When $\pi(n)$ divides n and when it does not

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Abstract

Let $\pi(n)$ denote the prime-counting function. In this paper we work with explicit formulas for $\pi(n)$ that are valid for infinitely many positive integers n, and we prove that if $n \geq 60184$ and $\mathrm{frac}(\ln n) = \ln n - \lfloor \ln n \rfloor > 0.5$, then $\pi(n)$ does not divide n. Based on this result, we show that if e is the base of the natural logarithm, a is a fixed integer ≥ 11 and n is any integer in the interval $[e^{a+0.5}, e^{a+1}]$, then $\pi(n) \nmid n$. In addition, we prove that if $n \geq 60184$ and $n/\pi(n)$ is an integer, then n is a multiple of $\lfloor \ln n - 1 \rfloor$ located in the interval $[e^{\lfloor \ln n - 1 \rfloor + 1}, e^{\lfloor \ln n - 1 \rfloor + 1.5}]$.

Keywords: *bounds on the prime-counting function, explicit formulas for the prime-counting function, intervals, prime numbers*

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0 Notation

Throughout this paper the number n is always a positive integer. Moreover, we use the following symbols:

- $\lfloor \ \rfloor$ (floor function)
- $\lceil \ \rceil$ (ceiling function)
- $\nmid$ (does not divide)
- frac() (fractional part)

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1 Introduction

Determining how prime numbers are distributed among natural numbers is one of the most difficult mathematical problems. This explains why the prime-counting function $\pi(n)$, which counts the number of primes less than or equal to a given number n, has been one of the main objects of study in Mathematics for centuries.

In [2] Gaitanas obtains an explicit formula for $\pi(n)$ that holds infinitely often. His proof is based on the fact that the function $f(n) = n/\pi(n)$ takes on every integer value greater than 1 (as proved by Golomb [3]) and on the fact that $x/(\ln x - 0.5) < \pi(x) < x/(\ln x - 1.5)$ for $x \geq 67$ (as shown by Rosser and Schoenfeld [4]). In this paper we find alternative expressions that are valid for infinitely many positive integers n, and we also prove that for $n \geq 60184$ if $\ln n - \lfloor \ln n \rfloor > 0.5$, then $n/\pi(n)$ is not an integer.

We will place emphasis on the following three theorems, which were proved by Golomb, Dusart, and Gaitanas respectively:

Theorem 1.1 [3]**.** The function $f(n) = n/\pi(n)$ takes on every integer value greater than 1. ■

Theorem 1.2 [1]**.** If n is an integer ≥ 60184, then

$$\frac{n}{\ln n - 1} < \pi(n) < \frac{n}{\ln n - 1.1}.$$ ■

Remark 1.3. Dusart's paper states that for $x \geq 60184$ we have $x/(\ln x - 1) \leq \pi(x) \leq x/(\ln x - 1.1)$, but since $\ln n$ is always irrational when n is an integer > 1, we can state his theorem the way we did. ◀

Theorem 1.4 [2]**.** The formula

$$\pi(n) = \frac{n}{\lfloor \ln n - 0.5 \rfloor}$$

is valid for infinitely many positive integers n. ■

2 Main theorems

We are now ready to prove our main theorems:

Theorem 2.1. The formula

$$\pi(n) = \frac{n}{\lfloor \ln n - 1 \rfloor}$$

holds for infinitely many positive integers n. ■

Proof. According to Theorem 1.2, for $n \geq 60184$ we have

$$\frac{n}{\ln n - 1} < \pi(n) < \frac{n}{\ln n - 1.1} \Rightarrow \frac{\ln n - 1.1}{n} < \frac{1}{\pi(n)} < \frac{\ln n - 1}{n}.$$

If we multiply by n, we get

$$\ln n - 1.1 < \frac{n}{\pi(n)} < \ln n - 1. \tag{1}$$

Since $\ln n - 1.1$ and $\ln n - 1$ are both irrational (for $n > 1$), inequality (1) implies that when $n/\pi(n)$ is an integer we must have

$$\frac{n}{\pi(n)} = \lfloor \ln n - 1 \rfloor = \lfloor \ln n - 1.1 \rfloor + 1 = \lceil \ln n - 1.1 \rceil = \lceil \ln n - 1 \rceil - 1. \tag{2}$$

Taking Theorem 1.2 and equality (2) into account, we can say that for every $n \geq 60184$ when $n/\pi(n)$ is an integer we must have

$$\frac{n}{\pi(n)} = \lfloor \ln n - 1 \rfloor \Rightarrow \pi(n) = \frac{n}{\lfloor \ln n - 1 \rfloor}.$$

Since Theorem 1.1 implies that $n/\pi(n)$ is an integer infinitely often, it follows that there are infinitely many positive integers n such that $\pi(n) = n/\lfloor \ln n - 1 \rfloor$. ■

In fact, the following theorem follows from Theorems 1.1, from Gaitana's proof of Theorem 1.4, and from the proof of Theorem 2.1:

Theorem 2.2. For every $n \geq 60184$ when $n/\pi(n)$ is an integer we must have

$$\frac{n}{\pi(n)} = \lfloor \ln n - 0.5 \rfloor = \lfloor \ln n - 1 \rfloor = \lfloor \ln n - 1.1 \rfloor + 1 = \lceil \ln n - 1.1 \rceil = \lceil \ln n - 1 \rceil - 1. \tag{3}$$

In other words, for $n \geq 60184$ when $n/\pi(n)$ is an integer we must have

$$\pi(n) = \frac{n}{\lfloor \ln n - 0.5 \rfloor} = \frac{n}{\lfloor \ln n - 1 \rfloor} = \frac{n}{\lfloor \ln n - 1.1 \rfloor + 1} = \frac{n}{\lceil \ln n - 1.1 \rceil} = \\ = \frac{n}{\lceil \ln n - 1 \rceil - 1}.$$ ■

Theorem 2.3. Let n be an integer ≥ 60184. If $\ln n - \lfloor \ln n \rfloor \geq 0.5$, then $\pi(n) \nmid n$ (that is to say, $n/\pi(n)$ is not an integer). ■

Proof. According to Theorem 2.2, for $n \geq 60184$ if $n/\pi(n)$ is an integer, then

$$\frac{n}{\pi(n)} = \lfloor \ln n - 0.5 \rfloor = \lfloor \ln n - 1 \rfloor.$$

In other words, for $n \geq 60184$ when $n/\pi(n)$ is an integer we have

$$\begin{aligned}
\lfloor \ln n - 0.5 \rfloor &= \lfloor \ln n - 1 \rfloor \\
\lfloor \ln n - 0.5 \rfloor &= \lfloor \ln n - 0.5 - 0.5 \rfloor \\
\operatorname{frac}(\ln n - 0.5) &\geq 0.5 \\
\ln n - 0.5 - \lfloor \ln n - 0.5 \rfloor &\geq 0.5 \\
\ln n - \lfloor \ln n - 0.5 \rfloor &\geq 1 \\
\operatorname{frac}(\ln n) &< 0.5 \\
\ln n - \lfloor \ln n \rfloor &< 0.5.
\end{aligned}$$

Suppose that P is the statement '$n/\pi(n)$ is an integer' and Q is the statement '$\ln n - \lfloor \ln n \rfloor < 0.5$'. According to propositional logic, the fact that $P \to Q$ implies that $\neg Q \to \neg P$. ■

Similar theorems can be proved by using Theorem 2.2 and equality (3).

Remark 2.4. When we mentioned Theorem 2.3 in the abstract, we replaced the expression $\ln n - \lfloor \ln n \rfloor \geq 0.5$ with $\ln n - \lfloor \ln n \rfloor > 0.5$ due to the fact that $\ln n$ is irrational when $n > 1$. ◀

Remark 2.5. Because $\ln n$ is irrational for $n > 1$, another way of stating Theorem 2.3 is by saying that if $n \geq 60184$ and the first digit to the right of the decimal point of $\ln n$ is 5, 6, 7, 8, or 9, then $\pi(n) \nmid n$.

Or we could also say that for $n \geq 60184$ if

$$n > e^{0.5 + \lfloor \ln n \rfloor},$$

then $\pi(n) \nmid n$. ◀

The following theorem follows from Theorem 2.3:

Theorem 2.6. Let e be the base of the natural logarithm. If a is any integer ≥ 11 and n is any integer contained in the interval $[e^{a+0.5}, e^{a+1}]$, then $\pi(n) \nmid n$. (The number e^r is irrational when r is a rational number $\neq 0$.) ■

Example 2.7. Take $a = 18$. If n is any integer in the interval $[e^{18.5}, e^{19}]$, then $\pi(n) \nmid n$. ◀

Theorem 2.8. If $n \geq 60184$ and $n/\pi(n)$ is an integer, then n is a multiple of $\lfloor \ln n - 1 \rfloor$ located in the interval $[e^{\lfloor \ln n-1 \rfloor+1}, e^{\lfloor \ln n-1 \rfloor+1.5}]$. ■

Proof. According to the proof of Theorem 2.3, if $n \geq 60184$ and $n/\pi(n)$ is an integer, then

$$\frac{n}{\pi(n)} = \lfloor \ln n - 1 \rfloor \Rightarrow n = \pi(n) \lfloor \ln n - 1 \rfloor$$

and

$$\mathrm{frac}(\ln n) = \ln n - \lfloor \ln n \rfloor < 0.5.$$

The fact that $\mathrm{frac}(\ln n) < 0.5$ implies that n is located in the interval

$$[e^k, e^{k+0.5}]$$

for some positive integer k. In other words, we have

$$e^k < n < e^{k+0.5} \Rightarrow k < \ln n < k + 0.5 \Rightarrow k - 1 < \ln n - 1 < k - 0.5.$$

This means that

$$\begin{aligned} k - 1 &= \lfloor \ln n - 1 \rfloor \\ k &= \lfloor \ln n - 1 \rfloor + 1, \end{aligned}$$

which proves the theorem. ■

Remark 2.9. In other words, if $n \geq 60184$ and $n/\pi(n)$ is an integer, then n is located in the interval $[e^k, e^{k+0.5}]$ for some positive integer k and n is divisible by $k - 1$. ◀

Question 2.10. Let b_0 be a sufficiently large positive integer and let b be any integer $\geq b_0$. In the interval $[e^b, e^{b+0.5}]$, is there always an integer n that is divisible by $\pi(n)$? ◀

A Better results

Theorem A.1. Let n be an integer ≥ 60184. If $\mathrm{frac}(\ln n) = \ln n - \lfloor \ln n \rfloor > 0.1$, then $\pi(n) \nmid n$ (that is to say, $n/\pi(n)$ is not an integer). ■

Proof. According to Theorem 2.2, for $n \geq 60184$ if $n/\pi(n)$ is an integer, then

$$\frac{n}{\pi(n)} = \lfloor \ln n - 1 \rfloor = \lceil \ln n - 1.1 \rceil.$$

In other words, for $n \geq 60184$ when $n/\pi(n)$ is an integer we have

$$\begin{aligned}
\lfloor \ln n - 1 \rfloor &= \lceil \ln n - 1.1 \rceil \\
\lfloor \ln n - 1 \rfloor &= \lceil \ln n - 1 - 0.1 \rceil \\
\operatorname{frac}(\ln n - 1) &\leq 0.1 \\
\ln n - 1 - \lfloor \ln n - 1 \rfloor &\leq 0.1 \\
\ln n - \lfloor \ln n - 1 \rfloor &\leq 1.1 \\
\operatorname{frac}(\ln n) &\leq 0.1 \\
\ln n - \lfloor \ln n \rfloor &\leq 0.1.
\end{aligned}$$

Suppose that P is the statement '$n/\pi(n)$ is an integer' and Q is the statement '$\ln n - \lfloor \ln n \rfloor \leq 0.1$'. The fact that $P \to Q$ implies that $\neg Q \to \neg P$. ■

Remark A.2. Because $\ln n$ is irrational for $n > 1$, Theorem A.1 implies that if $n \geq 60184$ and $\pi(n)$ divides n, then the first digit to the right of the decimal point of $\ln n$ can only be 0. In other words, if $n \geq 60184$ and the first digit to the right of the decimal point of $\ln n$ is 1, 2, 3, 4, 5, 6, 7, 8, or 9, then $\pi(n)$ does not divide n. ◀

The following theorem follows from Theorem A.1:

Theorem A.3. Let e be the base of the natural logarithm. If a is any integer ≥ 11 and n is any integer contained in the interval $[e^{a+0.1}, e^{a+1}]$, then $\pi(n) \nmid n$. ■

Example A.4. Take $a = 18$. If n is any integer in the interval $[e^{18.1}, e^{19}]$, then $\pi(n) \nmid n$. ◀

Remark A.5. If $n \geq 60184$ and $n/\pi(n)$ is an integer, then n is located in the interval $[e^k, e^{k+0.1}]$ for some positive integer k. ◀

Question A.6. Let b_0 be a sufficiently large positive integer and let b be any integer $\geq b_0$. In the interval $[e^b, e^{b+0.1}]$, is there always an integer n that is divisible by $\pi(n)$? ◀

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