On a unit that has to be an integral of the delta-function: one cannot detach mathematics from physics here!

Emanuel Gluskin

Kinneret College in the Jordan Valley (on the Sea of Galilee), 15132 Israel.
http://www.ee.bgu.ac.il/~gluskin/ ; gluskin@ee.bgu.ac.il

Abstract: The integral of the delta-function is 1, but when does '1' have to be interpreted as an integral of the delta-function? In order to make an interpretation of the volumes of figures of different dimensions more homogeneous, we follow a line of thought that leads us "back" to the original physical arguments from which the concept of delta-function arose.

What is this '1'?

The following consideration of a geometrical "reduction" to a less-dimensional object, obtained by a differentiation of a volume, leads us to an interesting observation regarding the role of the delta-function in mathematical physics, which seems to be interesting for the education of both mathematicians and physicists.

Consider first a parallelepiped with edges of the length *a*,*b*,*c*, having the volume

$$V = abc \, .$$

Singling out (say, by upward orientation) the edge of length *a*, we interpret the square *b-c* having the area $S = bc$ as a "basis" of the figure, and consider that

$$\frac{dV(a)}{da} = S \, . \qquad (1)$$

The geometrical meaning of this equality obviously is that when moving along (or in parallel to) '*a*', the area (basis) *S* is *sweeping over the volume V.*

Similarly, considering a 2D rectangle *a-b*, having the area *S* = *ab* (i.e. a "2D volume") we select '*b*' as the basis of the rectangle, and notice that

$$\frac{dS(a)}{da} = b \, , \qquad (2)$$

which means that when moving along '*a*', *segment 'b' is sweeping area S.*

Finally, taking a segment of straight line, having length *a*, we consider the equality:

$$\frac{da}{da} = 1 \, . \qquad (3)$$

For this configuration, it is also geometrically obvious that there is a point, i.e. the "basis" for the segment, that sweeps the segment. However, what is the direct geometrical (physical) meaning of this '1' in (3)? Equations (1-3) (see also Fig.1) sequentially give *S*, *b,* and 1. *What* is it physically, in the latter configuration, which equals 1?

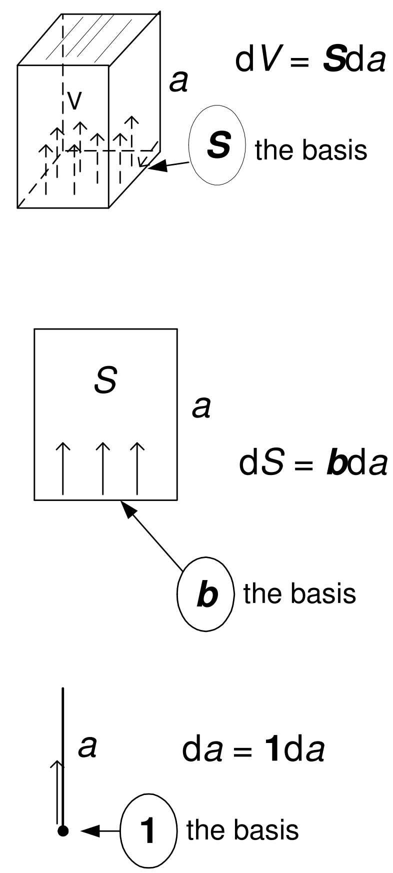

Fig. 1: The cases of N=3, N=2 and N=1. For N=3, we have area ***S*** as the integral measure of the base; for N=2, length ***b*** as such integral measure, and for N=1, **1** as the measure. There is no other thinkable measurement for the basis for N=1, but that of integration of a δ-function associated with some *physical transfer* to the geometrical object named "point". This is a heuristically important (perhaps, the simplest) case to be kept in mind, when one cannot detach mathematics from physics.

Since each of the values '*S*' and '*b*', -- an area and of the length of a segment, -- appearing in the right-hand sides of (1) and (2), are associated with a *physical measurement*, we have to find such a physical explanation also for (3).

Of course, one can object to this very purpose by saying that, mathematically formally, the derivative in (3) equals the ratio a/a, and thus, if there is any measurement at all, it can associated only with the principle (an axiom) of exhaustion introduced by Archimedes, according to which '*a*' contains *one* '*a*'. The logical problem with this explanation is that it appears here as something *ad hoc*, -- as the jump to a mathematically formal position that was not used for (1) and (2). That is, the visually good interpretation of *sweeping*, possible when one more dimension is added, disappears.

We insist, however, that it is possible to keep the line of thought developed for the 2D and 3D cases, also for the 1-D case. It is just necessary to understand that in this case we cannot avoid physical consideration, i.e. have to require this "1" to originate from something *measurable*.

The physical explanation

Physically, a "line" is a trace of a light spot, or the movement of an electrical charge (i.e., electrical current), or the trace of the tip of a pencil on a paper, etc.. Thus, creation of the line must start from accumulation of some material at the "point" that *sweeps* the segment, spending (using up) this material. With this physical interpretation, the mathematical '1' of (3) obtains the meaning of an expression of the type

$$\int \delta(x)dx \qquad (4)$$

where $\delta(x)$ is the "distribution" [1] of the *accumulated physical quantity*, which allows one to register (measure) this "point" (or, rather, something found at this point), and number 1 becomes the result of this measurement/registration, just as the obviously measurable area *S* of the basis appearing in (1), is. That is, '1' in (3) is some physical *measure* associated with the geometrical point. This is somewhat unusual for geometry where '1' usually means the length of a unit interval/segment, however, in our opinion, this is a correct geometrical/physical interpretation of '1' in (3), and we have here one more support for using the delta-function, -- this important mathematical object that was not known at the old times when the basic concepts of geometry, including the concept of "point", were introduced.

This interpretation seems useful for both mathematicians and physicists. However, while physicists always stress the necessity in improving their mathematical education and knowledge, mathematicians rarely express an interest in returning to the physical basics of the concepts they use, in order to thus (and not just axiomatically) rethink these concepts. As a rare good exception, I have found in the mathematical monograph [2]:

"*A mathematician with a general knowledge of analysis may find it useful to begin his study of classical potential theory by looking at its physical origins. Sections … give in part heuristic arguments based on physical considerations. These heuristic arguments suggest mathematical theorems and provide the mathematician with the problem of finding the proper hypotheses and mathematical proofs*."

In the historical regard, it is also interesting to note that the "sweeping" of a given figure by a lower-dimension figure, has relation to the old (e.g. [3]) idea of "movement in geometry".

Acknowledgement: That I started the argument using *derivatives*, is because when teaching electrical engineering students the course of electromagnetic fields, I start the mathematical introduction with the question of why, when differentiating the volume

of a ball, $V(R) = (4/3)\pi R^3$, one obtains the area of the ball's surface, $S = 4\pi R^2$. Professor Chandler Davis advised me not to avoid use of *differentials*, which I realized in Fig. 1.

Reference

[1] L. Schwartz, "Methods Mathematiques" pour les sciences physiques", Hermann, Paris 1961.

[2] J. Wermer, "*Potential theory*", New York, Springer, 1981.

[3] Alfonso (the Arabic pseudo-name of Avner) "*Mĕĭashshēr ʻăḳōb̲*" ("*Straightening a Curve*", in Hebrew), Nauka, Chief Office of Eastern literature, Moskva, 1983. (A Hebrew middle-ages manuscript, whose original is kept in British Library, worked out and supplied by an historical investigation, in Russian, by G.M Gluskin(a).)