The Proof for Non-existence of Perfect Cuboid

Bambore Dawit

odawit@yahoo.com

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Abstract

This paper shows the non-existence of perfect cuboid by using two tools, the first is representing Pythagoras triplets by two numbers and the second is realizing the impossibility of two similar equations for the same problem at the same time in different ways and the variables of one is relatively less than the other. When we express all Pythagoras triplets in perfect cuboid problem and rearrange it we can get a single equation that can express perfect cuboid. Unfortunately perfect cuboid has more than two similar equations that can express it and contradict one another.

Perfect cuboid problem is an extension of the Euler brick problem. The question is:- Is there a rectangular prism with integer sides and diagonals including space diagonal?

Proof: We can prove non-existance of perfect cuboid, proof by contradiction. let there is a perfect cuboid:

Let us see first how to make two variables constraint for any Pythagoras triplet.

First we need to make two variables constraint for any three perfect squares in arithmetic progression.

let us see How?

$$h < d < i$$
$$h^2 + i^2 = 2d^2$$
$$(i+d)(i-d) = (d+h)(d-h)$$

Let

$$t = i + d$$
$$j = i - d$$
$$u = d + h$$
$$v = d - h$$
$$tj = uv$$
$$t - j = u + v$$
$$t = j + u + v$$
$$j(j + u + v) = uv$$
$$u = \frac{j(j+v)}{v-j}$$

Let

$$k = v - j$$
$$u = \frac{j(2j+k)}{k} = \frac{2j^2}{k} + j$$
$$t = k + 2j + \frac{j(2j+k)}{k} = \frac{2j^2}{k} + 3j + k$$
$$h = \frac{u-v}{2} = \frac{\frac{2j^2}{k} - k}{2} = \frac{j^2}{k} - \frac{k}{2}$$
$$d = \frac{u+v}{2} = \frac{\frac{2j^2}{k} + 2j + k}{2} = \frac{j^2}{k} + j + \frac{k}{2}$$
$$i = \frac{t+j}{2} = \frac{\frac{2j^2}{k} + 3j + k + j}{2} = \frac{j^2}{k} + 2j + \frac{k}{2}$$

Since h, d and i are positive integers the final constraints must be set as follows.

$$h = \frac{2j^2}{k} - k$$
$$d = \frac{2j^2}{k} + 2j + k$$
$$i = \frac{2j^2}{k} + 4j + k$$

We have to remind that here well the constraints are positive integers if the three perfect squares are positive integers and the constraints have a common factor u if the three perfect squares have the common factor u also. This is valid for the constraints used in the whole proof.

We know that we can swamp any three perfect squares in arithmetic progression to corresponding Pythagoras triplets.

When we swamp three perfect squares in arithmetic progression to Pythagoras triplets.

$$(b-a)^2 + (b+a)^2 = 2d^2$$
$$2a^2 + 2b^2 = 2d^2$$
$$a^2 + b^2 = d^2$$

Let

$$i = b - a$$

$h = b + a$ then

$$a = \frac{h-i}{2}$$
$$b = \frac{h+i}{2}$$

$$a = 2j + k$$
$$b = \frac{2j^2}{k} + 2j$$
$$d = \frac{2j^2}{k} + 2j + k$$

$x = \frac{2j^2}{k}$ then

$k = \frac{2j^2}{x}$ there for we can make another constraints for the same pythagoras triplet in diffrent way.

$$a = 2j + x$$
$$b = \frac{2j^2}{x} + 2j$$

$d = \frac{2j^2}{x} + 2j + x$

Let a, b and c are the edges of rectangular prisim and $a < b < c$ then we have four different ways to define perfect cuboid complety and separetly by equations.

$a^2+b^2=d^2$	$a^2+b^2=d^2$	$a^2+b^2=d^2$	$a^2+b^2=d^2$
$a^2+c^2=e^2$	$a^2+c^2=e^2$	$a^2+c^2=e^2$	$a^2+c^2=e^2$
$b^2+c^2=f^2$	$b^2+c^2=f^2$	$b^2+c^2=f^2$	$b^2+c^2=f^2$
$a^2+f^2=g^2$	$b^2+e^2=g^2$	$c^2+d^2=g^2$	$a^2+b^2+c^2=g^2$

In our case we are going to use column one and column three to check if a perfect cuboid is exists. The first way to describe perfect cuboid by equation is : -

$$a^2+b^2=d^2$$
$$a^2+c^2=e^2$$
$$b^2+c^2=f^2$$
$$c^2+d^2=g^2$$

$$a^2+b^2=d^2$$
$$a = \frac{2j^2}{x} + 2j$$
$$b = 2j + x$$
$$d = \frac{2j^2}{x} + 2j + x$$

$$a^2+c^2=e^2$$
$$a = \frac{2l^2}{y} + 2l$$
$$c = 2l + y$$
$$e = \frac{2l^2}{y} + 2l + y$$

$$b^2+c^2=f^2$$
$$b = \frac{2n^2}{z} + 2n$$
$$c = 2n + z$$
$$f = \frac{2n^2}{z} + 2n + z$$

$$c^2+d^2=g^2$$
$$c=2r+s$$
$$d=\frac{2r^2}{s}+2r$$
$$g=\frac{2r^2}{s}+2r+s$$

$$\frac{2j^2}{x}+2j=\frac{2l^2}{y}+2l$$
$$\frac{j^2}{x}+j=\frac{l^2}{y}+l$$
$$2j+x=\frac{2n^2}{z}+2n$$
$$2l+y=2n+z=2r+s$$
$$\frac{2j^2}{x}+2j+x=\frac{2r^2}{s}+2r$$

$\frac{2j^2}{x}+2j=\frac{2l^2}{y}+2l$..eq 1.3.1

$2j+x=\frac{2n^2}{z}+2n$...eq 2.3.1

$2l+y=2n+z=2r+s$...eq 3.3.1

$\frac{2j^2}{x}+2j+x=\frac{2r^2}{s}+2r$ .... eq 4.3.1

$j=\frac{n^2}{z}+n-\frac{x}{2}$ .... from eq 2.3.1

$l=n+\frac{z}{2}-\frac{y}{2}$ ....from eq 3.3.1

$r=n+\frac{z}{2}-\frac{s}{2}$ ....from eq 3.3.1

When We substitute j, l and t in eq 1.3.1 and eq 4.3.1

$\frac{(n+\frac{z}{2})^2}{y}-\frac{y}{4}=\frac{(\frac{n^2}{z}+n)^2}{x}-\frac{x}{4}$.................................eq 1.4.1

$\frac{(n+\frac{z}{2})^2}{s}-\frac{s}{4}=\frac{(\frac{n^2}{z}+n)^2}{x}+\frac{x}{4}$.................................eq 2.4.1

When we substruct eq 1.4.1 from eq 2.4.1

$$x=2\frac{(n+\frac{z}{2})^2}{s}-2\frac{(n+\frac{z}{2})^2}{y}+\frac{y}{2}$$

When we substitute x in eq 1.4.1 or eq 2.4.1 and rearrange it:

$(\frac{n^2}{z}+n)^2=(\frac{2(n+z/2)^2}{s}-\frac{s}{2}-\frac{2(n+z/2)^2}{y}+\frac{y}{2})(\frac{(n+z/2)^2}{s}-\frac{s}{4}+\frac{2(n+z/2)^2}{y}-\frac{y}{2})$ .... eq 1.5.1

If we have Positive integer solution for eq 1.5.1 then there exist perfect cuboid. And if we have positive integer solution then there is no solution relatively less than the variables in it. If there is similar equation for the problem with relatively less variables that diqualifies the existence of perfect cuboid.

the second way to express perfect cuboid is

$$a^2+b^2=d^2$$
$$a^2+c^2=e^2$$
$$b^2+c^2=f^2$$
$$a^2+f^2=g^2$$

$$a^2+b^2=d^2$$
$$a=2j+k$$
$$b=\frac{2j^2}{k}+2j$$
$$d=\frac{2j^2}{k}+2j+k$$

$$a^2+c^2=e^2$$
$$a=2l+m$$
$$c=\frac{2l^2}{m}+2l$$
$$e=\frac{2l^2}{m}+2l+m$$

$$b^2+c^2=f^2$$
$$b=2n+o$$
$$c=\frac{2n^2}{o}+2n$$
$$f=\frac{2n^2}{o}+2n+o$$

$$a^2+f^2=g^2$$
$$a=2p+q$$
$$f=\frac{2p^2}{q}+2p$$
$$g=\frac{2p^2}{q}+2p+q$$

$2j+k=2l+m=2p+q$..eq 1.3.2

$\frac{2j^2}{k}+2j=2n+o$...eq 2.3.2
$\frac{l^2}{m}+l=\frac{n^2}{o}+n$..eq 3.3.2
$\frac{2n^2}{o}+2n+o=\frac{2p^2}{q}+2p$ .... eq 4.3.2

$l=j+\frac{k}{2}-\frac{m}{2}$ .... from eq 1.3.2
$p=j+\frac{k}{2}-\frac{q}{2}$ .... from eq 1.3.2
$n=\frac{j^2}{k}+j-\frac{o}{2}$ .... from eq 2.3.2
When We substitute l, n and p in eq 3.3.2 and eq 4.3.2
$\frac{(j+\frac{k}{2})^2}{m}-\frac{m}{4}=\frac{(\frac{j^2}{k}+j)^2}{o}-\frac{o}{4}$...........................eq 1.4.2
$\frac{(j+\frac{k}{2})^2}{q}-\frac{q}{4}=\frac{(\frac{j^2}{k}+j)^2}{o}+\frac{o}{4}$............................eq 2.4.2
When we substruct eq 1.4.2 from eq 2.4.2
$o=2\frac{(j+\frac{k}{2})^2}{q}-\frac{q}{2}-2\frac{(j+\frac{k}{2})^2}{m}+\frac{m}{2}$
When we substitute o in eq 1.4.2 or eq 2.4.2 and rearrange it:
$(\frac{j^2}{k}+j)^2=(\frac{2(j+k/2)^2}{q}-\frac{q}{2}-\frac{2(j+k/2)^2}{m}+\frac{m}{2})(\frac{(j+k/2)^2}{q}-\frac{q}{4}+\frac{2(j+k/2)^2}{m}-\frac{m}{2})$ .... eq 1.5.2
$z=f-b$ and $k=d-b\rightarrow k<z$
$y=e-a$ and $m=e-c\rightarrow m<y$
$s=g-d$ and $q=g-f\rightarrow q<s$
$2j=a+b-d$ and $2n=b+c-f$
$a+b-d<b+c-f\rightarrow j<n$

eq 1.5.1 and 1.5.2 are similar (the first solution for 1.5.1 and 1.5.2 are the same) and they can desribe perfect cuboid at the same time the variables in eq 1.5.2 is relatively less than from the variables 1.5.2.

This contradicts the existence of eq 1.5.1 for integer solution from the logic and there is no perfect cuboid. Q.E.D

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