c(r_0) = \alpha$ irrespective of the values of r_0 and n, in which case (B25) reduces to,

$$K_S = \frac{1}{2\alpha^2} \tag{B26}$$

Thus, (B26) is *entirely independent of the direction vectors* U^i and V^i, and of ϑ. Thus, $r = r_0$ produces an isotropic point, which again shows that the Schwarzschild form cannot be extended. Comparing (B26) to (A34) gives,

$$K_S = \frac{K}{2} \tag{B27}$$

Hence, at $r = r_0$ the Riemannian curvature of the Schwarzschild form is half the Gaussian curvature of the spherical surface in the spatial section of the Schwarzschild form.

(B26) is the negative of (B14): at $r = r_0$ the Riemannian curvature of the Schwarzschild form is the negative of the Riemannian curvature of the spatial section thereof. (B27) is another curvature invariant for the Schwarzschild form.

(B25) depends on ϑ. When $\vartheta = 0$ and $\theta = \pi$, (B25) becomes,

$$K_S = \frac{2\alpha(R_c-\alpha)W_{0101} - \alpha R_c(R_c-\alpha)^2 W_{0202} + \alpha R_c^3 W_{1212}}{-2R_c^3(R_c-\alpha)W_{0101} - 2R_c^4(R_c-\alpha)^2 W_{0202} + 2R_c^6 W_{1212}}$$

$$W_{ijkl} = \begin{vmatrix} U^i & U^j \\ V^i & V^j \end{vmatrix} \begin{vmatrix} U^k & U^l \\ V^k & V^l \end{vmatrix}$$

$$R_c = \left(\left|r - r_o\right|^n + \alpha^n\right)^{1/n}$$

$$r, r_0 \in \Re, \quad n \in \Re^+ \qquad \text{(B25b)}$$

When θ = $\pi/2$, (B25) becomes,

$$K_S = \frac{2\alpha(R_c-\alpha)W_{0101} - \alpha R_c(R_c-\alpha)^2 W_{0202} - \alpha R_c(R_c-\alpha)^2 W_{0303} + \alpha R_c^3 W_{1212} + \alpha R_c^3 W_{1313} - 2\alpha R_c^4(R_c-\alpha)W_{2323}}{-2R_c^3(R_c-\alpha)W_{0101} - 2R_c^4(R_c-\alpha)^2 W_{0202} - 2R_c^4(R_c-\alpha)^2 W_{0303} + 2R_c^6 W_{1212} + 2R_c^6 W_{1313} + 2R_c^7(R_c-\alpha)W_{2323}}$$

$$W_{ijkl} = \begin{vmatrix} U^i & U^j \\ V^i & V^j \end{vmatrix} \begin{vmatrix} U^k & U^l \\ V^k & V^l \end{vmatrix}$$

$$R_c = \left(\left|r - r_o\right|^n + \alpha^n\right)^{1/n}$$

$$r, r_0 \in \Re, \quad n \in \Re^+ \qquad \text{(B25c)}$$

Riemannian curvature of the Reissner-Nordström form

The Reissner-Nordström form is,

$$ds^2 = \left(1 - \frac{\alpha}{R_c} + \frac{q^2}{R_c^2}\right)dt^2 - \left(1 - \frac{\alpha}{R_c} + \frac{q^2}{R_c^2}\right)^{-1} dR_c^2 - R_c^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

$$R_c = \left(\left|r - r_0\right|^n + \xi^n\right)^{1/n}$$

$$\xi = \frac{\alpha}{2} + \sqrt{\frac{\alpha^2}{4} - q^2} \qquad q^2 < \frac{\alpha^2}{4}$$

$$r, r_0 \in \Re, \quad n \in \Re^+ \qquad \text{(B28)}$$

The metric tensor is diagonal,

$$g_{ik} = \begin{bmatrix} \left(1 - \frac{\alpha}{R_c} + \frac{q^2}{R_c^2}\right) & 0 & 0 & 0 \\ 0 & -\left(1 - \frac{\alpha}{R_c} + \frac{q^2}{R_c^2}\right)^{-1} & 0 & 0 \\ 0 & 0 & -R_c^2 & 0 \\ 0 & 0 & 0 & -R_c^2 \sin^2\theta \end{bmatrix} \qquad \text{(B29)}$$

The components of the metric tensor are,

$$g_{00} = \left(1 - \frac{\alpha}{R_c} + \frac{q^2}{R_c^2}\right) \quad g_{11} = -\left(1 - \frac{\alpha}{R_c} + \frac{q^2}{R_c^2}\right)^{-1} \quad g_{22} = -R_c^2 \quad g_{33} = -R_c^2 \sin^2\theta \qquad \text{(B30)}$$

Make the following assignments,

$$x^0 = t \qquad x^1 = R_c \qquad x^2 = \theta \qquad x^3 = \varphi$$

There are 28 Christoffel symbols of the second kind to consider. Calculation shows that there are only 9 non-zero such terms,

$$\Gamma^0_{01} = \frac{\left(\alpha R_c - 2q^2\right)}{2R_c\left(R_c^2 - \alpha R_c + q^2\right)} \qquad \Gamma^1_{00} = \frac{\left(R_c^2 - \alpha R + q^2\right)\left(\alpha R_c - 2q^2\right)}{2R_c^5}$$

$$\Gamma^1_{11} = \frac{-\left(\alpha R_c - 2q^2\right)}{2R_c\left(R_c^2 - \alpha R_c + q^2\right)} \qquad \Gamma^2_{21} = \frac{1}{R_c} \qquad \Gamma^3_{31} = \frac{1}{R_c} \qquad \Gamma^3_{32} = \cot\theta$$

$$\Gamma^1_{22} = \frac{-\left(R_c^2 - \alpha R_c + q^2\right)}{R_c} \qquad \Gamma^1_{33} = \frac{-\left(R_c^2 - \alpha R_c + q^2\right)\sin^2\theta}{R_c} \qquad \Gamma^2_{33} = -\sin\theta\cos\theta$$

There are 20 components of the Riemann-Christoffel curvature tensor to consider. Calculation shows that there are only 6 non-zero such terms,

$$R_{0101} = \frac{\left(\alpha R_c - 3q^2\right)}{R_c^4} \qquad R_{0202} = \frac{-\left(R_c^2 - \alpha R_c + q^2\right)\left(\alpha R_c - 2q^2\right)}{2R_c^4}$$

$$R_{0303} = \frac{-\left(R_c^2 - \alpha R_c + q^2\right)\left(\alpha R_c - 2q^2\right)\sin^2\theta}{2R_c^4}$$

$$R_{1212} = \frac{\left(\alpha R_c - 2q^2\right)}{2\left(R_c^2 - \alpha R_c + q^2\right)} \qquad R_{1313} = \frac{\left(\alpha R_c - 2q^2\right)\sin^2\theta}{2\left(R_c^2 - \alpha R_c + q^2\right)} \qquad R_{2323} = -\left(\alpha R_c - q^2\right)\sin^2\theta$$

Since the metric is diagonal the only non-zero G_{ijkl} are,

$$G_{0101} = -1 \qquad G_{0202} = -\left(R_c^2 - \alpha R_c + q^2\right) \qquad G_{0303} = -\left(R_c^2 - \alpha R_c + q^2\right)\sin^2\theta$$

$$G_{1212} = \frac{R_c^4}{\left(R_c^2 - \alpha R_c + q^2\right)} \qquad G_{1313} = \frac{R_c^4\sin^2\theta}{\left(R_c^2 - \alpha R_c + q^2\right)} \qquad G_{2323} = R_c^4\sin^2\theta$$

The Riemannian curvature for the Reissner-Nordström form is,

$$K_S = \frac{A+B+C}{D+E+F}$$

$$A = 2\left(R_c^2 - \alpha R_c + q^2\right)\left(\alpha R_c - 3q^2\right)W_{0101} - \left(R_c^2 - \alpha R_c + q^2\right)^2\left(\alpha R_c - 2q^2\right)W_{0202}$$

$$B = -\left(R_c^2 - \alpha R_c + q^2\right)^2\left(\alpha R_c - 2q^2\right)\sin^2\theta\, W_{0303} + R_c^4\left(\alpha R_c - 2q^2\right)W_{1212}$$

$$C = R_c^4\left(\alpha R_c - 2q^2\right)\sin^2\theta\, W_{1313} - 2R_c^4\left(\alpha R_c - q^2\right)\left(R_c^2 - \alpha R_c + q^2\right)\sin^2\theta\, W_{2323}$$

$$D = -2R_c^4\left(R_c^2 - \alpha R_c + q^2\right)W_{0101} - 2R_c^4\left(R_c^2 - \alpha R_c + q^2\right)^2 W_{0202}$$

$$E = -2R_c^4\left(R_c^2 - \alpha R_c + q^2\right)^2\sin^2\theta\, W_{0303} + 2R_c^8 W_{1212}$$

$$F = 2R_c^8\sin^2\theta\, W_{1313} + 2R_c^8\left(R_c^2 - \alpha R_c + q^2\right)\sin^2\theta\, W_{2323}$$

$$W_{ijkl} = \begin{vmatrix} U^i & U^j \\ V^i & V^j \end{vmatrix}\begin{vmatrix} U^k & U^l \\ V^k & V^l \end{vmatrix} \qquad R_c = \left(\left|r - r_0\right|^n + \xi^n\right)^{\frac{1}{n}}$$

$$\xi = \frac{\alpha}{2} + \sqrt{\frac{\alpha^2}{4} - q^2} \qquad q^2 < \frac{\alpha^2}{4}$$

$$r, r_0 \in \Re \qquad n \in \Re^+$$

(B31)

$R_c(r_0) = \xi$ irrespective of the values of r_0 and n, in which case (B31) reduces to,

$$K_S(r_0) = \frac{\alpha\xi - 2q^2}{2\xi^4}$$

$$\xi = \frac{\alpha}{2} + \sqrt{\frac{\alpha^2}{4} - q^2} \qquad q^2 < \frac{\alpha^2}{4} \tag{32}$$

where ξ is given by (B31). (B32) is *entirely independent of the direction vectors* U^i and V^i, and of ϑ. Thus, $r = r_0$ produces an isotropic point, which again shows that the Reissner-Nordström form cannot be extended. By (B31), (B32) is,

$$K_S(r_0) = \frac{\left(\alpha^2 + \alpha\sqrt{\alpha^2 - 4q^2} - 4q^2\right)}{4\left(\frac{\alpha}{2} + \frac{\sqrt{\alpha^2 - 4q^2}}{2}\right)^4}$$

Comparing (B32) with (B21) it is noted that at $r = r_0$ (B32) is the negative of the Riemannian curvature of the spatial section of the Reissner-Nordström form. Note also that if $q = 0$, then expressions (B31) and B(32) reduce to those for the Schwarzschild form, expressions (B25) and (B26) respectively. (B32) is an invariant for the Reissner-Nordström form.

(B31) depends upon ϑ. When $\vartheta = 0$ and $\vartheta = \pi$, (B31) becomes,

$$K_S = \frac{2\left(R_c^2 - \alpha R_c + q^2\right)\left(\alpha R_c - 3q^2\right)W_{0101} - \left(R_c^2 - \alpha R_c + q^2\right)^2\left(\alpha R_c - 2q^2\right)W_{0202} + R_c^4\left(\alpha R_c - 2q^2\right)W_{1212}}{-2R_c^4\left(R_c^2 - \alpha R_c + q^2\right)W_{0101} - 2R_c^4\left(R_c^2 - \alpha R_c + q^2\right)^2 W_{0202} + 2R_c^8 W_{1212}}$$

$$W_{ijkl} = \begin{vmatrix} U^i & U^j \\ V^i & V^j \end{vmatrix}\begin{vmatrix} U^k & U^l \\ V^k & V^l \end{vmatrix}$$

$$R_c = \left(\left|r - r_0\right|^n + \xi^n\right)^{1/n}$$

$$\xi = \frac{\alpha}{2} + \sqrt{\frac{\alpha^2}{4} - q^2} \qquad q^2 < \frac{\alpha^2}{4}$$

$$r, r_0 \in \Re \qquad n \in \Re^+$$

(B31b)

When $\vartheta = \pi/2$, (B31) becomes,

$$K_S = \frac{A + B + C}{D + E + F}$$

$$A = 2\left(R_c^2 - \alpha R_c + q^2\right)\left(\alpha R_c - 3q^2\right)W_{0101} - \left(R_c^2 - \alpha R_c + q^2\right)^2\left(\alpha R_c - 2q^2\right)W_{0202}$$

$$B = -\left(R_c^2 - \alpha R_c + q^2\right)^2\left(\alpha R_c - 2q^2\right)W_{0303} + R_c^4\left(\alpha R_c - 2q^2\right)W_{1212}$$

$$C = R_c^4\left(\alpha R_c - 2q^2\right)W_{1313} - 2R_c^4\left(\alpha R_c - q^2\right)\left(R_c^2 - \alpha R_c + q^2\right)W_{2323}$$

$$D = -2R_c^4\left(R_c^2 - \alpha R_c + q^2\right)W_{0101} - 2R_c^4\left(R_c^2 - \alpha R_c + q^2\right)^2 W_{0202}$$

$$E = -2R_c^4\left(R_c^2 - \alpha R_c + q^2\right)^2 W_{0303} + 2R_c^8 W_{1212}$$

$$F = 2R_c^8 W_{1313} + 2R_c^8\left(R_c^2 - \alpha R_c + q^2\right)W_{2323}$$

$$W_{ijkl} = \begin{vmatrix} U^i & U^j \\ V^i & V^j \end{vmatrix}\begin{vmatrix} U^k & U^l \\ V^k & V^l \end{vmatrix} \qquad R_c = \left(\left|r - r_0\right|^n + \xi^n\right)^{1/n}$$

$$\xi = \frac{\alpha}{2} + \sqrt{\frac{\alpha^2}{4} - q^2} \qquad q^2 < \frac{\alpha^2}{4}$$

$$r, r_0 \in \Re \qquad n \in \Re^+$$

(B31c)

Similar results can be obtained for (A1), reaffirming that (A1) cannot be extended, in accordance with (A17).

APPENDIX C – THE ACCELERATION INVARIANT

Doughty [81] obtained the following expression for the acceleration *β* of a point along a radial geodesic for the static spherically symmetric metrics,

$$\beta = \frac{\sqrt{-g_{11}}\left(-g^{11}\right)\left|\dfrac{\partial g_{00}}{\partial r}\right|}{2g_{00}} \tag{C1}$$

Since the Hilbert and Reissner-Nordström metrics are particular cases of respective infinite sets of equivalent solutions generated by expressions (A17) when *α* = 0, expression (C1) becomes,

$$\beta = \frac{\sqrt{-g_{11}}\left(-g^{11}\right)\left|\dfrac{\partial g_{00}}{\partial R_c}\right|}{2g_{00}}$$

$$R_c = \left(\left|r - r_0\right|^n + \xi^n\right)^{1/n} \qquad r, r_0 \in \Re \qquad n \in \Re^+$$

$$\xi = \frac{\alpha}{2} + \sqrt{\frac{\alpha^2}{4} - q^2} \qquad q^2 < \frac{\alpha^2}{4}$$

(C2)

In (C2),

$$g^{11} = \frac{1}{g_{11}} \tag{C3}$$

Then by (B16),

$$g^{11} = \frac{1}{g_{11}} = -\left(1 - \frac{\alpha}{R_c} + \frac{q^2}{R_c^2}\right)$$

$$\frac{\partial g_{00}}{\partial R_c} = \frac{\partial}{\partial R_c}\left(1 - \frac{\alpha}{R_c} + \frac{q^2}{R_c^2}\right) = \frac{\alpha R_c - 2q^2}{R_c^3}$$

Consequently, the acceleration is given by,

$$\beta = \frac{\left|\alpha R_c - 2q^2\right|}{2R_c^2\sqrt{R_c^2 - \alpha R_c + q^2}}$$

$$\xi = \frac{\alpha}{2} + \sqrt{\frac{\alpha^2}{4} - q^2} \qquad q^2 < \frac{\alpha^2}{4} \tag{C4}$$

Since $q^2 < \alpha^2/4$, (C4) becomes,

$$\beta = \frac{\alpha R_c - 2q^2}{2R_c^2\sqrt{R_c^2 - \alpha R_c + q^2}}$$

$$\xi = \frac{\alpha}{2} + \sqrt{\frac{\alpha^2}{4} - q^2} \qquad q^2 < \frac{\alpha^2}{4}$$

$$R_c = \left(\left|r - r_0\right|^n + \xi^n\right)^{1/n} \qquad r, r_0 \in \Re \quad n \in \Re^+ \tag{C5}$$

In all cases, whether or not $q = 0$, $r \to r_0 \Rightarrow \beta \to \infty$, which constitutes an invariant condition, and therefore reaffirms that the Schwarzschild and Reissner-Nordström forms cannot be extended, and hence do not to produce black holes.

Expression (C1) appears at first glance to be a first-order intrinsic differential invariant since it is superficially composed of only the components of the metric tensor and their first derivatives. This is however, not so, because expression (C1) applies only to the *radial direction*, i.e. to the motion of a point along a radial geodesic. In other words, (C1) involves a direction vector. Consequently, although (C1) is a first-order differential invariant, it is not intrinsic. First-order differential invariants exist, but first-order intrinsic differential invariants do not exist [91, 92]. That (C1) involves a direction vector is amplified by the Killing vector. Let X_a be a first-order tensor (i.e. a covariant vector). Then for it to be a Killing vector it must satisfy Killing's equations,

$$X_{a;b} + X_{b;a} = 0 \tag{C6}$$

where $X_{a;b}$ denotes the covariant derivative of X_a.

The condition for hypersurface orthogonality is [38, 45],

$$X_{[a} X_{c;b]} = 0 \tag{C7}$$

Conditions (C6) and (C7) determine a unique timelike Killing vector that fixes the direction of time [44]. By means of this Killing vector the four-velocity v^i is [45],

$$v^a = \frac{X^a}{\sqrt{X_a X^a}} \tag{C8}$$

The absolute derivative of the four-velocity along its own direction gives the four-acceleration β^a [45],

$$\beta^a = \frac{Dv^a}{du} \tag{C9}$$

The norm of the four-acceleration is [45],

$$\beta = \sqrt{-\beta_a \beta^a} \tag{C10}$$

Applying (C6) through (C10) to the Reissner-Nordström form from (A17) yields (C5),

$$\beta = \frac{\alpha R_c - 2q^2}{2R_c^2 \sqrt{R_c^2 - \alpha R_c + q^2}}$$

$$\xi = \frac{\alpha}{2} + \sqrt{\frac{\alpha^2}{4} - q^2} \qquad q^2 < \frac{\alpha^2}{4} \tag{C5}$$

Consequently, expression (C1) is not intrinsic; it is a first-order differential invariant which is constructed with the metric and an associated direction vector, as the limitation of (C1) to motion of a point along a radial geodesic implies. Recall that first-order intrinsic differential invariants do not exist [91, 92].

When $q = 0$ (C5) reduces to,

$$\beta = \frac{\alpha}{2R_c^2 \sqrt{1 - \frac{\alpha}{R_c}}} \tag{C11}$$

which can of course be calculated directly from (C1) for the Schwarzschild form (32) from (A17). In all cases $r \to r_0 \Rightarrow \beta \to \infty$, which constitutes an invariant condition, and therefore reaffirms once again that the Schwarzschild and Reissner-Nordström forms cannot be extended and therefore cannot produce black holes.

APPENDIX D – ISOTROPIC COORDINATE FORMS

Let Q_n and M_n be two metric spaces of dimension n with metrics g_{ik} and $\hat{g}_{ik}$ respectively. Let Q_n and M_n be described by the same set of coordinates (variables) x^i. The spaces with their metrics can be represented by the notation (Q_n, g_{ik}) and (M_n, $\hat{g}_{ik}$). If the two metrics are related by means of a smooth positive valued function Γ^2 of the x^i such that $\hat{g}_{ik} = \Gamma^2 g_{ik}$ then the correspondence between Q_n and M_n is called *conformal* and the metric spaces are called *conformal spaces*. Thus Γ^2 maps Q_n into M_n, denoted by,

$$\Gamma^2 : (Q_n, g_{ik}) \to (M_n, \hat{g}_{ik}) \tag{D1}$$

If g_{ik} is the Euclidean metric then M_n is said to be *conformally flat*. Conformal maps preserve angles, such as those between two arbitrary linearly independent vectors $\mathbf{U}_q$ and $\mathbf{V}_q$ of dimension n in Q_n. However, conformal maps do not necessarily preserve curvatures; in other words, the Riemannian curvature, for instance, at some point P_q in Q_n determined with two linearly independent vectors $\mathbf{U}_q$ and $\mathbf{V}_q$, is generally not the same at the corresponding point P_m in M_n with corresponding vectors $\mathbf{U'}_m$ and $\mathbf{V'}_m$. The magnitudes of the said corresponding vectors are proportional to $\mathbf{U}_q$ and $\mathbf{V}_q$ respectively, due to the conformal map or transformation, but the angle between them does not change. Furthermore, the components of the Riemann-Christoffel curvature tensor at some point P_q in Q_n do not generally have the same values as the corresponding components of the Riemann-Christoffel curvature tensor at the corresponding point P_m in M_n. Dimension n = 1 is trivial and dimension n = 2 metric spaces are conformal to any other.

Every particular metric of the metric ground-form (7) for 3-dimensional spherically symmetric metric spaces can be conformally represented in Euclidean 3-space. This simply means that expression (7) can be replaced by the following equivalent general metric ground-form,

$$ds^2 = H^2(\rho)\left[d\rho^2 + \rho^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)\right] \tag{D2}$$

because

$$A^2(k)dk^2 + k^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right) = H^2(\rho)\left[d\rho^2 + \rho^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)\right] \tag{D3}$$

means that

$$H(\rho)\rho = k \qquad H(\rho)d\rho = A(k)dk \qquad \frac{d\rho}{\rho} = A(k)\frac{dk}{k} \tag{D4}$$

If $A(k)$ is known, then from the last of these three relations ρ can be determined as a function of k, and if k is in turn a function of say r then ρ is determined as a function of r. Then by the first relation, H is determined as a function of r. Thus both metrics can be rendered in terms of the very same x^i. Note that the variables in the right side metric of (D3) are,

$$\hat{x}^1 = \rho \qquad \hat{x}^2 = \theta \qquad \hat{x}^3 = \varphi \tag{D5}$$

The variables in the left side metric of (D3) are,

$$x^1 = k \qquad x^2 = \theta \qquad x^3 = \varphi \tag{D6}$$

Thus,

$$\hat{x}^2 = x^2 \qquad \hat{x}^3 = x^3 \tag{D7}$$

Then by the last expression in (D4) $\hat{x}^1$ can be determined as a function of x^1. Thus, both metrics can be expressed in terms of the very same variables x^i. Note that (D1) is also a positive-definite quadratic form, as it must, and that (D3) satisfies the necessary and sufficient conformal condition,

$$\hat{g}_{ik} = \Gamma^2 g_{ik} \tag{D8}$$

The part in the square brackets of (D2) and (D3) is just the metric for Euclidean 3-space in spherical coordinates and so (D2) is said to be a conformal representation with Euclidean 3-space of the metric on the left side of (D3), and so the left side of (D3) is said to be 'conformally flat'.

This essentially constitutes the so-called 'isotropic coordinates' for the Schwarzschild form.

Theorem 4: *A Riemann space is flat if and only if its Riemannian curvature is zero at all points.*

Recall from Appendix B that the Riemannian curvature is a generalisation to dimensions $n > 2$ of the Gaussian curvature of a surface ($n = 2$). If the Gaussian curvature of a surface is zero it is a flat surface (i.e. it is the plane surface). The Riemannian curvature for Euclidean 3-space is zero everywhere, and so, likewise, this space is flat, by **Theorem 4**.

The isotropic Schwarzschild form

The astrophysical scientists render Hilbert's solution (2) in isotropic coordinates by setting [17, 38, 83, 84, 95],

$$r = \rho\left(1 + \frac{Gm}{2c^2\rho}\right)^2 \tag{D9}$$

Using (D3) Hilbert's metric (2) in isotropic coordinates is,

$$ds^2 = c^2\left(1 - \frac{Gm}{2c^2\rho}\right)^2\left(1 + \frac{Gm}{2c^2\rho}\right)^{-2} dt^2 - \left(1 + \frac{Gm}{2c^2\rho}\right)^4 \left[d\rho^2 + \rho^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)\right] \tag{D10}$$

wherein $\rho = \sqrt{x^2 + y^2 + z^2}$, owing to which (D10) is sometimes written as [83, 97],

$$ds^2 = c^2\left(1 - \frac{Gm}{2c^2\rho}\right)^2\left(1 + \frac{Gm}{2c^2\rho}\right)^{-2} dt^2 - \left(1 + \frac{Gm}{2c^2\rho}\right)^4 \left(dx^2 + dy^2 + dz^2\right)$$

$$\rho = \sqrt{x^2 + y^2 + z^2} \tag{D11}$$

Note that the spatial section of (D10) has precisely the metric form of (D2), where

$$H^2(\rho) = \left(1 + \frac{Gm}{2c^2\rho}\right)^4$$

I have shown elsewhere [79] that the infinite set of equivalent isotropic Schwarzschild forms is generated by (using *c* = 1),

$$ds^2 = \left(1 - \frac{\alpha}{4h}\right)^2\left(1 + \frac{\alpha}{4h}\right)^{-2} dt^2 - \left(1 + \frac{\alpha}{4h}\right)^4 \left[dh^2 + h^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)\right]$$

$$h = \left[\left|\rho - \rho_0\right|^n + \left(\frac{\alpha}{4}\right)^n\right]^{1/n} \qquad \rho, \rho_0 \in \Re, \quad n \in \Re^+ \tag{D12}$$

wherein ρ_0 and n are entirely arbitrary constants. Accordingly, the transformation from the Schwarzschild form (32) to isotropic coordinates is by means of,

$$R_c = h\left(1 + \frac{\alpha}{4h}\right)^2 \tag{D13}$$

where h is given by (D12) and R_c by (32), or (A17) when a = 0 and q = 0.

Since (D12) is equivalent to expressions (32) (and (A17) when a = q = 0), the curvature invariants for (D12) must correspond to curvature invariants for (32), but are not necessarily the same. To see that (D12) produces corresponding curvature invariants first consider the spatial section of (D12), given by,

$$ds^2 = \left(1 + \frac{\alpha}{4h}\right)^4 dh^2 + h^2\left(1 + \frac{\alpha}{4h}\right)^4 \left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

$$h = \left[\left|\rho - \rho_0\right|^n + \left(\frac{\alpha}{4}\right)^n\right]^{1/n}$$

$$\rho, \rho_0 \in \Re, \quad n \in \Re^+ \tag{D14}$$

This is a positive-definite quadratic form, as it must. The radius R_p for (D14) is given by [79],

$$R_p = h + \frac{\alpha}{2}\ln\left(\frac{4h}{\alpha}\right) - \frac{\alpha^2}{8h} + \frac{\alpha}{4} \tag{D15}$$

Now according to (D14),

$$h(\rho_0) = \frac{\alpha}{4} \quad \forall\, \rho_0 \quad \forall\, n \tag{D16}$$

in which case the radius (D15) is precisely zero, as it must.

Gaussian curvature of the surface in the spatial section of the isotropic Schwarzschild form

The surface in the spatial section of (D12) and (D14) is described by,

$$ds^2 = h^2\left(1 + \frac{\alpha}{4h}\right)^4 \left(d\theta^2 + \sin^2\theta\, d\varphi^2\right) \tag{D17}$$

Since metrics of dimension 2 are conformal to any other, and in accordance with Appendix A, the Gaussian curvature $\hat{K}$ of (D17) is given by,

$$\hat{K} = \frac{1}{h^2\left(1 + \frac{\alpha}{4h}\right)^4} \tag{D18}$$

This is a positive constant for any given admissible value for *h* and so (D17), by **Definition 2** (see section **VI**), is a spherical surface.

By (D16), at $h(\rho_0)$ (D18) takes the value,

$$\hat{K} = \frac{1}{\alpha^2} \tag{D19}$$

which is the very same invariant given by (A34) for the Schwarzschild form (32). Indeed, by (D13),

$$h(\rho_0)\left(1 + \frac{\alpha}{4h(\rho_0)}\right)^2 = \frac{\alpha}{4}\left(1 + \frac{\alpha}{4\frac{\alpha}{4}}\right)^2 = \alpha = R_c(r_0) \tag{D20}$$

which holds for all arbitrary ρ_0 for all arbitrary r_0 for all arbitrary *n*. Thus,

$$\left[h(\rho_0) \Rightarrow \left(\hat{K} = \frac{1}{\alpha^2}\right)\right] \Leftrightarrow \left[R_c(r_0) \Rightarrow \left(K = \frac{1}{\alpha^2}\right)\right] \tag{D21}$$

(isotropic Schwarzschild form) (Schwarzschild form)

By (D19) or (D21) every metric in the infinite set of equivalent metrics generated by (D12) produces the same invariant Gaussian curvature (D19), as they must, and this invariant is precisely the same as for the Schwarzschild form (A34), which is in its turn an invariant produced by every metric in the infinite set generated by (A17) when a = 0 and q = 0 in the latter.

Riemannian curvature of the spatial section of the isotropic Schwarzschild form

Just as the Gaussian curvature of the surface in the spatial section of the isotropic Schwarzschild form produces a corresponding curvature invariant to that of the Schwarzschild form, so must the Riemannian curvature of the spatial section (see Appendix B). The spatial section of the isotropic Schwarzschild form is,

$$ds^2 = \left(1+\frac{\alpha}{4h}\right)^4 dh^2 + h^2\left(1+\frac{\alpha}{4h}\right)^4\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right) \tag{D22}$$

The metric tensor is diagonal,

$$\hat{g}_{ik} = \begin{bmatrix} \left(1+\frac{\alpha}{4h}\right)^4 & 0 & 0 \\ 0 & h^2\left(1+\frac{\alpha}{4h}\right)^4 & 0 \\ 0 & 0 & h^2\left(1+\frac{\alpha}{4h}\right)^4 \sin^2\theta \end{bmatrix} \tag{D23}$$

The components of the metric tensor are,

$$\hat{g}_{11} = \left(1+\frac{\alpha}{4h}\right)^4 \qquad \hat{g}_{22} = h^2\left(1+\frac{\alpha}{4h}\right)^4 \qquad \hat{g}_{33} = h^2\left(1+\frac{\alpha}{4h}\right)^4 \sin^2\theta \tag{D24}$$

Since (D23) is diagonal equations (12) can be applied for determination of the Christoffel symbols of the second kind. There are 15 Christoffel symbols of the second kind to be considered. Calculation determines that there are only 7 non-zero such terms, viz,

$$\Gamma^1_{11} = \frac{-2\alpha}{h(4h+\alpha)} \qquad \Gamma^2_{21} = \frac{(4h-\alpha)}{h(4h+\alpha)} \qquad \Gamma^3_{31} = \frac{(4h-\alpha)}{h(4h+\alpha)} \qquad \Gamma^3_{32} = \cot\theta$$

$$\Gamma^1_{22} = \frac{-h(4h-\alpha)}{(4h+\alpha)} \qquad \Gamma^1_{33} = \frac{-h(4h-\alpha)\sin^2\theta}{(4h+\alpha)} \qquad \Gamma^2_{33} = -\sin\theta\cos\theta$$

(D25)

Since n = 3 there are 6 terms of the Riemann-Christoffel curvature tensor to be considered (see Appendix B). Calculation determines that there are only 3 non-zero such terms, viz,

$$\hat{R}_{1212} = -\frac{2\alpha(4h+\alpha)^2}{4^3 h^3} \qquad \hat{R}_{1313} = -\frac{2\alpha(4h+\alpha)^2 \sin^2\theta}{4^3 h^3}$$

$$\hat{R}_{2323} = \frac{\alpha(4h+\alpha)^2 \sin^2\theta}{4^2 h} \tag{D26}$$

Since the metric tensor is diagonal the only non-zero G_{ijkl} terms in the denominator for the Riemannian curvature are,

$$\hat{G}_{1212} = \frac{(4h+\alpha)^8}{4^8 h^6} \qquad \hat{G}_{1313} = \frac{(4h+\alpha)^8 \sin^2\theta}{4^8 h^6} \qquad \hat{G}_{2323} = \frac{(4h+\alpha)^8 \sin^2\theta}{4^8 h^4} \tag{D27}$$

The Riemannian curvature $\hat{K}_S$ is then given by,

$$\hat{K}_S = \frac{-2\alpha(4h+\alpha)^2\left(\hat{W}_{1212} + \hat{W}_{1313}\sin^2\theta\right) + 4\alpha h^2(4h+\alpha)^2 \hat{W}_{2323}\sin^2\theta}{\frac{(4h+\alpha)^8}{4^5 h^3}\left(\hat{W}_{1212} + \hat{W}_{1313}\sin^2\theta\right) + \frac{(4h+\alpha)^8}{4^5 h}\hat{W}_{2323}\sin^2\theta}$$

$$\hat{W}_{ijkl} = \begin{vmatrix} \hat{U}^i & \hat{U}^j \\ \hat{V}^i & \hat{V}^j \end{vmatrix} \begin{vmatrix} \hat{U}^k & \hat{U}^l \\ \hat{V}^k & \hat{V}^l \end{vmatrix}$$

$$h = \left[|\rho - \rho_0|^n + \left(\frac{\alpha}{4}\right)^n \right]^{1/n}$$

$$\rho, \rho_0 \in \Re \quad n \in \Re^+ \tag{D28}$$

In (D28) the $\hat{W}_{ijkl}$ are determined by the linearly independent direction vectors $\hat{U}^i$ and $\hat{V}^i$ which correspond to U^i and V^i in the Schwarzschild form, due to the conformal mapping of the spatial section of the Schwarzschild form.

When $\rho = \rho_0$, $h = \alpha/4$, for all ρ_0 for all n, and the Riemannian curvature becomes,

$$\hat{K}_S = \frac{-8\left(\hat{W}_{1212} + \hat{W}_{1313}\sin^2\theta\right) + \alpha^2 \hat{W}_{2323}\sin^2\theta}{16\alpha^2\left(\hat{W}_{1212} + \hat{W}_{1313}\sin^2\theta\right) + \alpha^4 \hat{W}_{2323}\sin^2\theta} \tag{D29}$$

Note that D(29) differs from (B14) due only to the terms in $\hat{W}_{2323}$ [i.e. if not for the $\hat{W}_{2323}$ terms the Riemannian curvature would be $-1/(2\alpha^2)$ as for the spatial section of the Schwarzschild form]. Moreover, (D28) depends upon ϑ and so at $\vartheta = 0$ and $\vartheta = \pi$ (D28) reduces to,

$$\hat{K}_S = \frac{-2 \cdot 4^5 \alpha h^3}{(4h+\alpha)^6} \tag{D29b}$$

$$h = \left[|\rho - \rho_0|^n + \left(\frac{\alpha}{4}\right)^n \right]^{1/n} \qquad \rho, \rho_0 \in \Re \quad n \in \Re^+$$

which is independent of the direction vectors $\hat{U}^i$ and $\hat{V}^i$. Hence, (D29b) produces isotropic points. Moreover, when $\rho = \rho_0$ in (D29b) the exact value for the spatial section of the Schwarzschild form [expression (B14)] results.

When $\vartheta = \pi/2$, the Riemannian curvature is,

$$\hat{K}_S = \frac{-2\alpha(4h+\alpha)^2\left(\hat{W}_{1212} + \hat{W}_{1313}\right) + 4\alpha h^2 (4h+\alpha)^2 \hat{W}_{2323}}{\frac{(4h+\alpha)^8}{4^5 h^3}\left(\hat{W}_{1212} + \hat{W}_{1313}\right) + \frac{(4h+\alpha)^8}{4^5 h}\hat{W}_{2323}}$$

$$\hat{W}_{ijkl} = \begin{vmatrix} \hat{U}^i & \hat{U}^j \\ \hat{V}^i & \hat{V}^j \end{vmatrix} \begin{vmatrix} \hat{U}^k & \hat{U}^l \\ \hat{V}^k & \hat{V}^l \end{vmatrix}$$

$$h = \left[|\rho - \rho_0|^n + \left(\frac{\alpha}{4}\right)^n \right]^{1/n} \qquad \rho, \rho_0 \in \Re \quad n \in \Re^+ \tag{D29c}$$

In this case, when $\rho = \rho_0$, the Riemannian curvature becomes,

$$K_S = \frac{-8\left(\hat{W}_{1212} + \hat{W}_{1313}\right) + \alpha^2 \hat{W}_{2323}}{16\alpha^2\left(\hat{W}_{1212} + \hat{W}_{1313}\right) + \alpha^4 \hat{W}_{2323}} \tag{D29d}$$

Thus,

$$\underbrace{\left[h(\rho_0) \Rightarrow \left(\hat{K}_S = \frac{-8\left(\hat{W}_{1212} + \hat{W}_{1313}\sin^2\theta\right) + \alpha^2 \hat{W}_{2323}\sin^2\theta}{16\alpha^2\left(\hat{W}_{1212} + \hat{W}_{1313}\sin^2\theta\right) + \alpha^4 \hat{W}_{2323}\sin^2\theta} \right) \right]}_{\text{(isotropic Schwarzschild form)}} \Leftrightarrow \underbrace{\left[R_c(r_0) \Rightarrow \left(K_S = -\frac{1}{2\alpha^2} \right) \right]}_{\text{(Schwarzschild form)}} \tag{D30}$$

This attests yet again that the isotropic Schwarzschild form cannot be extended.

Riemannian curvature of the isotropic Schwarzschild form

From (D12) the metric tensor for the isotropic Schwarzschild form is diagonal,

$$\hat{g}_{ik} = \begin{bmatrix} \left(\frac{4h-\alpha}{4h+\alpha}\right)^2 & 0 & 0 & 0 \\ 0 & -\left(1+\frac{\alpha}{4h}\right)^4 & 0 & 0 \\ 0 & 0 & -h^2\left(1+\frac{\alpha}{4h}\right)^4 & 0 \\ 0 & 0 & 0 & -h^2\left(1+\frac{\alpha}{4h}\right)^4 \sin^2\theta \end{bmatrix} \quad \text{(D31)}$$

The components of the metric tensor are,

$$\hat{g}_{00} = \left(\frac{4h-\alpha}{4h+\alpha}\right)^2 \qquad \hat{g}_{11} = -\left(1+\frac{\alpha}{4h}\right)^4 \qquad \hat{g}_{22} = -h^2\left(1+\frac{\alpha}{4h}\right)^4 \qquad \hat{g}_{33} = -h^2\left(1+\frac{\alpha}{4h}\right)^4 \sin^2\theta \quad \text{(D32)}$$

Consequently equations (12) for determination of the Christoffel symbols of the second kind can be applied. There are 28 Christoffel symbols of the second kind to consider. Calculation determines that there are only 9 non-zero such terms, viz,

$$\Gamma^0_{01} = \frac{8\alpha}{\left(16h^2-\alpha^2\right)} \qquad \Gamma^1_{11} = \frac{-2\alpha}{h(4h+\alpha)} \qquad \Gamma^2_{21} = \frac{(4h-\alpha)}{h(4h+\alpha)}$$

$$\Gamma^3_{31} = \frac{(4h-\alpha)}{h(4h+\alpha)} \qquad \Gamma^3_{32} = \cot\theta \qquad \Gamma^1_{00} = \frac{2\cdot 4^5 \alpha h^4 (4h-\alpha)}{(4h+\alpha)^7}$$

$$\Gamma^1_{22} = \frac{-h(4h-\alpha)}{(4h+\alpha)} \qquad \Gamma^1_{33} = \frac{-h(4h-\alpha)\sin^2\theta}{(4h+\alpha)} \qquad \Gamma^2_{33} = -\sin\theta\cos\theta \quad \text{(D33)}$$

There are 20 Riemann-Christoffel curvature tensor terms to consider. Calculation determines that there are only 6 non-zero such terms, viz,

$$\hat{R}_{0101} = \frac{16\alpha(4h-\alpha)^2}{h(4h+\alpha)^4} \qquad \hat{R}_{0202} = \frac{-8\alpha h(4h-\alpha)^2}{(4h+\alpha)^4} \qquad \hat{R}_{0303} = \frac{-8\alpha h(4h-\alpha)^2\sin^2\theta}{(4h+\alpha)^4}$$

$$\hat{R}_{1212} = \frac{2\alpha(4h+\alpha)^2}{4^3 h^3} \qquad \hat{R}_{1313} = \frac{2\alpha(4h+\alpha)^2\sin^2\theta}{4^3 h^3} \qquad \hat{R}_{2323} = \frac{-\alpha(4h+\alpha)^2\sin^2\theta}{4^2 h} \quad \text{(D34)}$$

Since the metric tensor is diagonal the only non-zero G_{ijkl} terms in the denominator for the Riemannian curvature are,

$$\hat{G}_{0101} = \frac{-(4h-\alpha)^2(4h+\alpha)^2}{4^4 h^4} \qquad \hat{G}_{0202} = \frac{-(4h-\alpha)^2(4h+\alpha)^2}{4^4 h^2}$$

$$\hat{G}_{0303} = \frac{-(4h-\alpha)^2(4h+\alpha)^2 \sin^2\theta}{4^4 h^2} \qquad \hat{G}_{1212} = \frac{(4h+\alpha)^8}{4^8 h^6}$$

$$\hat{G}_{1313} = \frac{(4h+\alpha)^8 \sin^2\theta}{4^8 h^6} \qquad \hat{G}_{2323} = \frac{(4h+\alpha)^8 \sin^2\theta}{4^8 h^4} \qquad \text{(D35)}$$

The Riemannian curvature is therefore,

$$\hat{K}_S = \frac{A+B}{C+D}$$

$$A = \frac{16\alpha(4h-\alpha)^2}{h(4h+\alpha)^4}\hat{W}_{0101} - \frac{8\alpha h(4h-\alpha)^2}{(4h+\alpha)^4}\hat{W}_{0202} - \frac{8\alpha h(4h-\alpha)^2 \sin^2\theta}{(4h+\alpha)^4}\hat{W}_{0303}$$

$$B = \frac{2\alpha(4h+\alpha)^2}{4^3 h^3}\hat{W}_{1212} + \frac{2\alpha(4h+\alpha)^2 \sin^2\theta}{4^3 h^3}\hat{W}_{1313} - \frac{\alpha(4h+\alpha)^2 \sin^2\theta}{4^2 h}\hat{W}_{2323}$$

$$C = \frac{-(4h-\alpha)^2(4h+\alpha)^2}{4^4 h^4}\hat{W}_{0101} - \frac{(4h-\alpha)^2(4h+\alpha)^2}{4^4 h^2}\hat{W}_{0202} - \frac{(4h-\alpha)^2(4h+\alpha)^2 \sin^2\theta}{4^4 h^2}\hat{W}_{0303}$$

$$D = \frac{(4h+\alpha)^8}{4^8 h^6}\hat{W}_{1212} + \frac{(4h+\alpha)^8 \sin^2\theta}{4^8 h^6}\hat{W}_{1313} + \frac{(4h+\alpha)^8 \sin^2\theta}{4^8 h^4}\hat{W}_{2323}$$

$$\hat{W}_{ijkl} = \begin{vmatrix} \hat{U}^i & \hat{U}^j \\ \hat{V}^i & \hat{V}^j \end{vmatrix} \begin{vmatrix} \hat{U}^k & \hat{U}^l \\ \hat{V}^k & \hat{V}^l \end{vmatrix}$$

$$h = \left[|\rho - \rho_0|^n + \left(\frac{\alpha}{4}\right)^n \right]^{1/n}$$

$$\rho, \rho_0 \in \Re \quad n \in \Re^+ \qquad \text{(D36)}$$

In (D36) the $\hat{W}_{ijkl}$ are determined by the linearly independent vectors $\hat{U}^i$ and $\hat{V}^i$ which correspond to U^i and V^i in the Schwarzschild form due to the conformal mapping of the spatial section of the Schwarzschild form.

When $\rho = \rho_0$, $h = \alpha/4$, for all ρ_0 and for all n, and the Riemannian curvature becomes,

$$\hat{K}_S(\rho_0) = \frac{8\left(\hat{W}_{1212} + \hat{W}_{1313}\sin^2\theta\right) - \alpha^2\hat{W}_{2323}\sin^2\theta}{16\alpha^2\left(\hat{W}_{1212} + \hat{W}_{1313}\sin^2\theta\right) + \alpha^4\hat{W}_{2323}\sin^2\theta} \qquad \text{(D37)}$$

Note that D(37) differs from (B27) due only to the terms in $\hat{W}_{2323}$ (i.e. if not for the $\hat{W}_{2323}$ terms the Riemannian curvature would be $1/(2\alpha^2)$ as for the Schwarzschild form). Moreover, (D37) depends upon ϑ and so at $\vartheta = 0$ and $\vartheta = \pi$ (D37) reduces to the exact value for the Schwarzschild form [expression (B27)]. Note also that (D37) is the negative of (D27) just as (B27) is the negative of (B15).

When $\vartheta = \pi/2$ the Riemannian curvature is,

$$\hat{K}_S = \frac{A+B}{C+D}$$

$$A = \frac{16\alpha(4h-\alpha)^2}{h(4h+\alpha)^4}\hat{W}_{0101} - \frac{8\alpha(4h-\alpha)^2}{(4h+\alpha)^4}\hat{W}_{0202} - \frac{8\alpha h(4h-\alpha)^2}{(4h+\alpha)^4}\hat{W}_{0303}$$

$$B = \frac{\alpha(4h+\alpha)^2}{32h^3}\hat{W}_{1212} + \frac{\alpha(4h+\alpha)^2}{32h^3}\hat{W}_{1313} - \frac{\alpha(4h+\alpha)^2}{16h}\hat{W}_{2323}$$

$$C = \frac{-(4h-\alpha)^2(4h+\alpha)^2}{4^4h^4}\hat{W}_{0101} - \frac{(4h-\alpha)^2(4h+\alpha)^2}{4^4h^2}\hat{W}_{0202} - \frac{(4h-\alpha)^2(4h+\alpha)^2}{4^4h^2}\hat{W}_{0303}$$

$$D = \frac{(4h+\alpha)^8}{4^8h^6}\hat{W}_{1212} + \frac{(4h+\alpha)^8}{4^8h^6}\hat{W}_{1313} + \frac{(4h+\alpha)^8}{4^8h^4}\hat{W}_{2323}$$

$$\hat{W}_{ijkl} = \begin{vmatrix} \hat{U}^i & \hat{U}^j \\ \hat{V}^i & \hat{V}^j \end{vmatrix}\begin{vmatrix} \hat{U}^k & \hat{U}^l \\ \hat{V}^k & \hat{V}^l \end{vmatrix}$$

$$h = \left[\left|\rho - \rho_0\right|^n + \left(\frac{\alpha}{4}\right)^n\right]^{\frac{1}{n}}$$

$$\rho, \rho_0 \in \Re \quad n \in \Re^+ \qquad \text{(D37b)}$$

Thus, the correspondence between the isotropic Schwarzschild form and the Schwarzschild form is,

$$\left[h(\rho_0) \Rightarrow \left(\hat{K}_S = \frac{8\left(\hat{W}_{1212} + \hat{W}_{1313}\sin^2\theta\right) - \alpha^2 \hat{W}_{2323}\sin^2\theta}{16\alpha^2\left(\hat{W}^{1212} + \hat{W}_{1313}\sin^2\theta\right) + \alpha^4 \hat{W}_{2323}\sin^2\theta} \right)\right] \Leftrightarrow \left[R_c(r_0) \Rightarrow \left(K_S = \frac{1}{2\alpha^2}\right)\right]$$

(isotropic Schwarzschild form) (Schwarzschild form)

(D38)

The acceleration invariant for the isotropic Schwarzschild form

Applying Doughty's [81] expression (C1) for the acceleration *β* of a point along a radial geodesic in (D12) gives,

$$\beta = \frac{128h^2}{(4h+\alpha)^3(4h-\alpha)} \tag{D39}$$

It follows from (D36) and (D39) that,

$$(\rho \to \rho_0) \Rightarrow \left(h \to \frac{\alpha}{4}\right) \Rightarrow (\beta \to \infty) \qquad \forall \rho_0 \ \forall n \tag{D40}$$

Thus, (D40) is an invariant for the isotropic Schwarzschild form just as for the Schwarzschild form [see (C5)].

The isotropic Reissner-Nordström form

The Reissner-Nordström solution is,

$$ds^2 = c^2\left(1 - \frac{2Gm}{c^2 r} + \frac{q^2}{r^2}\right)dt^2 - \left(1 - \frac{2Gm}{c^2 r} + \frac{q^2}{r^2}\right)^{-1} dr^2 - r^2\left(d\theta^2 + \sin^2\theta \, d\varphi^2\right)$$

$$0 \leq r \tag{D41}$$

The astrophysical scientists render the Reissner-Nordström solution in isotropic coordinates by setting,

$$r = \rho\left[\left(1 + \frac{Gm}{2c^2\rho}\right)^2 - \frac{q^2}{4\rho^2}\right] \tag{D42}$$

Using (D42) the Reissner-Nordström metric in isotropic coordinates is,

$$ds^2 = c^2 \frac{\left[1 - \left(\frac{Gm}{2c^2\rho}\right)^2 + \frac{q^2}{4\rho^2}\right]^2}{\left(1 + \frac{Gm}{2c^2\rho} + \frac{q}{2\rho}\right)^2 \left(1 + \frac{Gm}{2c^2\rho} - \frac{q}{2\rho}\right)^2} dt^2 -$$

$$- \left(1 + \frac{Gm}{2c^2\rho} + \frac{q}{2\rho}\right)^2 \left(1 + \frac{Gm}{2c^2\rho} - \frac{q}{2\rho}\right)^2 \left[d\rho^2 + \rho^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)\right]$$

(D43)

wherein $\rho = \sqrt{x^2 + y^2 + z^2}$, owing to which (D43) is sometimes written as,

$$ds^2 = c^2 \frac{\left[1 - \left(\frac{Gm}{2c^2\rho}\right)^2 + \frac{q^2}{4\rho^2}\right]^2}{\left(1 + \frac{Gm}{2c^2\rho} + \frac{q}{2\rho}\right)^2 \left(1 + \frac{Gm}{2c^2\rho} - \frac{q}{2\rho}\right)^2} dt^2 -$$

$$- \left(1 + \frac{Gm}{2c^2\rho} + \frac{q}{2\rho}\right)^2 \left(1 + \frac{Gm}{2c^2\rho} - \frac{q}{2\rho}\right)^2 \left(dx^2 + dy^2 + dz^2\right)$$

$$\rho = \sqrt{x^2 + y^2 + z^2} \qquad \text{(D44)}$$

Note that the spatial section of (D43) has precisely the metric form of (D2), where

$$H^2(\rho) = \left(1 + \frac{Gm}{2c^2\rho} + \frac{q}{2\rho}\right)^2 \left(1 + \frac{Gm}{2c^2\rho} - \frac{q}{2\rho}\right)^2$$

I now adduce the generator of the infinite set of equivalent isotropic Reissner-Nordström forms (using *c* = 1),

$$ds^2 = \frac{\left[1 - \frac{\alpha^2}{16h^2} + \frac{q^2}{4h^2}\right]^2}{\left(1 + \frac{\alpha}{4h} + \frac{q}{2h}\right)^2 \left(1 + \frac{\alpha}{4h} - \frac{q}{2h}\right)^2} dt^2 - \left(1 + \frac{\alpha}{4h} + \frac{q}{2h}\right)^2 \left(1 + \frac{\alpha}{4h} - \frac{q}{2h}\right)^2 \left[dh^2 + h^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)\right]$$

$$h = \left[\left|\rho - \rho_0\right|^n + \omega^n\right]^{1/n}$$

$$\omega = \frac{\sqrt{\alpha^2 - 4q^2}}{4} \qquad 4q^2 < \alpha^2$$

$$\rho, \rho_0 \in \Re \quad n \in \Re^+ \tag{D45}$$

wherein ρ_0 and n are entirely arbitrary constants. Accordingly, the transformation from the Reissner-Nordström form (B25) to isotropic coordinates is by means of,

$$R_c = h\left[\left(1 + \frac{\alpha}{4h}\right)^2 - \frac{q^2}{4h^2}\right] \tag{D46}$$

where h is the function of ρ given by (D45) and R_c by (A17) when $a = 0$ in the latter.
The radius R_p for (D41) is,

$$R_p = \int\left(1 + \frac{\alpha}{4h} + \frac{q}{2h}\right)\left(1 + \frac{\alpha}{4h} - \frac{q}{2h}\right)dh \quad = \quad h + \frac{\alpha}{2}\ln\left(\frac{4h}{\sqrt{\alpha^2 - 4q^2}}\right) - \frac{\left(\alpha^2 - 4q^2\right)}{16h} \tag{D47}$$

Note that (D47) is zero at $\rho = \rho_0$, $\forall\,\rho_0\ \forall\,n$, as it must.

The acceleration invariant for the isotropic Reissner-Nordström form

Applying Doughty's [81] expression (C1) for the acceleration *β* of a point along a radial geodesic in (D45) gives,

$$\beta = \frac{8h^2\left[64(4h + \alpha + 2q)(4h + \alpha - 2q)h - 16\left(16h^2 - \alpha^2 + 4q^2\right)(4h + \alpha)\right]}{(4h + \alpha + 2q)^2(4h + \alpha - 2q)^2\left(16h^2 - \alpha^2 + 4q^2\right)} \tag{D48}$$

It follows from (D45) and (D48) that,

$$(\rho \to \rho_0) \Rightarrow \left(h \to \frac{\sqrt{\alpha^2 - 4q^2}}{4}\right) \Rightarrow (\beta \to \infty) \qquad \forall\,\rho_0\ \forall\,n \tag{D49}$$

Thus, (D49) is an invariant for the isotropic Reissner-Nordström form, in similar fashion as for the Schwarzschild form (C5), the Reissner-Nordström form, and the isotropic Schwarzschild form (D39). Note that if $q = 0$ then expression (D49) reduces to that for the acceleration of a point along a radial geodesic in the isotropic Schwarzschild form [see (D39)].

Gaussian curvature of the surface in the spatial section of the isotropic Reissner-Nordström form

The surface in the spatial section of (D45) is described by,

$$ds^2 = \left(1 + \frac{\alpha}{4h} + \frac{q}{2h}\right)^2 \left(1 + \frac{\alpha}{4h} - \frac{q}{2h}\right)^2 h^2 \left(d\theta^2 + \sin^2\theta \, d\varphi^2\right) \qquad \text{(D50)}$$

Since metrics of dimension 2 are conformal to any other, and in accordance with Appendix A, the Gaussian curvature $\hat{K}$ of (D50) is given by,

$$\hat{K} = \frac{1}{h^2 \left(1 + \frac{\alpha}{4h} + \frac{q}{2h}\right)^2 \left(1 + \frac{\alpha}{4h} - \frac{q}{2h}\right)^2} \qquad \text{(D51)}$$

This is a positive constant for any given admissible value for *h* and so (D50), by **Definition 2** (see section **VI**), is a spherical surface.

By (D45), at $h(\rho_0)$ (D51) takes the value,

$$\hat{K} = \frac{16\left(\alpha^2 - 4q^2\right)}{\left[\left(\alpha + \sqrt{\alpha^2 - 4q^2}\right)^2 - 4q^2\right]^2} \qquad \text{(D52)}$$

(D52) corresponds to (A33) for the Reissner-Nordström form. Indeed, by (D45) and (D46),

$$h(\rho_0)\left[\left(1 + \frac{\alpha}{4h(\rho_0)}\right)^2 - \frac{q^2}{4h^2(\rho_0)}\right] = \frac{\alpha}{2} + \frac{\sqrt{\alpha^2 - 4q^2}}{2} = R_c(r_0) \qquad \text{(D53)}$$

which is just (B17), and holds for all arbitrary ρ_0 for all arbitrary r_0 for all arbitrary *n*. Thus,

$$\left[h(\rho_0) \Rightarrow \left(\hat{K} = \frac{16\left(\alpha^2 - 4q^2\right)}{\left[\left(\alpha + \sqrt{\alpha^2 - 4q^2}\right)^2 - 4q^2\right]^2}\right)\right] \Leftrightarrow \left[R_c(r_0) \Rightarrow K = \left(\frac{\alpha}{2} + \frac{\sqrt{\alpha^2 - 4q^2}}{2}\right)^{-2}\right] \qquad \text{(D54)}$$

(isotropic Reissner-Nordstrom förm) (Reissner-Nordström form)

By (D52) or (D54) every metric in the infinite set of equivalent metrics generated by (D45) produces the same invariant Gaussian curvature (D52), as they must, and this invariant corresponds to that for the Reissner-Nordstrom form (A33), which is in its turn an invariant produced by every metric in the infinite set generated by (A17) when *a* = 0 in the latter.

Riemannian curvature of the spatial section of the isotropic Reissner-Nordström form

Since (D45) is equivalent to expressions (A17) when *a* = 0, the curvature invariants for (D45) must correspond to curvature invariants for (A17) when *a* = 0, but are not necessarily the same. To see that (D45) produces corresponding curvature invariants first consider the spatial section of (D45), given by,

$$ds^2 = \left(1+\frac{\alpha}{4h}+\frac{q}{2h}\right)^2\left(1+\frac{\alpha}{4h}-\frac{q}{2h}\right)^2\left[dh^2 + h^2\left(d\theta^2+\sin^2\theta\, d\varphi^2\right)\right]$$

$$h = \left[\left|\rho-\rho_0\right|^n + \omega^n\right]^{\frac{1}{n}}$$

$$\omega = \frac{\sqrt{\alpha^2-4q^2}}{4} \qquad 4q^2 < \alpha^2$$

$$\rho, \rho_0 \in \Re \quad n \in \Re^+ \tag{D55}$$

This is a positive-definite quadratic form, as it must for spherical symmetry. The metric tensor is diagonal,

$$\hat{g}_{ik} = \begin{bmatrix} \left(1+\frac{\alpha}{4h}+\frac{q}{2h}\right)^2\left(1+\frac{\alpha}{4h}-\frac{q}{2h}\right)^2 & 0 & 0 \\ 0 & h^2\left(1+\frac{\alpha}{4h}+\frac{q}{2h}\right)^2\left(1+\frac{\alpha}{4h}-\frac{q}{2h}\right)^2 & 0 \\ 0 & 0 & h^2\left(1+\frac{\alpha}{4h}+\frac{q}{2h}\right)^2\left(1+\frac{\alpha}{4h}-\frac{q}{2h}\right)^2\sin^2\theta \end{bmatrix}$$

(D56)

The components of the metric tensor are,

$$\hat{g}_{11} = \left(1+\frac{\alpha}{4h}+\frac{q}{2h}\right)^2\left(1+\frac{\alpha}{4h}-\frac{q}{2h}\right)^2 \qquad \hat{g}_{22} = h^2\left(1+\frac{\alpha}{4h}+\frac{q}{2h}\right)^2\left(1+\frac{\alpha}{4h}-\frac{q}{2h}\right)^2$$

$$\hat{g}_{33} = h^2\left(1+\frac{\alpha}{4h}+\frac{q}{2h}\right)^2\left(1+\frac{\alpha}{4h}-\frac{q}{2h}\right)^2 \sin^2\theta$$

(D57)

Make the following assignments,

$$x^1 = h \qquad x^2 = \theta \qquad x^3 = \varphi \tag{D58}$$

Since (D56) is diagonal, the Christoffel symbols of the second kind can be calculated using the relations (B6). There are 15 Christoffel symbols of the second kind to consider. Calculation determines that there are only 7 non-zero such terms, viz,

$$\Gamma^1_{11} = \frac{2\left(4q^2-4\alpha h-\alpha^2\right)}{h(4h+\alpha+2q)(4h+\alpha-2q)} \qquad \Gamma^2_{21} = \frac{\left(16h^2-\alpha^2+4q^2\right)}{h(4+\alpha+2q)(4h+\alpha-2q)} \qquad \Gamma^3_{31} = \Gamma^2_{21}$$

$$\Gamma^3_{32} = \cot\theta \qquad \Gamma^1_{22} = \frac{-h\left(16h^2-\alpha^2+4q^2\right)}{(4h+\alpha+2q)(4h+\alpha-2q)} \qquad \Gamma^1_{33} = \Gamma^1_{22}\sin^2\theta$$

$$\Gamma^2_{33} = -\sin\theta\cos\theta$$

(D59)

There are 6 components of the Riemann-Christoffel curvature tensor to consider. Calculation determines that there are only 3 non-zero such terms, viz,

$$\hat{R}_{1212} = \frac{(4h+\alpha)\left(16h^2-\alpha^2+4q^2\right)-4h\left[(4h+\alpha)^2-4q^2\right]}{32h^3}$$

$$\hat{R}_{1313} = \hat{R}_{1212}\sin^2\theta$$

$$\hat{R}_{2323} = \frac{\left\{4h\left[(4h+\alpha)^2-4q^2\right]-(4h+\alpha)\left(16h^2-\alpha^2+4q^2\right)\right\}+16q^2h}{4^2h}\sin^2\theta \tag{D60}$$

Since the metric is diagonal the only non-zero G_{ijkl} are,

$$\hat{G}_{1212} = \frac{(4h+\alpha+2q)^4(4h+\alpha-2q)^4}{4^8h^6} \qquad \hat{G}_{1313} = \hat{G}_{1212}\sin^2\theta$$

$$\hat{G}_{2323} = \frac{(4h+\alpha+2q)^4(4h+\alpha-2q)^4\sin^2\theta}{4^8h^4}$$

(D61)

The Riemannian curvature $\hat{K}$ is given by,

$$\hat{K}_S = \frac{\hat{R}_{1212}\left(\hat{W}_{1212} + \hat{W}_{1313}\sin^2\theta\right) + \hat{R}_{2323}\hat{W}_{2323}}{\hat{G}_{1212}\left(\hat{W}_{1212} + \hat{W}_{1313}\sin^2\theta\right) + \hat{G}_{2323}\hat{W}_{2323}}$$

$$\hat{W}_{ijkl} = \begin{vmatrix} \hat{U}^i & \hat{U}^j \\ \hat{V}^i & \hat{V}^j \end{vmatrix} \begin{vmatrix} \hat{U}^k & \hat{U}^l \\ \hat{V}^k & \hat{V}^l \end{vmatrix}$$

$$h = \left[\left|\rho - \rho_0\right|^n + \omega^n\right]^{1/n}$$

$$\omega = \frac{\sqrt{\alpha^2 - 4q^2}}{4} \qquad 4q^2 < \alpha^2$$

$$\rho, \rho_0 \in \Re \quad n \in \Re^+ \qquad \text{(D62)}$$

wherein the $\hat{R}_{ikjl}$ and the $\hat{G}_{ijkl}$ are given by expressions (D60) and (D61) respectively. In (D62) the $\hat{W}_{ijkl}$ are determined by the linearly independent vectors $\hat{U}^i$ and $\hat{V}^i$ which correspond to U^i and V^i in the spatial section of the Reissner-Nordström form, due to the conformal mapping thereof. If q = 0, (D62), by means of (D60) and (D61), reduces to (D28) for the spatial section of the isotropic Schwarzschild form.

When $\rho = \rho_0$, $h = \omega = \sqrt{\alpha^2 - 4q^2}/4$ for (D62), for all ρ_0 and for all n, and the Riemannian curvature becomes,

$$\hat{K}_S = \frac{-\left[(4\omega+\alpha)^2 - 4q^2\right]\left(\hat{W}_{1212} + \hat{W}_{1313}\sin^2\theta\right) + 2\omega^2(4\omega+\alpha)^2\hat{W}_{2323}\sin^2\theta}{\frac{2\left[(4\omega+\alpha)^2 - 4q^2\right]^4}{4^7\omega^4}\left(\hat{W}_{1212} + \hat{W}_{1313}\sin^2\theta\right) + \frac{2\left[(4\omega+\alpha)^2 - 4q^2\right]^4}{4^7\omega^2}\hat{W}_{2323}\sin^2\theta}$$

$$\hat{W}_{ijkl} = \begin{vmatrix} \hat{U}^i & \hat{U}^j \\ \hat{V}^i & \hat{V}^j \end{vmatrix} \begin{vmatrix} \hat{U}^k & \hat{U}^l \\ \hat{V}^k & \hat{V}^l \end{vmatrix}$$

$$\omega = \frac{\sqrt{\alpha^2 - 4q^2}}{4} \qquad 4q^2 < \alpha^2 \qquad \text{(D63)}$$

If q = 0 then (D63) reduces to expression (D29) for the spatial section of the isotropic Schwarzschild form.

Eq.(D62) depends upon ϑ and so at $\vartheta = 0$ and $\vartheta = \pi$ (D62) reduces to,

$$\hat{K}_S = \frac{4^6 h^3 \left\{ (4h+\alpha)(16h^2 - \alpha^2 + 4q^2) - 4h\left[(4h+\alpha)^2 - 4q^2 \right] \right\}}{2\left[(4h+\alpha)^2 - 4q^2 \right]^4}$$

$$h = \left[|\rho - \rho_0|^n + \omega^n \right]^{1/n}$$

$$\omega = \frac{\sqrt{\alpha^2 - 4q^2}}{4} \qquad 4q^2 < \alpha^2 \qquad \rho, \rho_0 \in \Re \quad n \in \Re^+ \tag{D64}$$

(D64) is independent of the direction vectors $\hat{U}^i$ and $\hat{V}^i$ and so $\vartheta = 0$ and $\vartheta = \pi$ produce isotropic points. When $\rho = \rho_0$ (D64) reduces to,

$$\hat{K}_S = -\frac{1}{2\alpha^2} \tag{D65}$$

which is precisely expression (B14) for the spatial section of the Schwarzschild form, and for the spatial section of the isotropic Schwarzschild form when $\vartheta = 0$ and $\vartheta = \pi$. Thus the spatial sections of the isotropic Schwarzschild form and the Reissner-Nordström form have the very same isotropic Riemannian curvature when $\rho = \rho_0$ and $\vartheta = 0$ or $\vartheta = \pi$, irrespective of the values of ρ_o and n, and this value is that for the spatial section of the Schwarzschild form when $\rho = \rho_0$, which is independent of ϑ in the latter form.

Thus, the correspondence between the spatial section of the Reissner-Nordström form and the spatial section of the isotropic Reissner-Nordström form is,

$$h(\rho_0) \Rightarrow \hat{K}_S = \frac{-8\left[\left(\sqrt{\alpha^2 - 4q^2} + a \right)^2 - 4q^2 \right] \left(\hat{W}_{1212} + \hat{W}_{1313} \sin^2\theta \right) + \left(\alpha^2 - 4q^2 \right) \left(\sqrt{\alpha^2 - 4q^2} + \alpha \right)^2 \hat{W}_{2323} \sin^2\theta}{\frac{\left[\left(\sqrt{\alpha^2 - 4q^2} + \alpha \right)^2 - 4q^2 \right]^4}{4\left(\alpha^2 - 4q^2 \right)^2} \left(\hat{W}_{1212} + \hat{W}_{1313} \sin^2\theta \right) + \frac{\left[\left(\sqrt{\alpha^2 - 4q^2} + \alpha \right)^2 - 4q^2 \right]^4}{4^3 \left(\alpha^2 - 4q^2 \right)} \hat{W}_{2323} \sin^2\theta} \Leftrightarrow$$

(spatial section isotropic Reissner-Nordström form)

$$\Leftrightarrow R_c(r_0) \Rightarrow K_S = -\frac{\left(\alpha^2 + \alpha\sqrt{\alpha^2 - 4q^2} - 4q^2 \right)}{4\left(\frac{\alpha}{2} + \frac{\sqrt{\alpha^2 - 4q^2}}{2} \right)^4}$$

(spatial section Reissner-Nordström form)

(D66)

Riemannian curvature of the isotropic Reissner-Nordström form

The isotropic Reissner-Nordström form is given by (D45). To facilitate the calculations rewrite (D45) in the following simplified form:

$$ds^2 = \frac{\left(16h^2 - \alpha^2 + 4q^2\right)^2}{(4h+\alpha+2q)^2(4h+a-2q)^2}dt^2 - \frac{(4h+\alpha+2q)^2(4h+a-2q)^2}{4^4 h^4}\left[dh^2 + h^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)\right]$$

$$h = \left[\left|\rho - \rho_0\right|^n + \omega^n\right]^{1/n}$$

$$\omega = \frac{\sqrt{\alpha^2 - 4q^2}}{4} \qquad 4q^2 < \alpha^2$$

$$\rho, \rho_0 \in \Re \quad n \in \Re^+ \tag{D67}$$

The metric tensor is diagonal,

$$\hat{g}_{ik} = \begin{bmatrix} \frac{\left(16h^2-\alpha^2+4q^2\right)^2}{(4h+a+2q)^2(4h+\alpha-2q)^2} & 0 & 0 & 0 \\ 0 & -\frac{(4h+\alpha+2q)^2(4h+\alpha-2q)^2}{4^4h^4} & 0 & 0 \\ 0 & 0 & -\frac{(4h+\alpha+2q)^2(4h+\alpha-2q)^2}{4^4h^2} & 0 \\ 0 & 0 & 0 & -\frac{(4h+\alpha+2q)^2(4h+\alpha-2q)^2\sin^2\theta}{4^4h^2} \end{bmatrix} \tag{D68}$$

The components of the metric tensor are,

$$\hat{g}_{00} = \frac{\left(16h^2 - \alpha^2 + 4q^2\right)^2}{(4h+\alpha+2q)^2(4h+\alpha-2q)^2} \qquad \hat{g}_{11} = -\frac{(4h+\alpha+2q)^2(4h+\alpha-2q)^2}{4^4h^4}$$

$$\hat{g}_{22} = -\frac{(4h+\alpha+2q)^2(4h+\alpha-2q)^2}{4^4h^2} \qquad \hat{g}_{33} = -\frac{(4h+\alpha+2q)^2(4h+\alpha-2q)^2}{4^4h^2}\sin^2\theta$$

(D69)

Make the following assignments,

$$x^0 = t \qquad x^1 = h \qquad x^2 = \theta \qquad x^3 = \varphi \tag{D70}$$

There are 28 Riemann-Christoffel symbols of the second kind to consider. Calculation determines that there are only 9 non-zero such terms,

$$\Gamma^0_{01} = \frac{-8\left\{4h\left[(4h+\alpha)^2 - 4q^2\right] - (4h+\alpha)\left(16h^2-\alpha^2+4q^2\right)\right\}}{(4h+\alpha+2q)(4h+\alpha-2q)\left(16h^2-\alpha^2+4q^2\right)}$$

$$\Gamma^1_{00} = \frac{2\cdot 4^5 h^4\left(16h^2-\alpha^2+4q^2\right)\left[4h(4h+\alpha+2q)(4h+\alpha-2q) - (4h+\alpha)\left(16h^2-\alpha^2+4q^2\right)\right]}{(4h+\alpha+2q)^5(4h+\alpha-2q)^5}$$

$$\Gamma^1_{11} = \frac{2\left(4q^2 - 4\alpha h - \alpha^2\right)}{h\left(4h+\alpha+2q\right)\left(4h+\alpha-2q\right)} \qquad \Gamma^2_{21} = \frac{\left(16h^2 - \alpha^2 + 4q^2\right)}{h\left(4+\alpha+2q\right)\left(4h+\alpha-2q\right)} \qquad \Gamma^3_{31} = \Gamma^2_{21}$$

$$\Gamma^3_{32} = \cot\theta \qquad \Gamma^1_{22} = \frac{-h\left(16h^2 - \alpha^2 + 4q^2\right)}{\left(4h+\alpha+2q\right)\left(4h+\alpha-2q\right)} \qquad \Gamma^1_{33} = \Gamma^1_{22}\sin^2\theta$$

$$\Gamma^2_{33} = -\sin\theta\cos\theta$$

(D71)

There are 20 components of the Riemann-Christoffel curvature tensor to consider. Calculation determines that there are only 6 non-zero such terms,

$$\hat{R}_{0101} = \frac{8\left(16h^2 - \alpha^2 + 4q^2\right)\left(32q^2 - 32\alpha h - 8\alpha^2\right) + 4^4 hY}{\left(4h+\alpha+2q\right)^3\left(4h+\alpha-2q\right)^3} + \frac{32\left(16h^2 - \alpha^2 + 4q^2\right)Y}{\left(4h+\alpha+2q\right)^3\left(4h+\alpha-2q\right)^4}$$

$$+\frac{32\left(16h^2 - \alpha^2 + 4q^2\right)Y}{\left(4h+\alpha+2q\right)^4\left(4h+\alpha-2q\right)^3} + \frac{16\left(16h^2 - \alpha^2 + 4q^2\right)\left(4q^2 - 4\alpha h - \alpha^2\right)Y}{h\left(4h+\alpha+2q\right)^4\left(4h+\alpha-2q\right)^4}$$

$$-\frac{64Y^2}{\left(4h+\alpha+2q\right)^4\left(4h+\alpha-2q\right)^4}$$

$$Y = \left\{4h\left[\left(4h+\alpha\right)^2 - 4q^2\right] - \left(4h+\alpha\right)\left(16h^2 - \alpha^2 + 4q^2\right)\right\}$$

$$\hat{R}_{0202} = \frac{-8h\left(16h^2 - \alpha^2 + 4q^2\right)^2 Y}{\left(4h+\alpha+2q\right)^4\left(4h+\alpha-2q\right)^4}$$

$$\hat{R}_{0303} = \hat{R}_{0202}\sin^2\theta$$

$$\hat{R}_{1212} = \frac{Y}{32h^3}$$

$$\hat{R}_{1313} = \hat{R}_{1212}\sin^2\theta$$

$$\hat{R}_{2323} = -\frac{\left\{4h\left[\left(4h+\alpha\right)^2 - 4q^2\right] - \left(4h+\alpha\right)\left(16h^2 - \alpha^2 + 4q^2\right)\right\} + 16q^2 h}{4^2 h}\sin^2\theta$$

(D72)

Since the metric is diagonal the only non-zero G_{ijkl} are,

$$\hat{G}_{0101} = -\frac{\left(16h^2 - \alpha^2 + 4q^2\right)^2}{4^4 h^4} \qquad \hat{G}_{0202} = -\frac{\left(16h^2 - \alpha^2 + 4q^2\right)^2}{4^4 h^2} \qquad \hat{G}_{0303} = \hat{G}_{0202}\sin^2\theta$$

$$\hat{G}_{1212} = \frac{(4h+\alpha+2q)^4(4h+\alpha-2q)^4}{4^8 h^6} \qquad \hat{G}_{1313} = \hat{G}_{1212}\sin^2\theta$$

$$\hat{G}_{2323} = \frac{(4h+\alpha+2q)^4(4h+\alpha-2q)^4\sin^2\theta}{4^8 h^4}$$

(D73)

The Riemannian curvature $\hat{K}$ is given by,

$$\hat{K}_S = \frac{\hat{R}_{0101}\hat{W}_{0101} + \hat{R}_{0202}\left(\hat{W}_{0202} + \hat{W}_{0303}\sin^2\theta\right) + \hat{R}_{1212}\left(\hat{W}_{1212} + \hat{W}_{1313}\sin^2\theta\right) + \hat{R}_{2323}\hat{W}_{2323}}{\hat{G}_{0101}\hat{W}_{0101} + \hat{G}_{0202}\left(\hat{W}_{0202} + \hat{W}_{0303}\sin^2\theta\right) + \hat{G}_{1212}\left(\hat{W}_{1212} + \hat{W}_{1313}\sin^2\theta\right) + \hat{G}_{2323}\hat{W}_{2323}}$$

$$\hat{W}_{ijkl} = \begin{vmatrix}\hat{U}^i & \hat{U}^j \\ \hat{V}^i & \hat{V}^j\end{vmatrix}\begin{vmatrix}\hat{U}^k & \hat{U}^l \\ \hat{V}^k & \hat{V}^l\end{vmatrix}$$

$$h = \left[\left|\rho - \rho_0\right|^n + \omega^n\right]^{1/n}$$

$$\omega = \frac{\sqrt{\alpha^2 - 4q^2}}{4} \qquad 4q^2 < \alpha^2$$

$$\rho, \rho_0 \in \Re \quad n \in \Re^+ \tag{D74}$$

wherein the $\hat{R}_{ikjl}$ and the $\hat{G}_{ijkl}$ are given by expressions (D72) and (D73) respectively. In (D74) the $\hat{W}_{ijkl}$ are determined by the linearly independent vectors $\hat{U}^i$ and $\hat{V}^i$ which correspond to U^i and V^i in the Reissner-Nordström form, due to the conformal mapping of the spatial section thereof. If q = 0, (D74), by means of (D72) and (D73), reduces to (D36) for the isotropic Schwarzschild form.

When $\rho = \rho_0$, $h = \omega = \sqrt{\alpha^2 - 4q^2}\,/4$ for (D74), for all ρ_0 and for all n, and the Riemannian curvature becomes,

$$\hat{K}_S = \frac{\dfrac{4\left(\alpha^2 - 4q^2 + \alpha\sqrt{\alpha^2 - 4q^2}\right)}{\left(\alpha^2 - 4q^2\right)}\left(\hat{W}_{1212} + \hat{W}_{1313}\sin^2\theta\right) - \dfrac{\left(\sqrt{\alpha^2 - 4q^2} + \alpha\right)^2}{4}\hat{W}_{2323}\sin^2\theta}{\dfrac{\left[\left(\sqrt{\alpha^2 - 4q^2} + \alpha\right)^2 - 4q^2\right]^4}{4^2\left(\alpha^2 - 4q^2\right)^3}\left(\hat{W}_{1212} + \hat{W}_{1313}\sin^2\theta\right) + \dfrac{\left[\left(\sqrt{\alpha^2 - 4q^2} + \alpha\right)^2 - 4q^2\right]^4}{4^4\left(\alpha^2 - 4q^2\right)^2}\hat{W}_{2323}\sin^2\theta} \tag{D75}$$

(D74) depends upon ϑ and so at ϑ = 0 and $\vartheta = \pi$ (D74) reduces to,

$$\hat{K}_S = \frac{\hat{R}_{0101}\hat{W}_{0101} + \hat{R}_{0202}\hat{W}_{0202} + \hat{R}_{1212}\hat{W}_{1212}}{\hat{G}_{0101}\hat{W}_{0101} + \hat{G}_{0202}\hat{W}_{0202} + \hat{G}_{1212}\hat{W}_{1212}}$$

$$\hat{W}_{ijkl} = \begin{vmatrix} \hat{U}^i & \hat{U}^j \\ \hat{V}^i & \hat{V}^j \end{vmatrix} \begin{vmatrix} \hat{U}^k & \hat{U}^l \\ \hat{V}^k & \hat{V}^l \end{vmatrix}$$

$$h = \left[\left|\rho - \rho_0\right|^n + \omega^n\right]^{1/n}$$

$$\omega = \frac{\sqrt{\alpha^2 - 4q^2}}{4} \qquad 4q^2 < \alpha^2$$

$$\rho, \rho_0 \in \Re \quad n \in \Re^+ \tag{D76}$$

wherein the $\hat{R}_{ikjl}$ and the $\hat{G}_{ijkl}$ are still given by expressions (D72) and (D73) respectively. If q = 0 then (D75) reduces to that for the isotropic Schwarzschild form (D37), and hence to (B27).

At $\vartheta = \pi/2$ the Riemannian curvature is,

$$\hat{K}_S = \frac{\hat{R}_{0101}\hat{W}_{0101} + \hat{R}_{0202}\left(\hat{W}_{0202} + \hat{W}_{0303}\right) + \hat{R}_{1212}\left(\hat{W}_{1212} + \hat{W}_{1313}\right) + \hat{R}_{2323}\hat{W}_{2323}}{\hat{G}_{0101}\hat{W}_{0101} + \hat{G}_{0202}\left(\hat{W}_{0202} + \hat{W}_{0303}\right) + \hat{G}_{1212}\left(\hat{W}_{1212} + \hat{W}_{1313}\right) + \hat{G}_{2323}\hat{W}_{2323}}$$

$$\hat{W}_{ijkl} = \begin{vmatrix} \hat{U}^i & \hat{U}^j \\ \hat{V}^i & \hat{V}^j \end{vmatrix} \begin{vmatrix} \hat{U}^k & \hat{U}^l \\ \hat{V}^k & \hat{V}^l \end{vmatrix}$$

$$h = \left[\left|\rho - \rho_0\right|^n + \omega^n\right]^{1/n}$$

$$\omega = \frac{\sqrt{\alpha^2 - 4q^2}}{4} \qquad 4q^2 < \alpha^2 \qquad \rho, \rho_0 \in \Re \quad n \in \Re^+ \tag{D77}$$

wherein the remainder of the $\hat{R}_{ikjl}$ and the $\hat{G}_{ijkl}$ are still given by expressions (D72) and (D73) respectively. Once again, if q = 0 then (D74) reduces to that for the isotropic Schwarzschild form (D).

Thus, the correspondence between the Reissner-Nordström form and the isotropic Reissner-Nordström form is,

$$h(\rho_0) \Rightarrow \hat{K}_S = \frac{\dfrac{4\left(\alpha^2 - 4q^2 + \alpha\sqrt{\alpha^2 - 4q^2}\right)}{\left(\alpha^2 - 4q^2\right)}\left(\hat{W}_{1212} + \hat{W}_{1313}\sin^2\theta\right) - \dfrac{\left(\sqrt{\alpha^2 - 4q^2} - 4q^2\right)^2}{4}\hat{W}_{2323}\sin^2\theta}{\dfrac{\left[\left(\sqrt{\alpha^2 - 4q^2} + \alpha\right)^2 - 4q^2\right]^4}{4^2\left(\alpha^2 - 4q^2\right)^3}\left(\hat{W}_{1212} + \hat{W}_{1313}\sin^2\theta\right) + \dfrac{\left[\left(\sqrt{\alpha^2 - 4q^2} + \alpha\right)^2 - 4q^2\right]^4}{4^4\left(\alpha^2 - 4q^2\right)^2}\hat{W}_{2323}\sin^2\theta} \Leftrightarrow$$

(isotropic Reissner-Nordström form)

$$\Leftrightarrow R_c(r_0) \Rightarrow K_S = \frac{\left(\alpha^2 + \alpha\sqrt{\alpha^2 - 4q^2} - 4q^2\right)}{4\left(\dfrac{\alpha}{2} + \dfrac{\sqrt{\alpha^2 - 4q^2}}{2}\right)^4}$$

(Reissner-Nordström form)

(D78)

Black holes are also inconsistent with the isotropic forms

Curiously, the proponents of black holes do not use Hilbert's solution in isotropic form to describe their associated black holes. The reason is simple; in (D4), they, amongst others, incorrectly call the quantity ρ the "*radial coordinate*" [17, 21, 96], the "*radius variable*" [84], "*the radius*" [97], and the "*distance r_1 from the origin*" [83], and thereby, for (D4) to observe Hilbert's $0 \leq r$, it requires according to (D3) that,

$$-\frac{Gm}{2c^2} \leq \rho \qquad \text{(D79)}$$

However, by (D8) through to (D12), ρ is neither the radius nor a distance in (D4), just as r is neither the radius nor a distance in Hilbert's solution. Treating ρ as the radius or a distance in (D4) leads to inconsistencies with the notions of black holes obtained from Hilbert's solution. To amplify this rewrite (D4) thus,

$$ds^2 = c^2\left(\frac{2c^2\rho - Gm}{2c^2\rho + Gm}\right)^2 dt^2 - \left(1 + \frac{Gm}{2c^2\rho}\right)^4\left[d\rho^2 + \rho^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)\right]$$

(D80)

When $\rho = Gm/2c^2$ the coefficient of dt^2 vanishes, but the metric is not singular. When $\rho = 0$ the coefficient of dt^2 is 1 and the coefficient of the spatial section is singular, but there is no corresponding value for r in (D3) and hence no corresponding value in Hilbert's solution. When $\rho = -Gm/2c^2$ the coefficient of dt^2 is singular, the coefficient of the spatial section vanishes, and the value of r in (D3) is 0. This again reveals the veracity of (D6) which alone is consistent. In other words, (D6) cannot be extended, which is natural since the Schwarzschild forms (32) cannot be extended.

Similarly black holes are not consistent with the isotropic Reissner-Nordström form, of course.

APPENDIX E – THE KRETSCHMANN SCALAR

The Kretschmann scalar is also known as the Riemann tensor scalar curvature invariant. The Kretschmann scalar for the Kerr-Newman form has been obtained by Henry [98]. Hence,

$$f=\frac{8}{\left(R_c^2+a^2\cos^2\theta\right)^6}\left[\begin{array}{c}\frac{3\alpha^2}{2}\left(R_c^6-15a^2R_c^4\cos^2\theta+15a^4R_c^2\cos^4\theta-a^6\cos^6\theta\right)-\\ -6\alpha q^2R_c\left(R_c^4-10a^2R_c^2\cos^2\theta+5a^4\cos^4\theta\right)+\\ +q^4\left(7R_c^4-34a^2R_c^2\cos^2\theta+7a^4\cos^4\theta\right)\end{array}\right] \quad \text{(E1)}$$

wherein [71-79],

$$R_c=\left(\left|r-r_0\right|^n+\xi^n\right)^{1/n} \qquad \rho,\rho_0\in\Re \quad n\in\Re^+$$

$$\xi=\frac{\alpha}{2}+\sqrt{\frac{\alpha^2}{4}-q^2-a^2\cos^2\theta} \qquad a^2+q^2<\frac{\alpha^2}{4} \quad \text{(E2)}$$

By means of (E2), at $r = r_0$ (E1) has the value,

$$f=\frac{8}{\left(\xi^2+a^2\cos^2\theta\right)^6}\left[\begin{array}{c}\frac{3\alpha^2}{2}\left(\xi^6-15a^2\xi^4\cos^2\theta+15a^4\xi^2\cos^4\theta-a^6\cos^6\theta\right)-\\ -6\alpha q^2\xi\left(\xi^4-10a^2\xi^2\cos^2\theta+5a^4\cos^4\theta\right)+\\ +q^4\left(7\xi^4-34a^2\xi^2\cos^2\theta+7a^4\cos^4\theta\right)\end{array}\right] \quad \text{(E3)}$$

wherein ξ is given by (E2). Thus, the Kretschmann scalar is again finite when $r = r_0$, irrespective of the values of r_0 and n. Note that (E1) and hence (E3) depend upon ϑ. When $\vartheta = 0$ and when $\vartheta = \pi$,

$$f = \frac{8}{\left(\xi^2 + a^2\right)^6}\left[\begin{array}{c}\frac{3\alpha^2}{2}\left(\xi^6 - 15a^2\xi^4 + 15a^4\xi^2 - a^6\right) - 6\alpha q^2\xi\left(\xi^4 - 10a^2\xi^2 + 5a^4\right) + \\ + q^4\left(7\xi^4 - 34a^2\xi^2 + 7a^4\right)\end{array}\right] \quad \text{(E4)}$$

wherein the corresponding value of ξ is given by (E2). When $\vartheta = \pi/2$ (E3) reduces to,

$$f = \frac{8}{\xi^8}\left[\frac{3\alpha^2\xi^2}{2} - 6\alpha q^2\xi + 7q^4\right] \quad \text{(E5)}$$

wherein the corresponding value of ξ is again given by (E2). Note that (E5) does not contain the 'angular momentum' term *a* and that (E5) is precisely that for the Reissner-Nordström form (see Section **VII**).

Expression (E3) reduces to the Kerr form when $q = 0$, thus,

$$f = \frac{4}{\left(\xi^2 + a^2\cos^2\theta\right)^6}\left[3\alpha^2\left(\xi^6 - 15a^2\xi^4\cos^2\theta + 15a^4\xi^2\cos^4\theta - a^6\cos^6\theta\right)\right] \quad \text{(E6)}$$

wherein the corresponding value of ξ is again given by (E2). This too depends upon the value of ϑ. When $\vartheta = 0$ and when $\vartheta = \pi$, (E6) becomes,

$$f = \frac{4}{\left(\xi^2 + a^2\right)^6}\left[3\alpha^2\left(\xi^6 - 15a^2\xi^4 + 15a^4\xi^2 - a^6\right)\right] \quad \text{(E7)}$$

When $\vartheta = \pi/2$ (E6) reduces to (using (E2) for the value of ξ),

$$f = \frac{12}{\alpha^4} \quad \text{(E8)}$$

which is precisely the scalar invariant for the Schwarzschild form. Similarly, when both $q = 0$ and $a = 0$ (E1) reduces to the scalar invariant for the Schwarzschild form.

The Kretschmann scalar is finite in every case and so there are in fact no curvature singularities anywhere, contrary to the claims routinely made by proponents of black holes.

The Kretschmann scalar for the isotropic Schwarzschild and Reissner-Nordstrom forms

Since a conformal transformation does not preserve the values of the components of the Riemann curvature tensor the isotropic form need not necessarily produce the very same Kretschmann scalar as for the standard forms, but must produce a corresponding value that is invariant, independent of the values of ρ_0 and n, as is also the case for the Riemannian curvature (see **Appendices B and D**).

Consider a metric of the following general form,

$$ds^2 = g_{00}dt^2 + g_{11}\,dk^2 + g_{22}\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right) \tag{E9}$$

where $g_{00,}$ g_{11} and g_{22} and are functions of only k. Denote derivatives as follows,

$$g_{ii}^{'} = \frac{\partial g_{ii}}{dk} \qquad g_{ii}^{''} = \frac{\partial^2 g_{ii}}{\partial k^2} \tag{E10}$$

In terms of the components of the metric tensor of (E9) and their derivatives, the only non-zero R_{ijkl} are calculated to be,

$$R_{0101} = \frac{g_{11}\left(g_{oo}^{'}\right)^2 - 2g_{00}g_{11}g_{oo}^{''} + g_{00}g_{00}^{'}g_{11}^{'}}{4g_{00}g_{11}}$$

$$R_{0202} = \frac{-g_{00}^{'}g_{22}^{'}}{4g_{11}}$$

$$R_{0303} = R_{0202}\sin^2\theta = \frac{-g_{00}^{'}g_{22}^{'}\sin^2\theta}{4g_{11}}$$

$$R_{1212} = \frac{g_{22}g_{11}^{'}g_{22}^{'} - 2g_{11}g_{22}g_{22}^{''} + g_{11}\left(g_{22}^{'}\right)^2}{4g_{11}g_{22}}$$

$$R_{1313} = R_{1212}\sin^2\theta = \frac{g_{22}g_{11}^{'}g_{22}^{'} - 2g_{11}g_{22}g_{22}^{''} + g_{11}\left(g_{22}^{'}\right)^2}{4g_{11}g_{22}}\sin^2\theta$$

$$R_{2323} = \frac{\left\lfloor 4g_{11}g_{22} - \left(g_{22}^{'}\right)^2\right\rfloor \sin^2\theta}{4g_{11}} \tag{E11}$$

The corresponding non-zero R^{ijkl} are calculated to be,

$$R^{0101} = \frac{R_{0101}}{(g_{00}g_{11})^2} \qquad R^{0202} = \frac{R_{0202}}{(g_{00}g_{22})^2} \qquad R^{0303} = \frac{R_{0303}}{(g_{00}g_{22})^2 \sin^4\theta}$$

$$R^{1212} = \frac{R_{1212}}{(g_{11}g_{22})^2} \qquad R^{1313} = \frac{R_{1313}}{(g_{11}g_{22})^2 \sin^4\theta} \qquad R^{2323} = \frac{R_{2323}}{(g_{22})^4 \sin^4\theta}$$

(E12)

Then, taking into account the symmetries of the suffixes of the Riemann-Christoffel curvature tensor, the Kretschmann scalar $f = R_{ijkl}R^{ijkl}$ is given by,

$$f = 4\left(R_{0101}R^{0101} + R_{0202}R^{0202} + R_{0303}R^{0303} + R_{1212}R^{1212} + R_{1313}R^{1313} + R_{2323}R^{2323}\right) \qquad \text{(E13)}$$

Putting (E11) and (E12) into (E13) yields,

$$f = \frac{2\left[-g_{22}'' + \frac{g_{11}'g_{22}'}{2g_{11}} + \frac{(g_{22}')^2}{2g_{22}}\right]^2}{(g_{11}g_{22})^2} + \frac{\left[\frac{(g_{00}')^2}{2g_{00}} + \frac{g_{00}'g_{11}'}{2g_{11}} - g_{00}''\right]^2}{(g_{00}g_{11})^2} + \frac{4\left[g_{22} - \frac{(g_{22}')^2}{4g_{11}}\right]^2}{(g_{22})^4} + \frac{(g_{00}'g_{22}')^2}{2(g_{00}g_{11}g_{22})^2}$$

(E14)

The isotropic Schwarzschild form has the form of (E9), and in particular the form,

$$ds^2 = Adt^2 - Bdh^2 - C\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right) \qquad \text{(E15)}$$

where A, B, C are all > 0 (except at $\rho = \rho_0$) and are all functions only of h, where,

$$h = \left[|\rho - \rho_0|^n + \left(\frac{\alpha}{4}\right)^n\right]^{1/n}$$

$$\rho, \rho_0 \in \Re \quad n \in \Re^+ \qquad \text{(E16)}$$

Then by (D32) and (E15),

$$g_{00} = A = \left(\frac{4h-\alpha}{4h+\alpha}\right)^2 \qquad g_{11} = -B = -\left(1 + \frac{\alpha}{4h}\right)^4 \qquad g_{22} == -C = -h^2\left(1+\frac{\alpha}{4h}\right)^4$$

(E17)

and (E14) takes the following form [80] where the derivatives are with respect to *h*,

$$f=\frac{2\left[C''-\frac{B'C'}{2B}-\frac{(C')^2}{2C}\right]^2}{B^2C^2}+\frac{\left[\frac{(A')^2}{2A}+\frac{A'B'}{2B}-A''\right]^2}{A^2B^2}+\frac{4\left[\frac{(C')^2}{4B}-C\right]^2}{C^4}+\frac{(A'C')^2}{2A^2B^2C^2}$$

(E18)

The Kretschmann scalar is thereby calculated from (E18) at,

$$f=3\cdot 4^{13}\frac{\alpha^2h^6}{(4h+\alpha)^{12}}$$

$$h=\left[\left|\rho-\rho_0\right|^n+\left(\frac{\alpha}{4}\right)^n\right]^{1/n}$$

$$\rho,\rho_0\in\Re\quad n\in\Re^+$$

(E19)

At $\rho = \rho_0$, $h = \alpha/4$ for all ρ_0 and for all n. Thus the Kretschmann scalar is then,

$$f=\frac{12}{\alpha^4} \tag{E20}$$

which is the very same finite value as that for the Schwarzschild form.

The isotropic Reissner-Nordstrom form also has the form of (E15). Its Kretschmann scalar is thereby calculated from (E18) at,

$$f=\frac{4^{13}h^6\left\{\left[\alpha(4h+\alpha)^2-4q^2(8h+\alpha)\right]^2+\left[\alpha(4h+\alpha)^2-4q^2(12h+\alpha)\right]^2+\left[\alpha(4h+\alpha)^2-4q^2(4h+\alpha)\right]^2\right\}}{(4h+\alpha+2q)^8(4h+\alpha-2q)^8}$$

$$h=\left[\left|\rho-\rho_0\right|^n+\xi^n\right]^{1/n}$$

$$\xi=\frac{\sqrt{\alpha^2-4q^2}}{4}\qquad 4q^2<\alpha^2\qquad \rho,\rho_0\in\Re\quad n\in\Re^+ \tag{E21}$$

At $\rho = \rho_0$, $h = \xi = \sqrt{\alpha^2 - 4q^2}\,/4$ for all ρ_0 and for all n. Thus the Kretschmann scalar is then,

$$f = \frac{4^{13}h^6\left\{\left[\alpha(4\xi+\alpha)^2 - 4q^2(8\xi+\alpha)\right]^2 + \left[\alpha(4\xi+\alpha)^2 - 4q^2(12\xi+\alpha)\right]^2 + \left[\alpha(4\xi+\alpha)^2 - 4q^2(4\xi+\alpha)\right]^2\right\}}{(4\xi+\alpha+2q)^8(4\xi+\alpha-2q)^8}$$

$$\xi = \frac{\sqrt{\alpha^2 - 4q^2}}{4} \qquad 4q^2 < \alpha^2 \tag{E22}$$

which is finite. If $q = 0$ then (E22) reduces to (E20) for the isotropic Schwarzschild form.

APPENDIX F – GEODESIC COMPLETENESS

A geodesic is a line in some space. In Euclidean space the geodesics are simply straight lines. This is because the Riemannian curvature of Euclidean space is zero. If the Riemannian curvature is not zero throughout the entire space, the space is not Euclidean and the geodesics are curved lines rather than straight lines. If a geodesic terminates at some point in the space it is said to be incomplete, and the manifold or space in which it abodes is also said to be geodesically incomplete. If no geodesic in some manifold is incomplete then the manifold is said to be geodesically complete. More specifically, according to O'Neill [35],

"*A semi-Riemannian manifold M for which every maximal geodesic is defined on the entire real line is said to be* geodesically complete *– or merely* complete*. Note that if even a single point p is removed from a complete manifold M then M – p is no longer complete, since geodesics that formerly went through p are now obliged to stop*."

Consider now Hilbert's solution (2) (see section **V**). In 1931, Hagihara [99] proved that all geodesics therein that do not run into the boundary at $r = 2Gm/c^2$ are complete. Owing to (A17) this is also the case at $r = r_0$ for all the solutions generated thereby. Owing to (D12) and (D45) this is also the case at $\rho = \rho_0$ for the isotropic forms. The geodesics terminate at the origin; the point from which the radius emanates; $R_p = 0$. In other words, Hagihara effectively proved that all geodesics that do not run into the origin $R_p = 0$ are complete. This once again attests that none of these spaces can be 'extended' to produce a black hole (also see [44]).

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