THE THEORY OF THE THREE-PHASE ELECTRIC MULTIPOLE

MIHAI Gheorghe

University of Craiova, Faculty of Electrotechnics, Romania

E-mail adress: **gmihai@elth.ucv.ro**

FOREWORD

The book "The theory of the three-phase electric multipole" represents the generalization of the electric quadripole theory to the three-phase electric circuits, wants to maintain both the definitions and formulae obtained in the theory of the electric quadripole, but in another mathematical symbolism adapted to the three-phase electric circuits with or without neutral wire.

In the book, the following are determined:

- *the matrix form of the fundamental parameters for three-phase electrical networks in star/triangle;*
- *the matrix form of the impedance parameters for three phase-electrical networks in star/triangle;*
- *the matrix form of the impedance parameters for three-phase electrical networks function of the symmetrical components of direct, reverse and homopolar components;*
- *the matrix form of the Telegrapher's equations for three-phase electrical networks.*

The mathematical model utilized has at the basis the theory of the linear operators.

The book is a new approach and it is addressed to engineers from research and design, PhD students, to who wants to perfect in this field.

CONTENT

1. INTRODUCTION

The theory of the electrical quadripole is a well set theory, with theoretical results confirmed practically in numerous applications from different domains of science and technology.

Analogously, the theory of the three-phase electric circuits has its specific computational means, such as: the method of the neutral displacement, the method of symmetrical components, the nodal analysis, etc.

The link between the theory of the electrical quadripole and the three phase electrical circuits' theory is done only in the chapter referring to the building of the electrical schemes of direct, reverse and homopolar succession.

In the present paper, we want to generalize the theory of the electrical quadripole to three phase electrical circuits. This theory will be called „the theory of the three phase electrical multipole".

The main purpose of this paper is to keep unchanged the mathematical formulae obtained in the case of the electrical quadripole theory and to change only the mathematical significance of the terms. An approach of the electrical three phase systems from the perspective of the three phase electrical multipoles misses in the literature.

By three phase electrical multipole, one will understand an electrical structure which posses three or four input terminals and three or four output terminals, fed with a three-phase voltage system. The three phase electrical multipole has three or four terminals, depending on the three phase electrical system if it has a neutral wire or not, *fig.1a, 1b.*

The interaction of the three phase electrical multipole with the exterior is completely characterized by the three voltages at the access terminals and by the three electrical currents coming from outside.

A group of three or four access terminals for which the algebraic sum of the currents is null, no matter which are the potentials of the multipoles's terminals is called a port of the three-phase electrical multipole.

A port with three or four terminals, at which the applied three phase voltages and the corresponding electrical currents are associated after the receptors' rule, is called an input port.

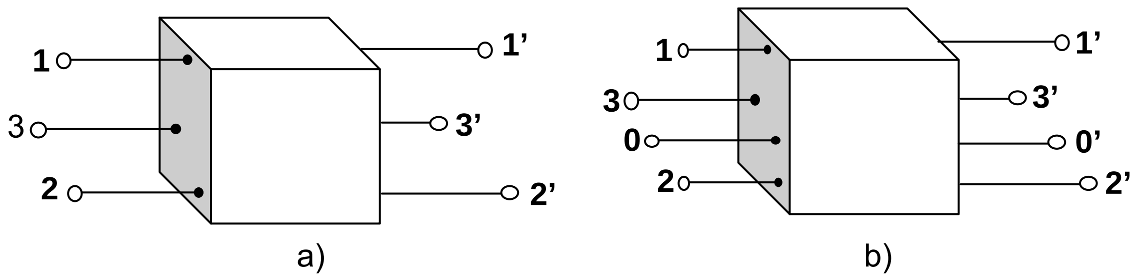

Figure 1. Symbolic representation for the three-phase electrical multipole:

a) Three-phase electrical multipole without neutral wire;

b) *Three-phase electrical multipole with neutral wire*;

At the input port, the complex power computed with these values is a received power.

A port with three or four terminals, at which the three phase voltages and the corresponding electrical currents are associated after the generators' rule, is called an output port.

At the output port, the complex power computed with these values is a delivered power.

2. EQUATIONS AND PARAMETERS OF THE PASSIVE, LINEAR AND RECIPROCAL THREE - PHASE MULTIPOLE

A three phase multipole is characterized by three input voltages $\underline{\mathbf{U}}_1, \underline{\mathbf{U}}_2, \underline{\mathbf{U}}_3$ and three output voltages $\underline{\mathbf{U}}_1', \underline{\mathbf{U}}_2', \underline{\mathbf{U}}_3'$. The two sets formed by the three elements, will be written as column matrixes:

$$\hat{\underline{\mathbf{U}}}_{\mathbf{input}} = \begin{pmatrix} \underline{\mathbf{U}}_1 \\ \underline{\mathbf{U}}_2 \\ \underline{\mathbf{U}}_3 \end{pmatrix} \qquad \hat{\underline{\mathbf{U}}}_{\mathbf{output}} = \begin{pmatrix} \underline{\mathbf{U}}_1' \\ \underline{\mathbf{U}}_2' \\ \underline{\mathbf{U}}_3' \end{pmatrix} \qquad (2.1)$$

Analogously, for the electrical currents, we will have two matrixes, the input current matrix formed by three elements, ($\underline{\mathbf{I}}_1, \underline{\mathbf{I}}_2, \underline{\mathbf{I}}_3$) and the output currents matrix ($\underline{\mathbf{I}}_1', \underline{\mathbf{I}}_2', \underline{\mathbf{I}}_3'$).

$$\hat{\underline{\mathbf{I}}}_{\mathbf{input}} = \begin{pmatrix} \underline{\mathbf{I}}_1 \\ \underline{\mathbf{I}}_2 \\ \underline{\mathbf{I}}_3 \end{pmatrix} \qquad \hat{\underline{\mathbf{I}}}'_{\mathbf{output}} = \begin{pmatrix} \underline{\mathbf{I}}_1' \\ \underline{\mathbf{I}}_2' \\ \underline{\mathbf{I}}_3' \end{pmatrix} \qquad (2.2)$$

In relations (2.1) and (2.2) it is obvious that the indexes *1, 2, 3* refer to phases *1, 2* and *3*.

From the four variables $\hat{\underline{\mathbf{U}}}_{\mathbf{input}}$, $\hat{\underline{\mathbf{U}}}_{\mathbf{output}}$, $\hat{\underline{\mathbf{I}}}_{\mathbf{input}}$ and $\hat{\underline{\mathbf{I}}}_{\mathbf{output,}}$, which characterize the interaction of the three phase electrical multipole with the exterior, only two of them are independent from the point of view of the internal structure of the three phase multipole. Two relations of the form are obtained:

$$\begin{cases} \mathbf{F}_1\left(\hat{\underline{\mathbf{I}}}_{\mathbf{input}}, \hat{\underline{\mathbf{I}}}_{\mathbf{output}}, \hat{\underline{\mathbf{U}}}_{\mathbf{input}}, \hat{\underline{\mathbf{U}}}_{\mathbf{output}}\right) = \mathbf{0} \\ \mathbf{F}_2\left(\hat{\underline{\mathbf{I}}}_{\mathbf{input}}, \hat{\underline{\mathbf{I}}}_{\mathbf{output}}, \hat{\underline{\mathbf{U}}}_{\mathbf{input}}, \hat{\underline{\mathbf{U}}}_{\mathbf{output}}\right) = \mathbf{0} \end{cases} \qquad (2.3)$$

Equations (2.3) are the equations of the three-phase multipole in implicit form. Supposing that the three-phase multipole is linear and passive, it results that equations (2.3) are linear and homogenous.

2.1. The fundamental form of the three-phase multipole equations and the fundamental parameters

In a three phase electrical network, the electromagnetic energy has a direction of propagation through the leader of three-phase multipole, so that the fundamental form is considered that expression in which the input quantities $\hat{\underline{\mathbf{U}}}_{\mathbf{input}}$ and $\hat{\underline{\mathbf{I}}}_{\mathbf{input}}$ are expressed linear with the output quantities $\hat{\underline{\mathbf{U}}}_{\mathbf{output}}$ and $\hat{\underline{\mathbf{I}}}_{\mathbf{output}}$:

$$\begin{cases} \underline{\hat{U}}_{input} = \underline{\hat{\hat{A}}} \cdot \underline{\hat{U}}_{output} + \underline{\hat{\hat{B}}} \cdot \underline{\hat{I}}_{output} \\ \underline{\hat{I}}_{input} = \underline{\hat{\hat{C}}} \cdot \underline{\hat{U}}_{output} + \underline{\hat{\hat{D}}} \cdot \underline{\hat{I}}_{output} \end{cases} \tag{2.4}$$

The following coefficients $\underline{\hat{\hat{A}}}, \underline{\hat{\hat{B}}}, \underline{\hat{\hat{C}}}, \underline{\hat{\hat{D}}}$ are called fundamental parameters of the three-phase multipole. From a mathematical perspective, they are matrixes (operators) $\mathbf{3} \times \mathbf{3}$.

3. THE CALCULUS OF THE IMPEDANCE OPERATORS FOR A THREE-PHASE ELECTRIC MULTIPOLE

3.1. Three-phase electric multipole in Δ connection

3.1.1. The calculus of the impedance operators

In *figure 3.1*., the three-phase electric multipole in *Δ* connection is presented.

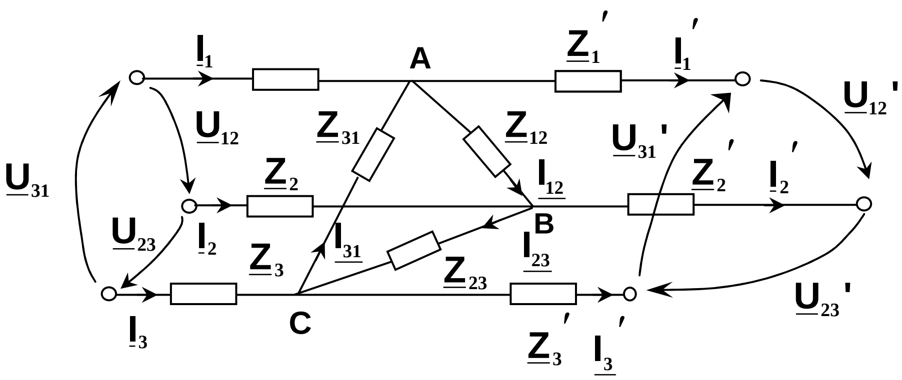

Figure 3.1. The three-phase electric multipole in Δ connection

In order to apply Kirchhoff's second law, we consider the prism associated to the three-phase multipole, defined by the input *1, 2, 3* respectively the output terminals *1', 2', 3', figure 3.2.*

On the side surfaces *11'2'2*, *22'3'3* and *11'3'3*, we consider the exterior normal vectors: $\overline{\mathbf{n}}_{12}$, $\overline{\mathbf{n}}_{23}$, $\overline{\mathbf{n}}_{31}$.

Accordingly, to the right handed screw rule, we associate to the direction of the normal vectors, the direction of the electrical current built on each side surface.

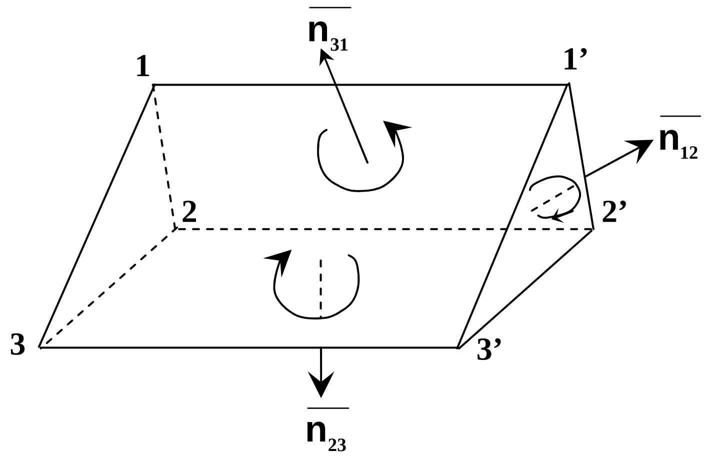

Figure 3.2. Explanation drawing for applying Kirchhoff's second law in space

For the input ports, the following equations are obtained:

$$\begin{cases} -\underline{\boldsymbol{U}}_{12} + \underline{\boldsymbol{I}}_1 \underline{\boldsymbol{Z}}_1 + \underline{\boldsymbol{I}}_{12} \underline{\boldsymbol{Z}}_{12} - \underline{\boldsymbol{I}}_2 \underline{\boldsymbol{Z}}_2 = \mathbf{0} \\ -\underline{\boldsymbol{U}}_{23} + \underline{\boldsymbol{I}}_2 \underline{\boldsymbol{Z}}_2 + \underline{\boldsymbol{I}}_{23} \underline{\boldsymbol{Z}}_{23} - \underline{\boldsymbol{I}}_3 \underline{\boldsymbol{Z}}_3 = \mathbf{0} \\ -\underline{\boldsymbol{U}}_{31} + \underline{\boldsymbol{I}}_3 \underline{\boldsymbol{Z}}_3 + \underline{\boldsymbol{I}}_{31} \underline{\boldsymbol{Z}}_{31} - \underline{\boldsymbol{I}}_1 \underline{\boldsymbol{Z}}_1 = \mathbf{0} \end{cases} \tag{3.1}$$

Analogously, the equations for the output ports are given:

$$\begin{cases} -\underline{\boldsymbol{U}'}_{12} + \underline{\boldsymbol{I}'}_1 \underline{\boldsymbol{Z}'}_1 - \underline{\boldsymbol{I}'}_{12} \underline{\boldsymbol{Z}'}_{12} - \underline{\boldsymbol{I}'}_2 \underline{\boldsymbol{Z}'}_2 = \mathbf{0} \\ -\underline{\boldsymbol{U}'}_{23} + \underline{\boldsymbol{I}'}_2 \underline{\boldsymbol{Z}'}_2 - \underline{\boldsymbol{I}'}_{23} \underline{\boldsymbol{Z}'}_{23} - \underline{\boldsymbol{I}'}_3 \underline{\boldsymbol{Z}'}_3 = \mathbf{0} \\ -\underline{\boldsymbol{U}'}_{31} + \underline{\boldsymbol{I}'}_3 \underline{\boldsymbol{Z}'}_3 - \underline{\boldsymbol{I}'}_{31} \underline{\boldsymbol{Z}'}_{31} - \underline{\boldsymbol{I}'}_1 \underline{\boldsymbol{Z}'}_1 = \mathbf{0} \end{cases} \tag{3.2}$$

For a three-phase electrical system, there are obvious equations:

$$\begin{cases} \underline{\boldsymbol{I}}_1 + \underline{\boldsymbol{I}}_2 + \underline{\boldsymbol{I}}_3 = \mathbf{0} \\ \underline{\boldsymbol{I}_1'} + \underline{\boldsymbol{I}'}_2 + \underline{\boldsymbol{I}'}_3 = \mathbf{0} \end{cases} \tag{3.3}$$

We apply Kirchhoff's first law in *A, B, C* nodes and we obtain:

$$\begin{cases} \underline{I}_1 - \underline{I'}_1 + \underline{I}_{31} - \underline{I}_{12} = 0 \\ \underline{I}_2 - \underline{I'}_2 + \underline{I}_{12} - \underline{I}_{23} = 0 \\ \underline{I}_3 - \underline{I'}_3 + \underline{I}_{23} - \underline{I}_{31} = 0 \end{cases} \quad (3.4)$$

For obtaining the equations of the three-phase electrical multipole, as a function only of the input and output voltages and currents, we will have to eliminate the currents $\underline{I}_{12}, \underline{I}_{23}, \underline{I}_{31}$ from equations (3.1) and (3.2).

In order to eliminate these currents, from the above-mentioned system of equations, we firstly consider, the transfiguration of the triangle with the vertices in the *A, B, C* nodes.

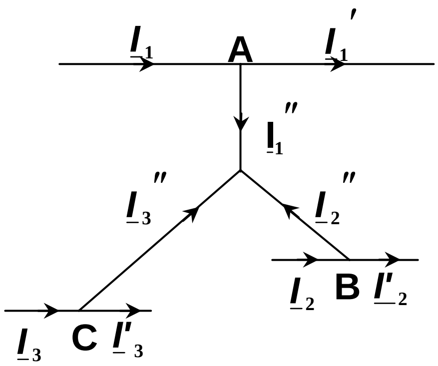

Figure 3.3. Explanatory drawing for Δ - Y transfiguration

Let currents $\underline{I}_1'', \underline{I}_2'', \underline{I}_3''$ coming from the nodes *A, B, C*, *figure 3.3.*

It is obvious that:

$$\underline{I}_1'' + \underline{I}_2'' + \underline{I}_3'' = 0 \quad (3.5)$$

From ***Δ -Y*** transfiguration conditions, the following relations are obtained:

$$\begin{cases} -\underline{I}_{31} + \underline{I}_{12} = \underline{I}_1'' \\ -\underline{I}_{12} + \underline{I}_{23} = \underline{I}_2'' \\ -\underline{I}_{23} + \underline{I}_{31} = \underline{I}_3'' \end{cases} \quad (3.6)$$

The phase diagram from *figure 3.4.* corresponds to relations (3.6).

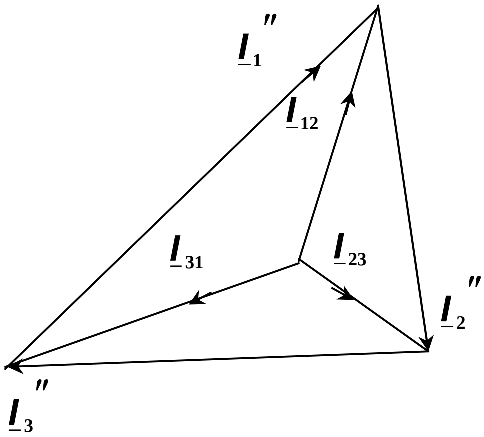

Figure 3.4. The currents' phase diagram at Δ- Y transfiguration

In *figure 3.5.* the currents $\underline{I}^*_{12}$, $\underline{I}^*_{23}$, $\underline{I}^*_{31}$ are built, which correspond to the case where all $\underline{Z}_{12}$, $\underline{Z}_{23}$, $\underline{Z}_{31}$ impedances are equal. These currents can be found on the medians of the triangle.

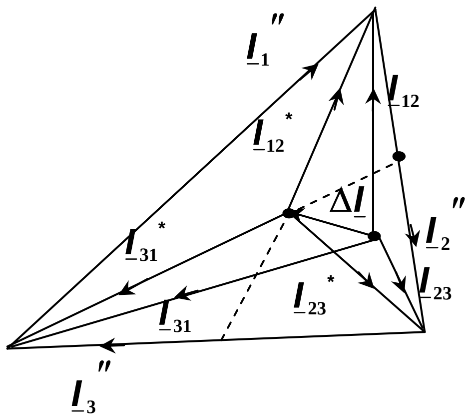

Fig.3.5. Explanatory drawing for determining the line currents

From the diagram, there can be set the link between the currents we must determine $\underline{I}_{12}$, $\underline{I}_{23}$, $\underline{I}_{31}$ and the $\underline{I}^*_{12}$, $\underline{I}^*_{23}$, $\underline{I}^*_{31}$, $\Delta\underline{I}$ currents.

There is obtained:

$$\begin{cases} \underline{I}_{12} = \underline{I}_{12}^{*} + \Delta\underline{I} \\ \underline{I}_{23} = \underline{I}_{23}^{*} + \Delta\underline{I} \\ \underline{I}_{31} = \underline{I}_{31}^{*} + \Delta\underline{I} \end{cases} \tag{3.7}$$

We add the three relations from (3.7) system and we take into account that:

$$\underline{I}_{12}^{*} + \underline{I}_{23}^{*} + \underline{I}_{31}^{*} = 0$$

We get that:

$$\Delta\underline{I} = \frac{\underline{I}_{12} + \underline{I}_{23} + \underline{I}_{31}}{3} \tag{3.8}$$

We apply Kirchhoff's second law on the contour delimited by the *A, B, C* nodes, *figure 3.1* and we obtain:

$$\underline{I}_{12}\underline{Z}_{12} + \underline{I}_{23}\underline{Z}_{23} + \underline{I}_{31}\underline{Z}_{31} = 0 \tag{3.9}$$

The first equation from (3.7) we multiply it with $\underline{Z}_{12}$, the second one with $\underline{Z}_{23}$ and the third one with $\underline{Z}_{31}$; then we add them and one finds:

$$\underline{I}_{12}\underline{Z}_{12} + \underline{I}_{23}\underline{Z}_{23} + \underline{I}_{31}\underline{Z}_{31} = \underline{I}_{12}^{*}\underline{Z}_{12} + \underline{I}_{23}^{*}\underline{Z}_{23} + \underline{I}_{31}^{*}\underline{Z}_{31} + \Delta\underline{I}\left(\underline{Z}_{12} + \underline{Z}_{23} + \underline{Z}_{31}\right)$$

Based on relation (3.9), one obtains:

$$\Delta\underline{I} = -\frac{\underline{I}_{12}^{*}\underline{Z}_{12} + \underline{I}_{23}^{*}\underline{Z}_{23} + \underline{I}_{31}^{*}\underline{Z}_{31}}{\underline{Z}_{12} + \underline{Z}_{23} + \underline{Z}_{31}} \tag{3.10}$$

Returning to relations (3.7) and replacing $\Delta\underline{I}$ from (3.10), we get:

$$\underline{I}_{12} = \frac{\left(\underline{I}^*_{12} - \underline{I}^*_{23}\right)\cdot \underline{Z}_{23} - \left(\underline{I}^*_{31} - \underline{I}^*_{12}\right)\cdot \underline{Z}_{31}}{\underline{Z}_{12} + \underline{Z}_{23} + \underline{Z}_{31}} \tag{3.11}$$

From the diagram in *figure 3.5,* it results that:

$$\begin{cases} \underline{I}^*_{12} - \underline{I}^*_{23} = -\underline{I}_2'' \\ \underline{I}^*_{31} - \underline{I}^*_{12} = -\underline{I}_1'' \end{cases} \tag{3.12}$$

So (3.11) becomes:

$$\underline{I}_{12} = \frac{-\underline{I}_2''\cdot \underline{Z}_{23} + \underline{I}_1''\cdot \underline{Z}_{31}}{\underline{Z}_{12} + \underline{Z}_{23} + \underline{Z}_{31}} \tag{3.13}$$

By cyclic permutations in relation (3.13), one gets:

$$\underline{I}_{23} = \frac{-\underline{I}_3''\cdot \underline{Z}_{31} + \underline{I}_2''\cdot \underline{Z}_{12}}{\underline{Z}_{12} + \underline{Z}_{23} + \underline{Z}_{31}} \tag{3.14}$$

$$\underline{I}_{31} = \frac{-\underline{I}_1''\cdot \underline{Z}_{12} + \underline{I}_3''\cdot \underline{Z}_{23}}{\underline{Z}_{12} + \underline{Z}_{23} + \underline{Z}_{31}} \tag{3.15}$$

The expressions of $\underline{I}_1''$, $\underline{I}_2''$, $\underline{I}_3''$ currents, can be obtained from *figure 3.3.*

$$\begin{cases} \underline{I}_1'' = \underline{I}_1 - \underline{I}_1' \\ \underline{I}_2'' = \underline{I}_2 - \underline{I}_2' \\ \underline{I}_3'' = \underline{I}_3 - \underline{I}_3' \end{cases} \tag{3.16}$$

So (3.13), (3.14), (3.15) relations become:

$$\begin{cases} \underline{I}_{12} = \dfrac{1}{\underline{Z}_t}\left[-\left(\underline{I}_2 - \underline{I}_2'\right)\cdot \underline{Z}_{23} + \left(\underline{I}_1 - \underline{I}_1'\right)\cdot \underline{Z}_{31}\right] \\ \underline{I}_{23} = \dfrac{1}{\underline{Z}_t}\left[-\left(\underline{I}_3 - \underline{I}_3'\right)\cdot \underline{Z}_{31} + \left(\underline{I}_2 - \underline{I}_2'\right)\cdot \underline{Z}_{12}\right] \\ \underline{I}_{31} = \dfrac{1}{\underline{Z}_t}\left[-\left(\underline{I}_1 - \underline{I}_1'\right)\cdot \underline{Z}_{12} + \left(\underline{I}_3 - \underline{I}_3'\right)\cdot \underline{Z}_{23}\right] \end{cases} \tag{3.17}$$

In which:

$$\underline{Z}_t = \underline{Z}_{12} + \underline{Z}_{23} + \underline{Z}_{31} \tag{3.18}$$

Relations (3.17) contain the transversal impedances of the multipole, as well as the input and output currents.

We replace the currents from (3.17) relation into (3.1) and (3.2) relations and one finds:

$$\begin{cases} \underline{U}_{12} = \underline{I}_1\left(\underline{Z}_1 + \dfrac{\underline{Z}_{12}}{\underline{Z}_t}\underline{Z}_{31}\right) - \underline{I}_2\left(\underline{Z}_2 + \dfrac{\underline{Z}_{12}}{\underline{Z}_t}\underline{Z}_{23}\right) - \underline{I}_1'\dfrac{\underline{Z}_{12}}{\underline{Z}_t}\underline{Z}_{31} + \underline{I}_2'\dfrac{\underline{Z}_{12}}{\underline{Z}_t}\underline{Z}_{23} \\ \underline{U}_{23} = \underline{I}_2\left(\underline{Z}_2 + \dfrac{\underline{Z}_{23}}{\underline{Z}_t}\underline{Z}_{12}\right) - \underline{I}_3\left(\underline{Z}_3 + \dfrac{\underline{Z}_{23}}{\underline{Z}_t}\underline{Z}_{31}\right) - \underline{I}_2'\dfrac{\underline{Z}_{23}}{\underline{Z}_t}\underline{Z}_{12} + \underline{I}_3'\dfrac{\underline{Z}_{23}}{\underline{Z}_t}\underline{Z}_{31} \\ \underline{U}_{31} = \underline{I}_1\left(\underline{Z}_1 + \dfrac{\underline{Z}_{31}}{\underline{Z}_t}\underline{Z}_{12}\right) - \underline{I}_3\left(\underline{Z}_3 + \dfrac{\underline{Z}_{31}}{\underline{Z}_t}\underline{Z}_{23}\right) - \underline{I}_1'\dfrac{\underline{Z}_{31}}{\underline{Z}_t}\underline{Z}_{12} - \underline{I}_3'\dfrac{\underline{Z}_{31}}{\underline{Z}_t}\underline{Z}_{23} \end{cases} \tag{3.19}$$

$$\begin{cases} \underline{U}_{12}' = \underline{I}_1\dfrac{\underline{Z}_{12}}{\underline{Z}_t}\underline{Z}_{31} - \underline{I}_2\dfrac{\underline{Z}_{12}}{\underline{Z}_t}\underline{Z}_{23} - \underline{I}_1'\left(\underline{Z}_1' + \dfrac{\underline{Z}_{12}}{\underline{Z}_t}\underline{Z}_{31}\right) + \underline{I}_2'\left(\underline{Z}_2' + \dfrac{\underline{Z}_{12}}{\underline{Z}_t}\underline{Z}_{23}\right) \\ \underline{U}_{23}' = \underline{I}_2\dfrac{\underline{Z}_{23}}{\underline{Z}_t}\underline{Z}_{12} - \underline{I}_3\dfrac{\underline{Z}_{23}}{\underline{Z}_t}\underline{Z}_{31} - \underline{I}_2'\left(\underline{Z}_2' + \dfrac{\underline{Z}_{223}}{\underline{Z}_t}\underline{Z}_{12}\right) + \underline{I}_3'\left(\underline{Z}_3' + \dfrac{\underline{Z}_{23}}{\underline{Z}_t}\underline{Z}_{31}\right) \\ \underline{U}_{31}' = \underline{I}_1\dfrac{\underline{Z}_{31}}{\underline{Z}_t}\underline{Z}_{12} - \underline{I}_3\dfrac{\underline{Z}_{31}}{\underline{Z}_t}\underline{Z}_{23} - \underline{I}_1'\left(\underline{Z}_1' + \dfrac{\underline{Z}_{31}}{\underline{Z}_t}\underline{Z}_{12}\right) - \underline{I}_3'\left(\underline{Z}_3' + \dfrac{\underline{Z}_{31}}{\underline{Z}_t}\underline{Z}_{23}\right) \end{cases} \tag{3.20}$$

As presented in chapter 1, the input and output operators for voltage and current are introduced:

$$\begin{cases} \hat{\underline{U}}_{input} = \begin{pmatrix} \underline{U}_{12} \\ \underline{U}_{23} \\ \underline{U}_{31} \end{pmatrix} \\ \hat{\underline{U}}_{output} = \begin{pmatrix} \underline{U}'_{12} \\ \underline{U}'_{23} \\ \underline{U}'_{31} \end{pmatrix} \end{cases} \qquad \begin{cases} \hat{\underline{I}}_{input} = \begin{pmatrix} \underline{I}_{1} \\ \underline{I}_{2} \\ \underline{I}_{3} \end{pmatrix} \\ \hat{\underline{I}}_{output} = \begin{pmatrix} \underline{I}'_{1} \\ \underline{I}'_{2} \\ \underline{I}'_{3} \end{pmatrix} \end{cases} \tag{3.21}$$

Accordingly to the above relations, we form the matrix relation between the operators:

$$\begin{pmatrix} \hat{\underline{U}}_{input} \\ \hat{\underline{U}}_{output} \end{pmatrix} = \begin{pmatrix} \hat{\hat{\underline{Z}}}_{11} & \hat{\hat{\underline{Z}}}_{12} \\ \hat{\hat{\underline{Z}}}_{21} & \hat{\hat{\underline{Z}}}_{31} \end{pmatrix} \begin{pmatrix} \hat{\underline{I}}_{input} \\ \hat{\underline{I}}_{output} \end{pmatrix} \tag{3.22}$$

By comparing relations (3.19), (3.20) and (3.22), there are obtained ***the expressions of the impedance operators of the three-phase electrical multipole:***

$$\hat{\hat{\underline{Z}}}_{11} = \begin{pmatrix} \underline{Z}_1 + \frac{\underline{Z}_{12}}{\underline{Z}_t}\underline{Z}_{31} & -\left(\underline{Z}_2 + \frac{\underline{Z}_{12}}{\underline{Z}_t}\underline{Z}_{23}\right) & \mathbf{0} \\ \mathbf{0} & \underline{Z}_2 + \frac{\underline{Z}_{23}}{\underline{Z}_t}\underline{Z}_{12} & -\left(\underline{Z}_3 + \frac{\underline{Z}_{23}}{\underline{Z}_t}\underline{Z}_{31}\right) \\ -\left(\underline{Z}_1 + \frac{\underline{Z}_{31}}{\underline{Z}_t}\underline{Z}_{12}\right) & \mathbf{0} & \underline{Z}_3 + \frac{\underline{Z}_{31}}{\underline{Z}_t}\underline{Z}_{23} \end{pmatrix} \tag{3.23}$$

$$\hat{\hat{\underline{Z}}}_{12} = \begin{pmatrix} -\frac{\underline{Z}_{12}}{\underline{Z}_t}\underline{Z}_{31} & \frac{\underline{Z}_{12}}{\underline{Z}_t}\underline{Z}_{23} & \mathbf{0} \\ \mathbf{0} & -\frac{\underline{Z}_{23}}{\underline{Z}_t}\underline{Z}_{12} & \frac{\underline{Z}_{23}}{\underline{Z}_t}\underline{Z}_{31} \\ \frac{\underline{Z}_{31}}{\underline{Z}_t}\underline{Z}_{12} & \mathbf{0} & -\frac{\underline{Z}_{31}}{\underline{Z}_t}\underline{Z}_{23} \end{pmatrix} \tag{3.24}$$

$$\hat{\hat{\underline{Z}}}_{21} = \begin{pmatrix} \frac{\underline{Z}_{12}}{\underline{Z}_t}\underline{Z}_{31} & -\frac{\underline{Z}_{12}}{\underline{Z}_t}\underline{Z}_{23} & 0 \\ 0 & \frac{\underline{Z}_{23}}{\underline{Z}_t}\underline{Z}_{12} & -\frac{\underline{Z}_{23}}{\underline{Z}_t}\underline{Z}_{31} \\ -\frac{\underline{Z}_{31}}{\underline{Z}_t}\underline{Z}_{12} & 0 & \frac{\underline{Z}_{31}}{\underline{Z}_t}\underline{Z}_{23} \end{pmatrix} \quad (3.25)$$

$$\hat{\hat{\underline{Z}}}_{22} = \begin{pmatrix} -\left(\underline{Z}_1' + \frac{\underline{Z}_{12}}{\underline{Z}_t}\underline{Z}_{31}\right) & \underline{Z}_2' + \frac{\underline{Z}_{12}}{\underline{Z}_t}\underline{Z}_{23} & 0 \\ 0 & -\left(\underline{Z}_2' + \frac{\underline{Z}_{23}}{\underline{Z}_t}\underline{Z}_{12}\right) & \underline{Z}_3' + \frac{\underline{Z}_{23}}{\underline{Z}_t}\underline{Z}_{31} \\ \underline{Z}_1' + \frac{\underline{Z}_{31}}{\underline{Z}_t}\underline{Z}_{12} & 0 & -\left(\underline{Z}_3' + \frac{\underline{Z}_{31}}{\underline{Z}_t}\underline{Z}_{23}\right) \end{pmatrix} \quad (3.26)$$

Relations (3.23), (3.24), (3.25) and (3.26) represent the operational form of the impedance parameters, for a three-phase electrical multipole in Δ connection. If examined the form of the impedance operators, one can notice that:

$$\hat{\hat{\underline{Z}}}_{12} = -\hat{\hat{\underline{Z}}}_{21} \quad (3.27)$$

which is the necessary reciprocal condition for three phase electrical multipoles. For a three-phase symmetrical multipole we have:

$$\underline{Z}_1 = \underline{Z}_1',\ \underline{Z}_2 = \underline{Z}_2',\ \underline{Z}_3 = \underline{Z}_3'$$

Then, replaced in relations (3.23) and (3.26) leads to:

$$\hat{\hat{\underline{Z}}}_{11} = -\hat{\hat{\underline{Z}}}_{22} \quad (3.28)$$

which is the identical condition formally speaking with the condition for a symmetrical four-port.

The general reciprocal condition (2.56) is fulfilled only if the three-phase multipole is symmetrical.

3.1.2. The calculus of the fundamental parameter operators: $\hat{\hat{\underline{A}}}$, $\hat{\hat{\underline{B}}}$, $\hat{\hat{\underline{C}}}$, $\hat{\hat{\underline{D}}}$

In the previous paragraph, we have determined the operator for the impedance parameters. The operator for the fundamental parameters can be obtained using the transformation formulae from the operator for the impedance parameters. In order to better understand how the operator for the fundamental parameters is determined in the case of a three-phase electrical multipole, we will use the direct approach, starting form Kirchhoff's laws, starting with fig.3.1.
In node *A* we apply Kirchhoff's first law:

$$\underline{I}_1 = \underline{I}_{12} - \underline{I}_{31} + \underline{I}_1' \tag{3.29}$$

By applying Kirchhoff's second law on loop *A1'2'BA,* respectively on loop *A1'3'CA*, one gets the following relations:

$$\underline{I}_{12} = \frac{\underline{U}_{12}' + \underline{I}_1'\,\underline{Z}_1' - \underline{I}_2'\,\underline{Z}_2'}{\underline{Z}_{12}} \tag{3.30}$$

$$\underline{I}_{31} = \frac{\underline{U}_{31}' + \underline{I}_3'\,\underline{Z}_3' - \underline{I}_1'\,\underline{Z}_1'}{\underline{Z}_{31}} \tag{3.31}$$

After replacing (3.30) and (3.31) in (3.29), we obtain after some manipulations:

$$\begin{aligned}\underline{I}_1 &= \underline{U}_{12}'\,\underline{Y}_{12} + \underline{U}_{23}'\cdot 0 + \underline{U}_{31}'\left(-\underline{Y}_{31}\right) + \\ &+ \underline{I}_1'\left(\underline{Z}_1'\,\underline{Y}_{12} + \underline{Z}_1'\,\underline{Y}_{31} + 1\right) + \underline{I}_2'\left(-\underline{Z}_2'\,\underline{Y}_{12}\right) + \underline{I}_3'\left(-\underline{Z}_3'\,\underline{Y}_{31}\right)\end{aligned} \tag{3.32}$$

Using cyclic permutations of the indexes, one gets the expressions for the other currents as well:

$$\underline{I}_2 = \underline{U}_{12}'\left(-\underline{Y}_{12}\right) + \underline{U}_{23}'\cdot\underline{Y}_{23} + \underline{U}_{31}'\cdot 0 + \underline{I}_1'\left(-\underline{Z}_1'\underline{Y}_{12}\right) + \\ + \underline{I}_2'\left(-\underline{Z}_2'\underline{Y}_{23} + \underline{Z}_2'\underline{Y}_{12} + 1\right) + \underline{I}_3'\left(-\underline{Z}_3'\underline{Y}_{23}\right) \tag{3.33}$$

$$\underline{I}_3 = \underline{U}_{12}'\cdot 0 + \underline{U}_{23}'\cdot\left(-\underline{Y}_{23}\right) + \underline{U}_{31}'\cdot\underline{Y}_{31} + \underline{I}_1'\left(-\underline{Z}_1'\underline{Y}_{31}\right) + \\ + \underline{I}_2'\left(-\underline{Z}_2'\underline{Y}_{23}\right) + \underline{I}_3'\left(\underline{Z}_3'\underline{Y}_{31} + \underline{Z}_3'\underline{Y}_{23} + 1\right) \tag{3.34}$$

From relations (3.32), (3.33) and (3.34), it can be noticed that all have a common property meaning they represent the input electrical currents for phases *1, 2, 3*, as a function of the line voltages at the output and the phase currents at the output. In conclusion, we can write the operational relation:

$$\hat{\underline{I}}_{input} = \hat{\hat{\underline{C}}}\hat{\underline{U}}_{output,l} + \hat{\hat{\underline{D}}}\hat{\underline{I}}_{output}$$

in which:

$$\hat{\underline{I}}_{input} = \begin{pmatrix} \underline{I}_1 \\ \underline{I}_2 \\ \underline{I}_3 \end{pmatrix}; \quad \hat{\underline{I}}_{output} = \begin{pmatrix} \underline{I}_1' \\ \underline{I}_2' \\ \underline{I}_3' \end{pmatrix}; \quad \hat{\underline{U}}_{output,l} = \begin{pmatrix} \underline{U}'_{12} \\ \underline{U}'_{23} \\ \underline{U}'_{31} \end{pmatrix} \tag{3.35}$$

$$\hat{\hat{\underline{C}}} = \begin{pmatrix} \underline{Y}_{12} & 0 & -\underline{Y}_{31} \\ -\underline{Y}_{12} & \underline{Y}_{23} & 0 \\ 0 & -\underline{Y}_{23} & \underline{Y}_{31} \end{pmatrix} \tag{3.36}$$

$$\hat{\hat{\underline{D}}} = \begin{pmatrix} \underline{Z}_1'\underline{Y}_{12} + \underline{Z}_1'\underline{Y}_{31} + 1 & -\underline{Z}_2'\underline{Y}_{12} & -\underline{Z}_3'\underline{Y}_{31} \\ -\underline{Z}_1'\underline{Y}_{12} & \underline{Z}_2'\underline{Y}_{23} + \underline{Z}_2'\underline{Y}_{12} + 1 & -\underline{Z}_3'\underline{Y}_{23} \\ -\underline{Z}_1'\underline{Y}_{31} & -\underline{Z}_2'\underline{Y}_{23} & \underline{Z}_3'\underline{Y}_{31} + \underline{Z}_3'\underline{Y}_{23} + 1 \end{pmatrix} \tag{3.37}$$

For obtaining the mathematical expression for operators $\hat{\underline{\hat{A}}}$ and $\hat{\underline{\hat{B}}}$, we will keep in mind that we must obtain a relation having the form:

$$\underline{\hat{U}}_{input,l} = \underline{\hat{\hat{A}}}\underline{\hat{U}}_{output,l} + \underline{\hat{\hat{B}}}\underline{\hat{I}}_{output} \tag{3.38}$$

In order to obtain a relation like the one in (3.38), we apply Kirchhoff's second law on the loop *1AB21*:

$$\underline{U}_{12} = \underline{I}_1\underline{Z}_1 + \underline{I}_{12}\underline{Z}_{12} - \underline{I}_2\underline{Z}_2 \tag{3.39}$$

From the expressions of the electrical currents (3.30), (3.32), (3.33), (3.34) and relation (3.39), one finds:

$$\begin{aligned}\underline{U}_{12} &= \underline{U'}_{12}\left(\underline{Y}_{12}\underline{Z}_1 + \underline{Y}_{12}\underline{Z}_2 + 1\right) + \underline{U'}_{32}\left(-\underline{Y}_{23}\underline{Z}_2\right) + \underline{U'}_{31}\left(-\underline{Y}_{31}\underline{Z}_1\right) + \\ &+ \underline{I'}_1\left[\underline{Z'}_1\left(\underline{Z}_1\underline{Y}_{12} + \underline{Z}_1\underline{Y}_{31} + \underline{Z}_2\underline{Y}_{12} + 1\right) + \underline{Z}_1\right] + \\ &+ \underline{I'}_2\left[\underline{Z'}_2\left(-\underline{Z}_1\underline{Y}_{12} - \underline{Z}_2\underline{Y}_{23} - \underline{Z}_2\underline{Y}_{12} - 1\right) - \underline{Z}_2\right] + \underline{I'}_3\left[\underline{Z'}_3\left(-\underline{Z}_1\underline{Y}_{31} + \underline{Z}_2\underline{Y}_{23}\right)\right]\end{aligned} \tag{3.40}$$

Analogously, the following relations result:

$$\begin{aligned}\underline{U}_{23} &= \underline{U'}_{12}\left(-\underline{Y}_{12}\underline{Z}_2\right) + \underline{U'}_{23}\left(\underline{Y}_{23}\underline{Z}_2 + \underline{Y}_{23}\underline{Z}_3 + 1\right) + \\ &+ \underline{U'}_{31}\left(-\underline{Y}_{31}\underline{Z}_3\right) + \underline{I'}_1\left[\underline{Z'}_1\left(-\underline{Z}_2\underline{Y}_{12} + \underline{Z}_3\underline{Y}_{31}\right)\right] + \\ &+ \underline{I'}_2\left[\underline{Z'}_2\left(\underline{Z}_2\underline{Y}_{23} + \underline{Z}_2\underline{Y}_{12} + \underline{Z}_3\underline{Y}_{23} + 1\right) + \underline{Z}_2\right] + \\ &+ \underline{I'}_3\left[\underline{Z'}_3\left(-\underline{Z}_2\underline{Y}_{23} - \underline{Z}_3\underline{Y}_{31} - \underline{Z}_3\underline{Y}_{23} - 1\right) - \underline{Z}_3\right]\end{aligned} \tag{3.41}$$

$$\begin{aligned}\underline{U}_{23} &= \underline{U'}_{12}\left(-\underline{Y}_{12}\underline{Z}_1\right) + \underline{U'}_{23}\left(-\underline{Y}_{23}\underline{Z}_3\right) + \underline{U'}_{31}\left(\underline{Y}_{31}\underline{Z}_3 + \underline{Y}_{31}\underline{Z}_1 + 1\right) + \\ &+ \underline{I'}_1\left[\underline{Z'}_1\left(-\underline{Z}_3\underline{Y}_{31} - \underline{Z}_1\underline{Y}_{12} - \underline{Z}_1\underline{Y}_{31} - 1\right) - \underline{Z}_1\right] + \underline{I'}_2\left[\underline{Z'}_2\left(-\underline{Z}_3\underline{Y}_{23} + \underline{Z}_1\underline{Y}_{12}\right)\right] + \\ &+ \underline{I'}_3\left[\underline{Z'}_3\left(\underline{Z}_3\underline{Y}_{31} + \underline{Z}_3\underline{Y}_{23} + \underline{Z}_1\underline{Y}_{31} + 1\right) + \underline{Z}_3\right]\end{aligned} \tag{3.42}$$

By comparing relations (3.40), (3.41), (3. 42) with relation (3.38), one gets:

$$\underline{\hat{U}}_{input,l} = \begin{pmatrix} \underline{U}_{12} \\ \underline{U}_{23} \\ \underline{U}_{31} \end{pmatrix}; \qquad \underline{\hat{U}}_{output,l} = \begin{pmatrix} \underline{U'}_{12} \\ \underline{U'}_{23} \\ \underline{U'}_{31} \end{pmatrix}; \qquad \underline{\hat{I}}_{output} = \begin{pmatrix} \underline{I'}_{1} \\ \underline{I'}_{2} \\ \underline{I'}_{3} \end{pmatrix} \tag{3.43}$$

$$\underline{\hat{\hat{A}}} = \begin{pmatrix} \underline{Y}_{12}\underline{Z}_1 + \underline{Y}_{12}\underline{Z}_2 + 1 & -\underline{Y}_{23}\underline{Z}_2 & -\underline{Y}_{31}\underline{Z}_1 \\ -\underline{Y}_{12}\underline{Z}_2 & \underline{Y}_{23}\underline{Z}_2 + \underline{Y}_{23}\underline{Z}_3 + 1 & -\underline{Y}_{31}\underline{Z}_3 \\ -\underline{Y}_{12}\underline{Z}_1 & -\underline{Y}_{23}\underline{Z}_3 & \underline{Y}_{31}\underline{Z}_3 + \underline{Y}_{31}\underline{Z}_1 + 1 \end{pmatrix} \tag{3.44}$$

Operator $\underline{\hat{\hat{B}}}$ has the form:

$$\underline{\hat{\hat{B}}} = \begin{pmatrix} \underline{B}_{11} & \underline{B}_{12} & \underline{B}_{13} \\ \underline{B}_{21} & \underline{B}_{22} & \underline{B}_{23} \\ \underline{B}_{31} & \underline{B}_{32} & \underline{B}_{33} \end{pmatrix} \tag{3.45}$$

in which $\boldsymbol{B_{11}}$, $\boldsymbol{B_{12}}$ ...$\boldsymbol{B_{33}}$ components have the expressions:

$$\begin{cases} \underline{B}_{11} = \underline{Z'}_1\left(\underline{Z}_1\underline{Y}_{12} + \underline{Z}_1\underline{Y}_{31} + \underline{Z}_2\underline{Y}_{12} + 1\right) + \underline{Z}_1 \\ \underline{B}_{12} = \underline{-Z'}_2\left(\underline{Z}_1\underline{Y}_{12} + \underline{Z}_2\underline{Y}_{23} + \underline{Z}_2\underline{Y}_{12} + 1\right) + \underline{Z}_2 \\ \underline{B}_{13} = \underline{Z'}_3\left(-\underline{Z}_1\underline{Y}_{31} + \underline{Z}_2\underline{Y}_{23}\right) \end{cases} \tag{3.46}$$

$$\begin{cases} \underline{B}_{21} = \underline{Z'}_1\left(-\underline{Z}_2\underline{Y}_{12} + \underline{Z}_3\underline{Y}_{31}\right) \\ \underline{B}_{22} = \underline{Z'}_2\left(\underline{Z}_2\underline{Y}_{23} + \underline{Z}_2\underline{Y}_{12} + +\underline{Z}_3\underline{Y}_{23} + 1\right) + \underline{Z}_2 \\ \underline{B}_{23} = -\underline{Z'}_3\left(\underline{Z}_2\underline{Y}_{23} + \underline{Z}_3\underline{Y}_{31} + \underline{Z}_3\underline{Y}_{23} + 1\right) + \underline{Z}_3 \end{cases} \tag{3.47}$$

$$\begin{cases} \underline{B}_{31} = -\underline{Z'}_1\left(\underline{Z}_3\underline{Y}_{31} + \underline{Z}_1\underline{Y}_{12} + \underline{Z}_1\underline{Y}_{31} + 1\right) + \underline{Z}_1 \\ \underline{B}_{32} = \underline{Z'}_2\left(-\underline{Z}_3\underline{Y}_{23} + \underline{Z}_1\underline{Y}_{12}\right) \\ \underline{B}_{33} = \underline{Z'}_3\left(\underline{Z}_3\underline{Y}_{31} + \underline{Z}_3\underline{Y}_{23} + \underline{Z}_1\underline{Y}_{31} + 1\right) + \underline{Z}_3 \end{cases} \tag{3.48}$$

3.2. Three-phase electric multipole in Y connection

3.2.1. Three-phase electric multipole in Y connection with neutral wire

3.2.1.1. The calculus of impedance operators

In *fig. 3.6.,* the three-phase electrical multipole in Y connection with neutral wire is presented.

In order to apply Kirchhoff's second law, we consider the fascicle of planes associated to the electrical multipole defined by the input terminals *1,2,3,0* and output terminals *1',2',3',0'*, as fig.3.7 shows.

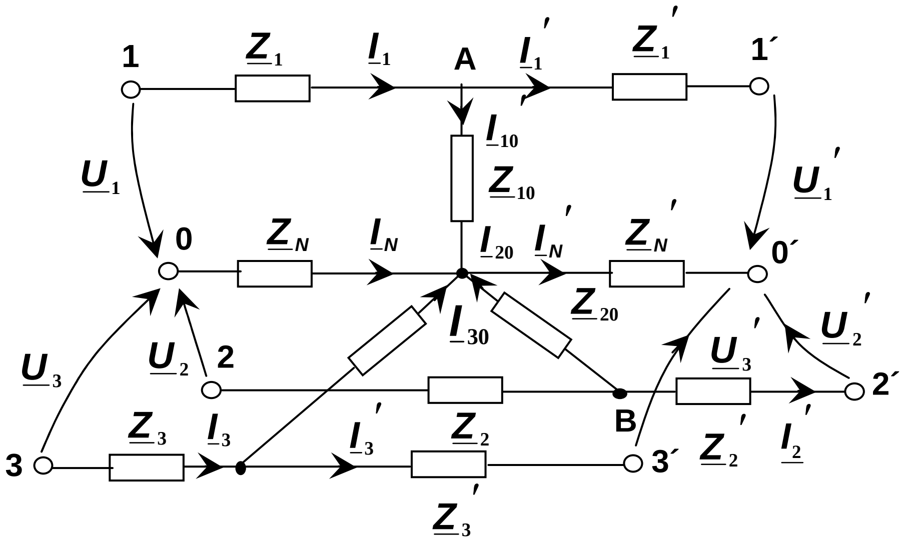

Fig.3.6. The three-phase electrical multipole in Y connection with neutral wire

We consider the normal vectors $\overline{n}_1$, $\overline{n}_2$, $\overline{n}_3$ on *(1,1',0',0), (2,2',0',0)* and *(3,3',0',0)* planes. The direction of the normal is chosen after the right handed screw rule. The direction of rotation of the right handed screw is the direction of the electrical currents built on each plane.

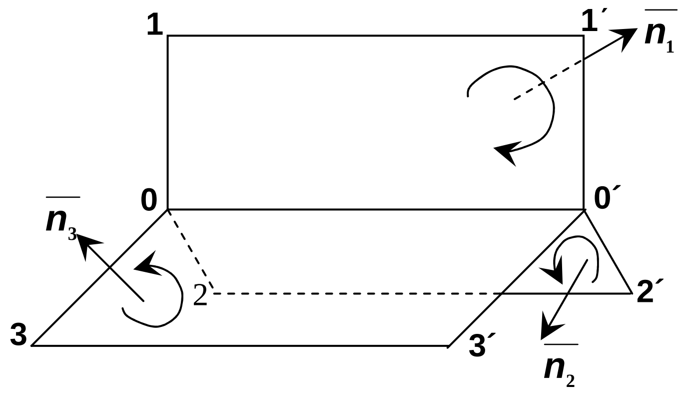

Fig.3.7. Explanation drawing for applying Kirchhoff's second law in space

For the input ports, the following equations can be obtained:

$$\begin{cases} -\underline{U}_1 + \underline{I}_1\underline{Z}_1 + \underline{I}_{10}\underline{Z}_{10} - \underline{I}_N\underline{Z}_N = 0 \\ -\underline{U}_2 + \underline{I}_2\underline{Z}_2 + \underline{I}_{20}\underline{Z}_{20} - \underline{I}_N\underline{Z}_N = 0 \\ -\underline{U}_3 + \underline{I}_3\underline{Z}_3 + \underline{I}_{30}\underline{Z}_{30} - \underline{I}_N\underline{Z}_N = 0 \end{cases} \quad (3.49)$$

$$\underline{I}_N\underline{Z}_N = \Delta\underline{U}$$

Analogously, there are obtained the equations for the output ports:

$$\begin{cases} \underline{U}'_1 + \underline{I}'_1\underline{Z}'_1 - \underline{I}_{10}\underline{Z}_{10} - \underline{I}'_N\underline{Z}'_N = 0 \\ \underline{U}'_2 + \underline{I}'_2\underline{Z}'_2 - \underline{I}_{20}\underline{Z}_{20} - \underline{I}'_N\underline{Z}'_N = 0 \\ \underline{U}'_3 + \underline{I}'_3\underline{Z}'_3 - \underline{I}_{30}\underline{Z}_{30} - \underline{I}'_N\underline{Z}'_N = 0 \end{cases} \quad (3.50)$$

$$\underline{I}'_N\underline{Z}'_N = \Delta\underline{U}'$$

We have relations between the phase currents and the neutral wire for the output and input terminals:

$$\begin{cases} \underline{I}_1 + \underline{I}_2 + \underline{I}_3 + \underline{I}_N = 0 \\ \underline{I}'_1 + \underline{I}'_2 + \underline{I}'_3 + \underline{I}'_N = 0 \end{cases} \quad (3.51)$$

We apply Kirchhoff's first law for *A, B, C* nodes and we find:

$$\begin{cases} \underline{I}_1 - \underline{I'}_1 = \underline{I}_{10} \\ \underline{I}_2 - \underline{I'}_2 = \underline{I}_{20} \\ \underline{I}_3 - \underline{I'}_3 = \underline{I}_{30} \end{cases} \tag{3.52}$$

We substitute (3.51) and (3.52) relations in (3.49) and (3.50) and we get:

$$\begin{cases} \underline{U}_1 = \underline{I}_1\left(\underline{Z}_1 + \underline{Z}_{10} + \underline{Z}_N\right) + \underline{I}_2\underline{Z}_N + \underline{I}_3\underline{Z}_N - \underline{I'}_1\underline{Z}_{10} \\ \underline{U}_2 = \underline{I}_1\underline{Z}_N + \underline{I}_2\left(\underline{Z}_2 + \underline{Z}_{20} + \underline{Z}_N\right) + \underline{I}_3\underline{Z}_N - \underline{I'}_2\underline{Z}_{20} \\ \underline{U}_1 = \underline{I}_1\underline{Z}_N + \underline{I}_2\underline{Z}_N + \underline{I}_3\left(\underline{Z}_3 + \underline{Z}_{30} + \underline{Z}_N\right) - \underline{I'}_3\underline{Z}_{30} \end{cases} \tag{3.53}$$

$$\begin{cases} \underline{U'}_1 = \underline{I}_1\underline{Z}_{10} + \underline{I'}_1\left(-\underline{Z'}_1 - \underline{Z}_{10} - \underline{Z}_N\right) - \underline{I'}_2\underline{Z}_N - \underline{I'}_3\underline{Z}_N \\ \underline{U'}_2 = \underline{I}_2\underline{Z}_{20} - \underline{I'}_1\underline{Z}_N + \underline{I'}_2\left(-\underline{Z'}_2 - \underline{Z}_{20} - \underline{Z}_N\right) - \underline{I'}_3\underline{Z}_N \\ \underline{U'}_1 = \underline{I}_3\underline{Z}_{30} - \underline{I'}_1\underline{Z}_N - \underline{I'}_2\underline{Z}_N + \underline{I'}_3\left(-\underline{Z'}_3 - \underline{Z}_{30} - \underline{Z}_N\right) \end{cases} \tag{3.54}$$

As presented in the previous chapter and last paragraph, the input and output operators for voltage and current are introduced:

$$\underline{\hat{U}}_{input} = \begin{pmatrix} \underline{U}_1 \\ \underline{U}_2 \\ \underline{U}_3 \end{pmatrix} \qquad \underline{\hat{U}}_{output} = \begin{pmatrix} \underline{U'}_1 \\ \underline{U'}_2 \\ \underline{U'}_3 \end{pmatrix}$$

$$\underline{\hat{I}}_{input} = \begin{pmatrix} \underline{I}_1 \\ \underline{I}_2 \\ \underline{I}_3 \end{pmatrix} \qquad \underline{\hat{U}}_{output} = \begin{pmatrix} \underline{I'}_1 \\ \underline{I'}_2 \\ \underline{I'}_3 \end{pmatrix} \tag{3.55}$$

Accordingly to the above relations, we form the matrix relation between the operators:

$$\begin{pmatrix} \underline{\hat{U}}_{input} \\ \underline{\hat{U}}_{output} \end{pmatrix} = \begin{pmatrix} \underline{\hat{\hat{Z}}}_{11} & \underline{\hat{\hat{Z}}}_{12} \\ \underline{\hat{\hat{Z}}}_{21} & \underline{\hat{\hat{Z}}}_{22} \end{pmatrix} \cdot \begin{pmatrix} \underline{\hat{I}}_{input} \\ \underline{\hat{I}}_{output} \end{pmatrix} \tag{3.56}$$

By comparing (3.53) and (3.54) relations with (3.56), one can find the expressions for the impedance operators of the three phase electrical multipole in ***Y*** connection:

$$\underline{\hat{\hat{Z}}}_{11} = \begin{pmatrix} \underline{Z}_1 + \underline{Z}_{10} + \underline{Z}_N & \underline{Z}_N & \underline{Z}_N \\ \underline{Z}_N & \underline{Z}_2 + \underline{Z}_{20} + \underline{Z}_N & \underline{Z}_N \\ \underline{Z}_N & \underline{Z}_N & \underline{Z}_3 + \underline{Z}_{30} + \underline{Z}_N \end{pmatrix} \tag{3.57}$$

$$\underline{\hat{\hat{Z}}}_{12} = \begin{pmatrix} -\underline{Z}_{10} & 0 & 0 \\ 0 & -\underline{Z}_{20} & 0 \\ 0 & 0 & -\underline{Z}_{30} \end{pmatrix} \tag{3.58}$$

$$\underline{\hat{\hat{Z}}}_{21} = \begin{pmatrix} \underline{Z}_{10} & 0 & 0 \\ 0 & -\underline{Z}_{20} & 0 \\ 0 & 0 & \underline{Z}_{30} \end{pmatrix} \tag{3.59}$$

$$\underline{\hat{\hat{Z}}}_{22} = \begin{pmatrix} -\underline{Z'}_1 - \underline{Z}_{10} - \underline{Z'}_N & -\underline{Z}_N & -\underline{Z}_N \\ -\underline{Z}_N & -\underline{Z'}_2 - \underline{Z}_{20} - \underline{Z'}_N & -\underline{Z}_N \\ -\underline{Z}_N & -\underline{Z}_N & -\underline{Z'}_3 - \underline{Z}_{30} - \underline{Z'}_N \end{pmatrix} \tag{3.60}$$

From examining the form of the impedance operators, one can observe that:

$$\underline{\hat{\hat{Z}}}_{12} = -\underline{\hat{\hat{Z}}}_{21} \tag{3.61}$$

which represents the necessary condition of reciprocity for the three-phase electrical multipoles. For a symmetric three-phase electrical multipole having: $\underline{Z}_1 = \underline{Z'}_1$; $\underline{Z}_2 = \underline{Z'}_2$; $\underline{Z}_3 = \underline{Z'}_3$; $\underline{Z}_N = \underline{Z'}_N$, it can be obtained:

$$\hat{\hat{\underline{Z}}}_{11} = -\hat{\hat{\underline{Z}}}'_{22} \tag{3.62}$$

Conditions (3.61) and (3.62) are formally identical with the conditions for a quadripole to be symmetrical and reciprocal.

The general reciprocal condition is fulfilled only if the three-phase multipole is symmetrical.

3.2.2. Three-phase electric multipole in Y connection without neutral wire

3.2.2.1. The calculus of the impedance operators

In *figure 3.8.,* a three-phase electrical multipole in **Y** connection without neutral wire is presented.

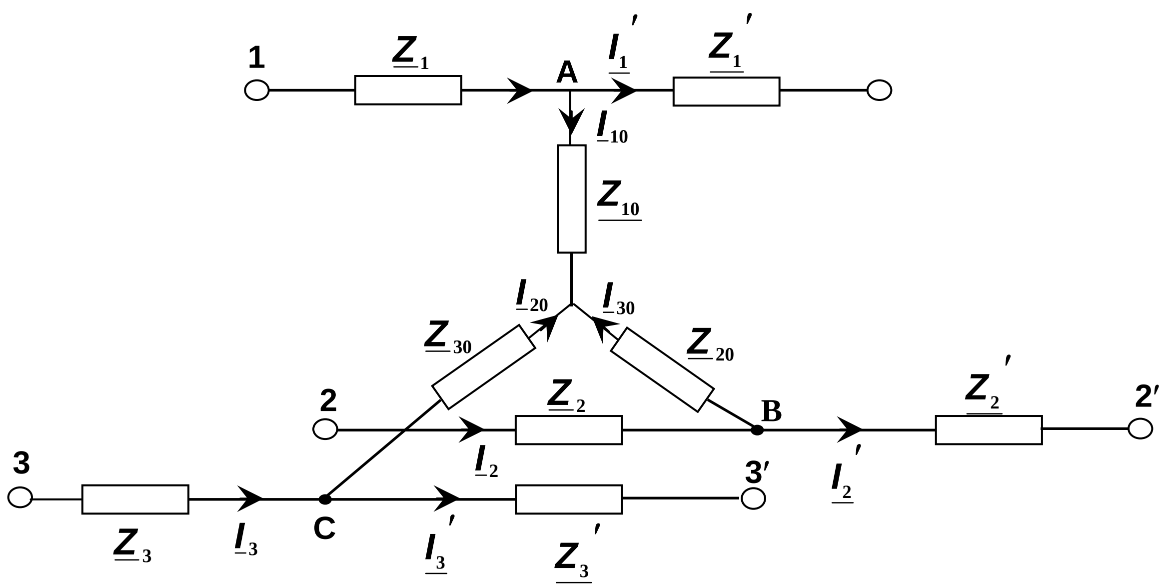

Fig. 3.8. A three-phase electrical multipole in Y connection without neutral wire

The three-phase electrical multipole is fed by a three phase voltage system $\underline{U}_{12}$, $\underline{U}_{23}$, $\underline{U}_{31}$. It can be noticed that the three-phase electrical multipole in **Y** connection without neutral wire can be obtained from the three-phase electrical multipole in Δ connection if there is transfigured the impedance triangle, $\underline{Z}_{12}$, $\underline{Z}_{23}$, $\underline{Z}_{31}$, between nodes *A, B, C, fig. 3.1*.

Impedances $\underline{Z}_{12}$, $\underline{Z}_{23}$, $\underline{Z}_{31}$ as a function of impedances $\underline{Z}_{10}$, $\underline{Z}_{20}$, $\underline{Z}_{31}$, have the expressions:

$$\begin{cases} \underline{Z}_{12} = \dfrac{\underline{Z}_{10}\underline{Z}_{20} + \underline{Z}_{20}\underline{Z}_{30} + \underline{Z}_{30}\underline{Z}_{10}}{\underline{Z}_{30}} \\ \underline{Z}_{23} = \dfrac{\underline{Z}_{10}\underline{Z}_{20} + \underline{Z}_{20}\underline{Z}_{30} + \underline{Z}_{30}\underline{Z}_{10}}{\underline{Z}_{10}} \\ \underline{Z}_{31} = \dfrac{\underline{Z}_{10}\underline{Z}_{20} + \underline{Z}_{20}\underline{Z}_{30} + \underline{Z}_{30}\underline{Z}_{10}}{\underline{Z}_{20}} \end{cases} \tag{3.63}$$

The form of the impedance operator for the three-phase electrical multipole in **Y** connection without neutral wire can be obtained from (3.23), (3.24), (3.25) and (3.26) in which relations (3.50) are replaced.

3.3. Determining the impedance operators function of symmetric components for a three phase-multipole, in Y connection with neutral wire

In previous paragraphs there have been deduced the expressions of the impedance operators' expressions for an electrical multipole fed by an asymmetric three-phase voltage system. Both the system of the electrical voltages and the one of the electrical corresponding currents can be decomposed into systems of direct, reverse and homopolar succession. For the phase voltages $\underline{U}_1$, $\underline{U}_2$, $\underline{U}_3$, there are obtained the direct, reverse and homopolar components of voltage.

$$\begin{cases} \underline{U}_h = \frac{1}{3}\left(\underline{U}_1 + \underline{U}_2 + \underline{U}_3\right) \\ \underline{U}_d = \frac{1}{3}\left(\underline{U}_1 + a\underline{U}_2 + a^2\underline{U}_3\right) \\ \underline{U}_c = \frac{1}{3}\left(\underline{U}_1 + a^2\underline{U}_2 + a\underline{U}_3\right) \end{cases} \tag{3.64}$$

The phase voltages at the input $\underline{U}_1$, $\underline{U}_2$, $\underline{U}_3$, have been calculated in section 3.2.1.1 as functions of the three phase electrical multipole parameters and have the following expressions:

$$\begin{cases} \underline{U}_1 = \underline{I}_1\left(\underline{Z}_1 + \underline{Z}_{10} + \underline{Z}_N\right) + \underline{I}_2\underline{Z}_N + \underline{I}_3\underline{Z}_N - \underline{I'}_1\underline{Z}_{10} \\ \underline{U}_2 = \underline{I}_1\underline{Z}_N + \underline{I}_2\left(\underline{Z}_2 + \underline{Z}_{20} + \underline{Z}_N\right) + \underline{I}_3\underline{Z}_N - \underline{I'}_2\underline{Z}_{20} \\ \underline{U}_3 = \underline{I}_1\underline{Z}_N + \underline{I}_2\underline{Z}_N + \underline{I}_3\left(\underline{Z}_3 + \underline{Z}_{30} + \underline{Z}_N\right) - \underline{I'}_3\underline{Z}_{30} \end{cases} \tag{3.65}$$

We factorize the input electrical currents $\underline{I}_1$, $\underline{I}_2$, $\underline{I}_3$ as a function of the corresponding symmetrical components $\underline{I}_h$, $\underline{I}_d$, $\underline{I}_i$:

$$\begin{cases} \underline{I}_1 = \underline{I}_h + \underline{I}_d + \underline{I}_i \\ \underline{I}_2 = \underline{I}_h + a^2\underline{I}_d + a\underline{I}_i \\ \underline{I}_3 = \underline{I}_h + a\underline{I}_d + a^2\underline{I}_i \end{cases} \tag{3.66}$$

First, we replace relations (3.65) in relations (3.64) and then we replace the result in relation (3.66):

$$\underline{U}_h = \frac{1}{3}\left[\left(\underline{Z}_1 + \underline{Z}_2 + \underline{Z}_3\right) + \left(\underline{Z}_{10} + \underline{Z}_{20} + \underline{Z}_{30}\right) + 9\underline{Z}_N\right]\underline{I}_h +$$

$$+\frac{1}{3}\left[\left(\underline{Z}_1+a^2\underline{Z}_2+a\underline{Z}_3\right)+\left(\underline{Z}_{10}+a^2\underline{Z}_{20}+a\underline{Z}_{30}\right)\right]\underline{I}_d+$$

$$+\frac{1}{3}\left[\left(\underline{Z}_1+a\underline{Z}_2+a^2\underline{Z}_3\right)+\left(\underline{Z}_{10}+a\underline{Z}_{20}+a^2\underline{Z}_{30}\right)\right]\underline{I}_i-$$

$$-\frac{1}{3}\left(\underline{I'}_1\underline{Z}_{10}+\underline{I'}_2\underline{Z}_{20}+\underline{I'}_3\underline{Z}_{30}\right) \qquad (3.67)$$

$$\underline{U}_d=\frac{1}{3}\left[\left(\underline{Z}_1+a\underline{Z}_2+a^2\underline{Z}_3\right)+\left(\underline{Z}_{10}+a\underline{Z}_{20}+a^2\underline{Z}_{30}\right)\right]\underline{I}_h+$$

$$+\frac{1}{3}\left[\left(\underline{Z}_1+\underline{Z}_2+\underline{Z}_3\right)+\left(\underline{Z}_{10}+\underline{Z}_{20}+\underline{Z}_{30}\right)\right]\underline{I}_d+$$

$$+\frac{1}{3}\left[\left(\underline{Z}_1+a^2\underline{Z}_2+a\underline{Z}_3\right)+\left(\underline{Z}_{10}+a^2\underline{Z}_{20}+a\underline{Z}_{30}\right)\right]\underline{I}_i-$$

$$-\frac{1}{3}\left(\underline{I'}_1\underline{Z}_{10}+a\underline{I'}_2\underline{Z}_{20}+a^2\underline{I'}_3\underline{Z}_{30}\right) \qquad (3.68)$$

$$\underline{U}_i=\frac{1}{3}\left[\left(\underline{Z}_1+a^2\underline{Z}_2+a\underline{Z}_3\right)+\left(\underline{Z}_{10}+a^2\underline{Z}_{20}+a\underline{Z}_{30}\right)\right]\underline{I}_h+$$

$$+\frac{1}{3}\left[\left(\underline{Z}_1+a\underline{Z}_2+a^2\underline{Z}_3\right)+\left(\underline{Z}_{10}+a\underline{Z}_{20}+a^2\underline{Z}_{30}\right)\right]\underline{I}_d+$$

$$+\frac{1}{3}\left[\left(\underline{Z}_1+\underline{Z}_2+\underline{Z}_3\right)+\left(\underline{Z}_{10}+\underline{Z}_{20}+\underline{Z}_{30}\right)\right]\underline{I}_i-$$

$$-\frac{1}{3}\left(\underline{I'}_1\underline{Z}_{10}+a^2\underline{I'}_2\underline{Z}_{20}+a\underline{I'}_3\underline{Z}_{30}\right) \qquad (3.69)$$

It can be noticed that the longitudinal impedances $\underline{Z}_1$, $\underline{Z}_2$, $\underline{Z}_3$ and the transversal impedances $\underline{Z}_{10}$, $\underline{Z}_{20}$, $\underline{Z}_{30}$, appear in the above expressions as a function of operator *"a".* After the grouping mode used in relations (3.67) – (3.69), we define the symmetrical components of the longitudinal and transversal impedances.

- ***The homopolar longitudinal impedance has the expression:***

$$\underline{Z}_{lh} = \frac{1}{3}\left(\underline{Z}_1 + \underline{Z}_2 + \underline{Z}_3\right) \quad (3.70)$$

- ***The homopolar transversal impedance has the expression:***

$$\underline{Z}_{th} = \frac{1}{3}\left(\underline{Z}_{10} + \underline{Z}_{20} + \underline{Z}_{30}\right) \quad (3.71)$$

- ***The direct longitudinal impedance*:**

$$\underline{Z}_{ld} = \frac{1}{3}\left(\underline{Z}_1 + a\underline{Z}_2 + a^2\underline{Z}_3\right) \quad (3.72)$$

- ***The direct transversal impedance*:**

$$\underline{Z}_{td} = \frac{1}{3}\left(\underline{Z}_{10} + a\underline{Z}_{20} + a^2\underline{Z}_{30}\right) \quad (3.73)$$

- ***The reverse longitudinal impedance is:***

$$\underline{Z}_{li} = \frac{1}{3}\left(\underline{Z}_1 + a^2\underline{Z}_2 + a\underline{Z}_3\right) \quad (3.74)$$

- ***The reverse transversal impedance*:**

$$\underline{Z}_{ti} = \frac{1}{3}\left(\underline{Z}_{10} + a^2\underline{Z}_{20} + a\underline{Z}_{30}\right) \quad (3.75)$$

In each right parenthesis from expressions (3.67) – (3.69) we have sums of the same types of impedances: homopolar, direct and reverse.

We define:

- ***Total homopolar impedance***

$$\underline{Z}_h = \left(\underline{Z}_{lh} + \underline{Z}_{th}\right) + 3\underline{Z}_N \quad (3.76)$$

- ***Total direct impedance:***

$$\underline{Z}_d = \left(\underline{Z}_{ld} + \underline{Z}d_{th}\right) \quad (3.77)$$

- ***Total reverse impedance:***

$$\underline{Z}_i = \left(\underline{Z}_{li} + \underline{Z}_{ti}\right) \quad (3.78)$$

Analogously, we compute the expressions as functions of the electrical currents at the output ports $\underline{I'}_1$, $\underline{I'}_2$, $\underline{I'}_3$.

- Calculus for the expression :

$$-\frac{1}{3}\left(\underline{I'}_1\underline{Z}_{10}+\underline{I'}_2\underline{Z}_{20}+\underline{I'}_3\underline{Z}_{30}\right) \tag{3.79}$$

From relation (3.67), we factorize the currents $\underline{I'}_1$, $\underline{I'}_2$ şi $\underline{I'}_3$, as function of the symmetrical components:

$$\begin{cases}\underline{I'}_1=\underline{I'}_h+\underline{I'}_d+\underline{I'}_i\\ \underline{I'}_2=\underline{I'}_h+\underline{a}^2\underline{I'}_d+\underline{a}\underline{I'}_i\\ \underline{I'}_3=\underline{I'}_h+\underline{a}\underline{I'}_d+\underline{a}^2\underline{I'}_i\end{cases} \tag{3.80}$$

We replace relations (3.80) in (3.79) and we obtain:

$$-\frac{1}{3}\left(\underline{I'}_1\underline{Z}_{10}+\underline{I'}_2\underline{Z}_{20}+\underline{I'}_3\underline{Z}_{30}\right)=-\underline{Z}_{th}\underline{I'}_h-\underline{Z}_{ti}\underline{I'}_d-\underline{Z}_{td}\underline{I'}_i \tag{3.81}$$

- Calculus for the expression:

$$-\frac{1}{3}\left(\underline{I'}_1\underline{Z}_{10}+\underline{a}\underline{I'}_2\underline{Z}_{20}+\underline{a}^2\underline{I'}_3\underline{Z}_{30}\right) \tag{3.82}$$

From relation (3.68), we apply the same algorithm, by replacing relation (3.83) in relation (3.82), and we obtain:

$$-\frac{1}{3}\left(\underline{I'}_1\underline{z}_{10}+\underline{a}\underline{I'}_2\underline{z}_{20}+\underline{a}^2\underline{I'}_3\underline{z}_{30}\right)=-\underline{z}_{td}\underline{I}_h-\underline{z}_{th}\underline{I'}_d-\underline{z}_{ti}\underline{I'}_i \tag{3.83}$$

- Calculus for the expression:

$$-\frac{1}{3}\left(\underline{I'}_1\underline{Z}_{10}+\underline{a}^2\underline{I'}_2\underline{Z}_{20}+\underline{a}\underline{I'}_3\underline{Z}_{30}\right)$$

From relation (3.70), we obtain:

$$-\underline{Z}_{ti}\underline{I}_h - \underline{Z}_{ti}\underline{I'}_d - \underline{Z}_{th}\underline{I'}_i \tag{3.84}$$

The final expression for (3.67), (3.68) and (3.69), as a function of (3.76), (3.77), (3.78), (3.81), (3.83) and (3.85) become:

$$\begin{cases} \underline{U}_h = \left(\underline{Z}_h + 3\underline{Z}_N\right)\underline{I}_h + \underline{Z}_i\underline{I}_d + \underline{Z}_d\underline{I}_i - \underline{Z}_{th}\underline{I'}_h - \underline{Z}_{ti}\underline{I'}_h - \underline{Z}_{tid}\underline{I'}_i \\ \underline{U}_d = \underline{Z}_d\underline{I}_h + \underline{Z}_h\underline{I}_d + \underline{Z}_i\underline{I}_i - \underline{Z}_{td}\underline{I'}_h - \underline{Z}_{th}\underline{I'}_d - \underline{Z}_{th}\underline{I'}_i \\ \underline{U}_i = \underline{Z}_i\underline{I}_h + \underline{Z}_d\underline{I}_d + \underline{Z}_h\underline{I}_i - \underline{Z}_{ti}\underline{I'}_h - \underline{Z}_{td}\underline{I'}_d - \underline{Z}_{th}\underline{I'}_i \end{cases} \tag{3.85}$$

Analogously for the output ports we obtain:

$$\begin{cases} -\underline{U'}_h = \left(\underline{Z'}_h + 3\underline{Z}_N\right)\underline{I'}_h + \underline{Z'}_i\underline{I'}_d + \underline{Z'}_d\underline{I'}_i - \underline{Z}_{th}\underline{I}_h - \underline{Z}_{ti}\underline{I}_h - \underline{Z}_{tid}\underline{I}_i \\ -\underline{U'}_d = \underline{Z'}_d\underline{I'}_h + \underline{Z'}_h\underline{I'}_d + \underline{Z'}_i\underline{I'}_i - \underline{Z}_{td}\underline{I}_h - \underline{Z}_{th}\underline{I}_d - \underline{Z}_{th}\underline{I}_i \\ -\underline{U'}_i = \underline{Z'}_i\underline{I'}_h + \underline{Z'}_d\underline{I'}_d + \underline{Z'}_h\underline{I'}_i - \underline{Z}_{ti}\underline{I}_h - \underline{Z}_{td}\underline{I}_d - \underline{Z}_{th}\underline{I}_i \end{cases} \tag{3.86}$$

in which:

$$\begin{cases} \underline{Z'}_{lh} = \dfrac{1}{3}\left(\underline{Z'}_1 + \underline{Z'}_2 + \underline{Z'}_3\right) \\ \underline{Z'}_{th} = \underline{Z}_{th} \end{cases} \tag{3.87}$$

$$\begin{cases} \underline{Z'}_{ld} = \dfrac{1}{3}\left(\underline{Z'}_1 + \underline{a}\underline{Z'}_2 + \underline{a}^2\underline{Z'}_3\right) \\ \underline{Z'}_{td} = \underline{Z}_{th} \end{cases} \tag{3.88}$$

$$\begin{cases} \underline{Z'}_{li} = \dfrac{1}{3}\left(\underline{Z'}_1 + \underline{a}^2\underline{Z'}_2 + \underline{a}\underline{Z'}_3\right) \\ \underline{Z'}_{li} = \underline{Z}_{li} \end{cases} \tag{3.89}$$

$$\begin{cases} \underline{Z'}_h = \underline{Z'}_{lh} + \underline{Z'}_{th} \\ \underline{Z'}_d = \underline{Z'}_{ld} + \underline{Z'}_{td} \\ \underline{Z'}_i = \underline{Z'}_{li} + \underline{Z'}_{ti} \end{cases} \tag{3.90}$$

We introduce the operator of the symmetric components for the input and output ports:

$$\begin{cases} \underline{\hat{U}}_{input.sim.} = \begin{pmatrix} \underline{U}_h \\ \underline{U}_d \\ \underline{U}_i \end{pmatrix} \\ \underline{\hat{U}}_{output.sim.} = \begin{pmatrix} \underline{U'}_h \\ \underline{U'}_d \\ \underline{U'}_i \end{pmatrix} \\ \underline{\hat{I}}_{input.sim} = \begin{pmatrix} \underline{I}_h \\ \underline{I}_d \\ \underline{I}_i \end{pmatrix} \\ \underline{\hat{I}}_{output.sim} = \begin{pmatrix} \underline{I'}_h \\ \underline{I'}_d \\ \underline{I'}_i \end{pmatrix} \end{cases} \tag{3.91}$$

We introduce the matrix relation between operators:

$$\begin{pmatrix} \underline{\hat{U}}_{input.sim} \\ \underline{\hat{U}}_{output.sim} \end{pmatrix} = \begin{pmatrix} \underline{\hat{\hat{Z}}}_{11s} & \underline{\hat{\hat{Z}}}_{12s} \\ \underline{\hat{\hat{Z}}}_{21s} & \underline{\hat{\hat{Z}}}_{22s} \end{pmatrix} \begin{pmatrix} \underline{\hat{I}}_{input.s} \\ \underline{\hat{I}}_{output.s} \end{pmatrix} \tag{3.92}$$

From expecting relations (3.85) and (3.86) with (3.92) it results:

$$\underline{\hat{\hat{Z}}}_{11S} = \begin{pmatrix} \underline{Z}_h + 3\underline{Z}_N & \underline{Z}_i & \underline{Z}_d \\ \underline{Z}_d & \underline{Z}_h & \underline{Z}_i \\ \underline{Z}_i & \underline{Z}_d & \underline{Z}_h \end{pmatrix} \tag{3.93}$$

$$\underline{\hat{\hat{Z}}}_{12S} = -\begin{pmatrix} \underline{Z}_{th} & \underline{Z}_{ti} & \underline{Z}_{td} \\ \underline{Z}_{td} & \underline{Z}_{th} & \underline{Z}_{ti} \\ \underline{Z}_{ti} & \underline{Z}_{td} & \underline{Z}_{th} \end{pmatrix} \tag{3.94}$$

$$\underline{\hat{\hat{Z}}}_{21S} = \begin{pmatrix} \underline{Z}_{th} & \underline{Z}_{ti} & \underline{Z}_{td} \\ \underline{Z}_{td} & \underline{Z}_{th} & \underline{Z}_{ti} \\ \underline{Z}_{ti} & \underline{Z}_{td} & \underline{Z}_{th} \end{pmatrix} \quad (3.95)$$

$$\underline{\hat{\hat{Z}}}_{22S} = -\begin{pmatrix} \underline{Z}'_h + 3\underline{Z}_N & \underline{Z}'_i & \underline{Z}'_d \\ \underline{Z}'_d & \underline{Z}'_h & \underline{Z}'_i \\ \underline{Z}'_i & \underline{Z}'_d & \underline{Z}'_h \end{pmatrix} \quad (3.96)$$

By examining the form of the symmetric impedance operators, there can be noticed that:

$$\underline{\hat{\hat{Z}}}_{12S} = -\underline{\hat{\hat{Z}}}_{21S}$$

This represents the reciprocal condition.

For a three-phase electrical multipole, with $\underline{Z}_1 = \underline{Z}'_1$, $\underline{Z}_2 = \underline{Z}'_2$ and $\underline{Z}_3 = \underline{Z}'_3$, one can obtain the symmetry condition, known from the electrical quadripoles:

$$\underline{\hat{\hat{Z}}}_{11S} = -\underline{\hat{\hat{Z}}}_{22S}$$

4. THREE-PHASE ELECTRIC LINES WITH DISTRIBUTED PARAMETERS

4.1. Telegraphers' equations for three-phase electric lines

A three-phase electrical line with distributed parameters represents a generalization of a three-phase electrical line with distributed parameters in which each phase is coupled with the other phases.

As in the case of the transmission line, we will introduce the following per unit length parameters for three-phase electrical lines :

- ***R_1, R_2, R_3, the resistance per unit length of the conductors that make phases 1, 2 and 3.***
- ***L_1, L_2, L_3 the inductance per unit length of the conductors themselves*.**
- ***L_{12}, L_{23}, L_{31}, the inductance per unit length of the couple between phases.***
- ***C_{12}, C_{23}, C_{31}, the capacitance per unit length of the coupling between phases*.**
- ***G_{12}, G_{23}, G_{31} the conductance per unit length of the losses between phases.***

In *figure 4.1* an infinite small transmission line of length *dx*, is presented for which the electrical parameters ($R_i dx$, $L_i dx$, $L_{ij} dx$, $C_{ij} dx$, $G_{ij} dx$ with $i, j = 1, 2, 3$**)** are considered lumped.

If we consider $u_{12}(x)$, $u_{23}(x)$, $u_{31}(x)$, respectively $i_1(x)$, $i_2(x)$, $i_3(x)$, the electrical voltage and current at the distance ***x*** from the beginning of the transmission line, then at the distance ***x+dx***, these quantities will be $u_{12}(x+dx)$, $u_{23}(x+dx)$, $u_{31}(x+dx)$, $i_1(x+dx)$, $i_2(x+dx)$, $i_3(x+dx)$.

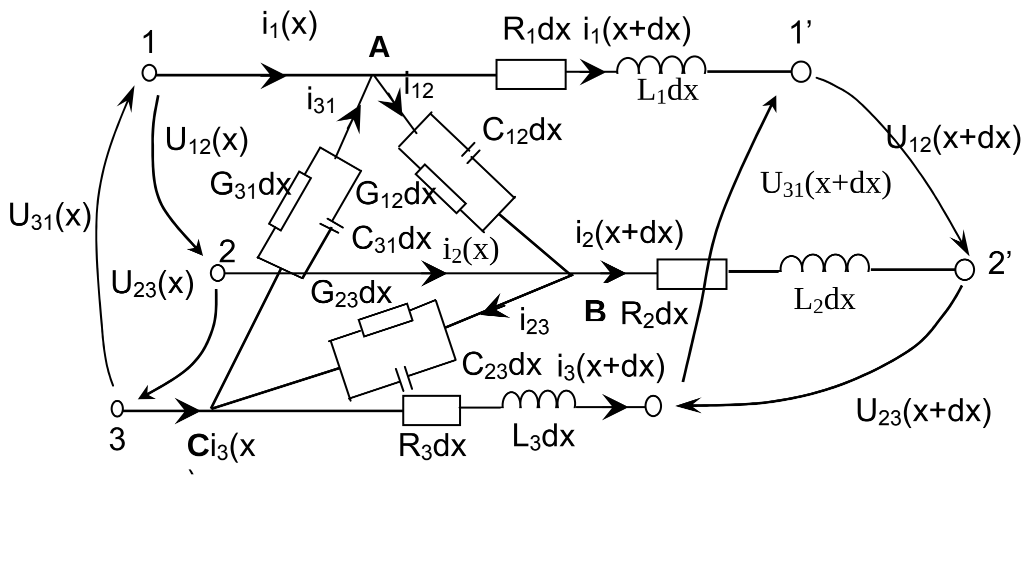

Fig.4.1. Infinite small transmission line, used for obtaining Telegrapher's equations for a three phase electrical line

We apply Kirchhoff's second law on the contour *"11'2'2"*:

$$u_{12}(x) - u_{12}(x+dx) = R_1 dx(i_1 + \frac{\partial i_1}{\partial x}dx) + L_1 dx \frac{\partial}{\partial t}(i_1 + \frac{\partial i_1}{\partial x}dx) +$$
$$+ L_{21} dx \frac{\partial}{\partial t}(i_2 + \frac{\partial i_2}{\partial x}dx) + L_{31} dx \frac{\partial}{\partial t}(i_3 + \frac{\partial i_3}{\partial x}dx) - R_2 dx(i_2 + \frac{\partial i_2}{\partial x}dx) -$$
$$- L_2 dx \frac{\partial}{\partial t}(i_2 + \frac{\partial i_2}{\partial x}dx) - L_{12} dx \frac{\partial}{\partial t}(i_1 + \frac{\partial i_1}{\partial x}dx) - L_{32} dx \frac{\partial}{\partial t}(i_3 + \frac{\partial i_3}{\partial x}dx)$$

The infinite small terms of second order are neglected and we obtain:

$$-\frac{\partial u_{12}}{\partial x} = R_1 i_1 + L_{11}\frac{\partial i_1}{\partial t} + L_{21}\frac{\partial i_2}{\partial t} + L_{31}\frac{\partial i_3}{\partial t} - R_2 i_2 - L_{22}\frac{\partial i_2}{\partial t} - L_{12}\frac{\partial i_1}{\partial t} - L_{32}\frac{\partial i_3}{\partial t} \tag{4.1}$$

Through circular permutations of the indexes, the other two equations are found:

$$-\frac{\partial u_{23}}{\partial x} = R_2 i_2 + L_{22}\frac{\partial i_2}{\partial t} + L_{32}\frac{\partial i_3}{\partial t} + L_{12}\frac{\partial i_1}{\partial t} - R_3 i_3 - L_{33}\frac{\partial i_3}{\partial t} - L_{23}\frac{\partial i_2}{\partial t} - L_{12}\frac{\partial i_1}{\partial t} \tag{4.2}$$

$$-\frac{\partial u_{31}}{\partial x} = R_3 i_3 + L_{33}\frac{\partial i_3}{\partial t} + L_{12}\frac{\partial i_1}{\partial t} + L_{23}\frac{\partial i_2}{\partial t} - R_1 i_1 - L_{11}\frac{\partial i_1}{\partial t} - L_{31}\frac{\partial i_3}{\partial t} - L_{23}\frac{\partial i_2}{\partial t} \tag{4.3}$$

Considering the currents' case, by applying Kirchhoff's first law in *A, B, C* nodes, it results that:

$$-i_1 + (i_1 + \frac{\partial i_1}{\partial x}dx) + C_{12} dx \frac{\partial u_{12}}{\partial t} + G_{12} u_{12} - C_{31} dx \frac{\partial u_{31}}{\partial t} - G_{31} u_{31} dx = 0$$

And the following equation is obtained:

$$-\frac{\partial \boldsymbol{i}_1}{\partial} = \boldsymbol{G}_{12}\boldsymbol{u}_{12} + \boldsymbol{C}_{12}\frac{\partial \boldsymbol{u}_{12}}{\partial \boldsymbol{t}} - \boldsymbol{G}_{31}\boldsymbol{u}_{31} - \boldsymbol{C}_{31}\frac{\partial \boldsymbol{u}_{31}}{\partial \boldsymbol{t}} \tag{4.4}$$

Analogously, the following equations are obtained:

$$-\frac{\partial \boldsymbol{i}_2}{\partial} = \boldsymbol{G}_{23}\boldsymbol{u}_{23} + \boldsymbol{C}_{23}\frac{\partial \boldsymbol{u}_{23}}{\partial \boldsymbol{t}} - \boldsymbol{G}_{12}\boldsymbol{u}_{12} - \boldsymbol{C}_{12}\frac{\partial \boldsymbol{u}_{12}}{\partial \boldsymbol{t}} \tag{4.5}$$

$$-\frac{\partial \boldsymbol{i}_3}{\partial} = \boldsymbol{G}_{31}\boldsymbol{u}_{31} + \boldsymbol{C}_{31}\frac{\partial \boldsymbol{u}_{31}}{\partial \boldsymbol{t}} - \boldsymbol{G}_{23}\boldsymbol{u}_{23} - \boldsymbol{C}_{23}\frac{\partial \boldsymbol{u}_{23}}{\partial \boldsymbol{t}} \tag{4.6}$$

Relations (4.1), (4.2), (4.3), (4.4), (4.5) and (4.6) form a system of equations with partial derivates for the three-phase electrical lines with distributed parameters which generalize Telegrapher's equations for transmission lines. The system of equations with partial derivates reveal the fact that both electrical voltage and current are functions of variables *x* and *t*, so we will have $\boldsymbol{u}_j(\boldsymbol{x},\boldsymbol{t})$ respectively $\boldsymbol{i}_j(\boldsymbol{x},\boldsymbol{t})$ (*j = 1,2,3* and they represent the phases).

For sinusoidal steady state we will look for a solution of the form:

$$\begin{cases} \boldsymbol{u}_{12}(\boldsymbol{x},\boldsymbol{t}) = \underline{\boldsymbol{U}}_{12}(\boldsymbol{x})\boldsymbol{e}^{j\omega t} \\ \boldsymbol{u}_{23}(\boldsymbol{x},\boldsymbol{t}) = \underline{\boldsymbol{U}}_{23}(\boldsymbol{x})\boldsymbol{e}^{j(\omega t-120^o)} \\ \boldsymbol{u}_{31}(\boldsymbol{x},\boldsymbol{t}) = \underline{\boldsymbol{U}}_{31}(\boldsymbol{x})\boldsymbol{e}^{j(\omega t-240^o)} \end{cases} \tag{4.7}$$

$$\begin{cases} i_{12}(x,t) = \underline{I}_1(x)e^{j\omega t} \\ i_2(x,t) = \underline{I}_2(x)e^{j(\omega t-120^o)} \\ i_{31}(x,t) = \underline{I}_3(x)e^{j(\omega t-240^o)} \end{cases} \tag{4.8}$$

By expressing the voltages and currents in complex, the system of equations expressed by relations (4.1) -(4.8) becomes:

$$\begin{cases} -\frac{d\underline{U}_{12}}{dx} = [R_1 + j\omega(L_{11} - L_{12})]\underline{I_1} + [-R_2 - j\omega(L_{22} - L_{21})]\underline{I_2} + [j\omega(L_{31} - L_{32})]\underline{I_3} \\ -\frac{d\underline{U}_{23}}{dx} = j\omega(L_{11} - L_{12})\underline{I_1} + [R_2 + j\omega(L_{22} - L_{23})]\underline{I_2} + [-R_3 - j\omega(L_{33} - L_{32})]\underline{I_3} \\ -\frac{d\underline{U}_{31}}{dx} = [-R_1 + j\omega(L_{11} - L_{13})]\underline{I_1} + j\omega(L_{23} - L_{21})\underline{I_2} + [R_{3+} j\omega(L_{33} - L_{31})]\underline{I_3} \end{cases} \tag{4.9}$$

$$\begin{cases} -\frac{d\underline{I}_1}{dx} = (G_{12} + j\omega C_{12})\underline{U}_{12} + \underline{U}_{23}\cdot 0 + (-G_{31} - j\omega C_{31})\underline{U}_{31} \\ -\frac{d\underline{I}_2}{dx} = (-G_{12} - j\omega C_{12})\underline{U}_{12} + (G_{23} + j\omega C_{23})\underline{U}_{23} + 0\cdot\underline{U}_{31} \\ -\frac{d\underline{I}_3}{dx} = 0\cdot\underline{U}_{12} + (-G_{23} - j\omega C_{23})\underline{U}_{23}\cdot 0 + (G_{31} + j\omega C_{31})\underline{U}_{31} \end{cases} \tag{4.10}$$

We derive the group of equations (4.9) in respect to *x* and we write it using matrixes:

$$\begin{pmatrix} -\frac{d^2\underline{U}_{12}}{dx^2} \\ -\frac{d^2\underline{U}_{23}}{dx^2} \\ -\frac{d^2\underline{U}_{31}}{dx^2} \end{pmatrix} =$$

$$= \begin{pmatrix} R_1 + j\omega(L_{11} - L_{21}) & [-R_2 - j\omega(L_{22} - L_{21})] & j\omega(L_{31} - L_{32}) \\ j\omega(L_{12} - L_{13}) & [R_2 + j\omega(L_{122} - L_{23})] & [-R_1 - j\omega(L_{33} - L_{32})] \\ -R_1 - j\omega(L_{11} - L_{13}) & j\omega(L_{23} - L_{21}) & R_3 + j\omega(L_{33} - L_{31}) \end{pmatrix}\cdot$$

$$\cdot \begin{pmatrix} \dfrac{d\underline{I}_1}{dx} \\ \dfrac{d\underline{I}_2}{dx} \\ \dfrac{d\underline{I}_3}{dx} \end{pmatrix} \tag{4.11}$$

$$\begin{pmatrix} \dfrac{d\underline{I}_1}{dx} \\ \dfrac{d\underline{I}_2}{dx} \\ \dfrac{d\underline{I}_3}{dx} \end{pmatrix} = \begin{pmatrix} G_{12} + j\omega C_{12} & 0 & -G_{31} - j\omega C_{31} \\ -G_{12} - j\omega C_{12} & -G_{23} - j\omega C_{23} & 0 \\ 0 & -G_{23} - j\omega C_{23} & -G_{31} - j\omega C_{31} \end{pmatrix} \cdot \begin{pmatrix} \underline{U}_{12} \\ \underline{U}_{23} \\ \underline{U}_{31} \end{pmatrix} \tag{4.12}$$

In relation (4.11), we note $\hat{\hat{\underline{Z}}}_{long}$ the right matrix and with $\hat{\hat{\underline{Y}}}_{trans}$ we note the matrix in the right side of relation (4.12).

$$\hat{\hat{\underline{Z}}}_{long} = \begin{pmatrix} R_1 + j\omega(L_{11} - L_{21}) & [-R_2 - j\omega(L_{22} - L_{21})] & j\omega(L_{31} - L_{32}) \\ j\omega(L_{12} - L_{13}) & [R_2 + j\omega(L_{122} - L_{23})] & [-R_1 - j\omega(L_{33} - L_{32})] \\ -R_1 - j\omega(L_{11} - L_{13}) & j\omega(L_{23} - L_{21}) & R_3 + j\omega(L_{33} - L_{31}) \end{pmatrix} \tag{4.13}$$

$$\hat{\hat{\underline{Y}}}_{trans} = \begin{pmatrix} G_{12} + j\omega C_{12} & 0 & -G_{31} - j\omega C_{31} \\ -G_{12} - j\omega C_{12} & -G_{23} - j\omega C_{23} & 0 \\ 0 & -G_{23} - j\omega C_{23} & -G_{31} - j\omega C_{31} \end{pmatrix} \tag{4.14}$$

From relations (4.11) and (4.12) and with notations (4.13) and (4.14), we get:

$$\frac{d^2\underline{\hat{U}}(x)_l}{dx^2}=\underline{\hat{\hat{Z}}}_{long}\underline{\hat{\hat{Y}}}_{trans}\underline{\hat{U}}(x)_l \tag{4.15}$$

Relation 4.15. represents ***the complex form of the Telegrapher's equations of second order generalized for three phase electrical lines, for voltages on the line.***

In relation (4.15) we introduce **the propagation (or transfer) constant** :

$$\underline{\gamma}=\sqrt{\underline{Z}_{long}\cdot\underline{Y}_{trans}}$$

In order to obtain Telegrapher's equations of second order generalized for three phase electrical lines, for phase currents, relations (4.9) are (4.10) rewritten using matrix form:

$$\begin{cases}-\dfrac{d\underline{\hat{U}}_l}{dx}=\underline{\hat{\hat{Z}}}_{long}\underline{\hat{I}}\\-\dfrac{d\underline{\hat{I}}}{dx}=\underline{\hat{\hat{Y}}}_{trans}\underline{\hat{U}}_l\end{cases} \tag{4.16}$$

The second equation from (4.16) is derived and introduced in the first equation, obtaining:

$$\frac{d^2\underline{\hat{I}}(x)}{dx}=\underline{\hat{\hat{Y}}}_{trans}\underline{\hat{\hat{Z}}}_{long}\underline{\hat{I}}(x) \tag{4.17}$$

It can be observed that in (4.16) and (4.17) operators $\underline{\hat{\hat{Z}}}_{long}$ and $\underline{\hat{\hat{Y}}}_{long}$ are switched which means that different expressions for voltages and currents will be obtained.

4.2. Multipolar equations

The general solution of the equation (4.15) is:

$$\underline{\hat{U}}(x) = e^{-\hat{\underline{\hat{\gamma}}}x}\underline{\hat{C}}_1 + e^{\hat{\underline{\hat{\gamma}}}x}\underline{\hat{C}}_2 \tag{4.18}$$

In which $\underline{\hat{C}}_1$ and $\underline{\hat{C}}_2$ are arbitrary constants written as column vector, generally complex, which are determined using conditions at the ends of the line.

For determining the current $\underline{\hat{I}}$, the general solution (4.18) of the voltage is introduced in relation (4.9), which is written in matrix form:

$$-\frac{d\underline{\hat{U}}}{dx} = \underline{\hat{\hat{Z}}}_{long}\underline{\hat{I}} \tag{4.19}$$

We obtain:

$$\underline{\hat{\hat{\gamma}}}\underline{\hat{C}}_1 e^{-\hat{\underline{\hat{\gamma}}}x} - \underline{\hat{\hat{\gamma}}}\underline{\hat{C}}_2 e^{\hat{\underline{\hat{\gamma}}}x} = \underline{\hat{\hat{Z}}'}_{long}\underline{\hat{\hat{I}}} \tag{4.20}$$

By multiplying on the left with $\underline{\hat{\hat{Z}}}^{-1}$, the electrical current expression is obtained:

$$\underline{\hat{I}} = \underline{\hat{\hat{Z}}}_{long}^{-1}\underline{\hat{\hat{\gamma}}}\left(e^{-\hat{\underline{\hat{\gamma}}}x}\underline{\hat{C}}_1 - e^{\hat{\underline{\hat{\gamma}}}x}\underline{\hat{C}}_2\right) \tag{4.21}$$

and

$$\underline{\hat{\hat{Z}}}_{long}^{-1}\underline{\hat{\hat{\gamma}}} = \underline{\hat{\hat{Z}}}_c^{-1} = \underline{\hat{\hat{Y}}}_c \tag{4.22}$$

The expression given by relation (4.22) is called characteristic admittance operator of the three-phase electrical line. It is a matrix and it generalizes the characteristic admittance of the transmission line that is a complex number.

In order to obtain the expression for the characteristic impedance matrix, relation (4.22) is inverted term by term and the following is obtained:

$$\hat{\hat{\underline{Z}}}_c = \underline{\gamma}^{-1}\hat{\hat{\underline{Z}}}'_{log} \tag{4.23}$$

The general solutions for the electrical voltage and current are:

$$\begin{cases} \hat{\underline{U}}(x) = e^{-\hat{\hat{\underline{\gamma}}}x}\hat{\underline{C}}_1 + e^{\hat{\hat{\underline{\gamma}}}x}\hat{\underline{C}}_2 \\ \hat{\underline{I}}(x) = \hat{\hat{\underline{Y}}}_c\left(e^{-\hat{\hat{\underline{\gamma}}}x}\hat{\underline{C}}_1 - e^{\hat{\hat{\underline{\gamma}}}x}\hat{\underline{C}}_2\right) \end{cases} \tag{4.24}$$

To determine the constants $\hat{\underline{C}}_1$ and $\hat{\underline{C}}_2$ that occur in equation (4.24), voltages and currents are assumed known at the beginning of the line.

For:

$x = 0$, $\hat{\underline{I}} = \hat{\underline{I}}_1$ and $\hat{\underline{U}} = \hat{\underline{U}}_1$.

It is obvious that $\hat{\underline{I}}_1$ and $\hat{\underline{U}}_1$ are three-column array elements.

We obtain:

$$\begin{cases} \hat{\underline{U}}_1 = \hat{\underline{C}}_1 + \hat{\underline{C}}_2 \\ \hat{\underline{I}}_1 = \hat{\hat{\underline{Y}}}_c\left(\hat{\underline{C}}_1 - \hat{\underline{C}}_2\right) \end{cases}$$

In order to determine the constants $\underline{\hat{C}}_1$ and $\underline{\hat{C}}_2$, the second relation above is multiplied on the left with $\underline{\hat{\hat{Y}}}_c^{-1}$.

For constants $\underline{\hat{C}}_1$ and $\underline{\hat{C}}_2$ the following expressions result:

$$\begin{cases} \underline{\hat{C}}_1 = \dfrac{\underline{\hat{U}}_1 + \underline{\hat{\hat{Z}}}_c \underline{\hat{I}}_1}{2} \\ \underline{\hat{C}}_2 = \dfrac{\underline{\hat{U}}_1 - \underline{\hat{\hat{Z}}}_c \underline{\hat{I}}_1}{2} \end{cases} \quad (4.25)$$

Based on relations (4.25), relations (4.24) become:

$$\begin{cases} \underline{\hat{U}}(x) = \left(ch\underline{\hat{\gamma}}x\right)\underline{\hat{U}}_1 - \left(sh\underline{\hat{\gamma}}x\right)\underline{\hat{\hat{Z}}}_c \underline{\hat{I}}_1 \\ \underline{\hat{I}}(x) = -\left(sh\underline{\hat{\gamma}}x\right)\underline{\hat{\hat{Y}}}_c \underline{\hat{U}}_1 + \left(ch\underline{\hat{\gamma}}x\right)\underline{\hat{I}}_1 \end{cases} \quad (4.26)$$

Equations (4.26) represent the multipolar equations of a piece of three-phase electrical line, situated between the input and a section at "**x**" distance.

If in equations (4.26) there is introduced $x = l$, there are obtained the equations at the output as a function of the input quantities:

$$\begin{cases} \underline{\hat{U}}_{output} = \underline{\hat{U}}(l) = \left(ch\underline{\hat{\gamma}}l\right)\cdot\underline{\hat{U}}_{input} - \left(sh\underline{\hat{\gamma}}l\right)\cdot\underline{\hat{\hat{Z}}}_c \underline{\hat{I}}_{input} \\ \underline{\hat{I}}_{output} = \underline{\hat{I}}(l) = -\left(sh\underline{\hat{\gamma}}l\right)\cdot\underline{\hat{\hat{Y}}}_c \underline{\hat{U}}_{input} + \left(ch\underline{\hat{\gamma}}l\right)\cdot\underline{\hat{I}}_{input} \end{cases} \quad (4.27)$$

Equation (4.27) can be written also using matrix form:

$$\begin{pmatrix} \hat{\underline{U}}_{output} \\ \hat{\underline{I}}_{output} \end{pmatrix} = \begin{pmatrix} ch\hat{\hat{\underline{\gamma}}}l & -\left(sh\hat{\hat{\underline{\gamma}}}l\right)\cdot \hat{\hat{\underline{Z}}}_c \\ -\left(sh\hat{\hat{\underline{\gamma}}}l\right)\cdot \hat{\hat{\underline{Y}}}_c & ch\hat{\hat{\underline{\gamma}}}l \end{pmatrix} \begin{pmatrix} \hat{\underline{U}}_{input} \\ \hat{\underline{I}}_{input} \end{pmatrix} \tag{4.28}$$

It can be observed that the squared matrix *2x2,* on the right side of relation (4.28), is the matrix of the fundamental operators, written with the help of the characteristic parameters $\hat{\hat{\underline{Z}}}_c$ and $\hat{\hat{\underline{g}}}_c = \hat{\underline{\gamma}}l$.

$$\hat{\hat{\underline{A}}} = ch\hat{\hat{\underline{\gamma}}}l \qquad \hat{\hat{\underline{B}}} = -\left(sh\hat{\hat{\underline{\gamma}}}l\right)\hat{\hat{\underline{Z}}}_c$$

$$\hat{\hat{\underline{C}}} = -sh\hat{\hat{\underline{\gamma}}}l \qquad D = ch\hat{\hat{\underline{\gamma}}}l$$

4.3. Refection coefficient

Let's analyze the general solutions (4.24) not just from the point of view of the multipolar equations, but also from the point of view of the forward and backward waves with which we are familiar from the transmission lines.

Referring to relations (4.24), we can write them like this:

$$\begin{cases} \underline{U}(x) = e^{-\hat{\hat{\underline{\gamma}}}x}\hat{\underline{C}}_1 + e^{\hat{\hat{\underline{\gamma}}}x}\hat{\underline{C}}_2 = \hat{\underline{U}}_d(x) + \hat{\underline{U}}_i(x) \\ \underline{I}(x) = \hat{\hat{\underline{Y}}}_c(e^{-\hat{\hat{\underline{\gamma}}}x}\hat{\underline{C}}_1 - e^{\hat{\hat{\underline{\gamma}}}x}\hat{\underline{C}}_2) = \hat{\underline{I}}_d(x) + \hat{\underline{I}}_i(x) \end{cases} \tag{4.29}$$

The forward voltage and current waves have the form:

$$\begin{cases} \hat{\underline{U}}_d(x) = e^{-\hat{\hat{\underline{\gamma}}}x}\hat{\underline{C}}_1 \\ \hat{\underline{I}}_d(x) = \hat{\hat{\underline{Y}}}_c e^{-\hat{\hat{\underline{\gamma}}}x}\hat{\underline{C}}_1 \end{cases} \tag{4.30}$$

And the backward voltage and current waves are:

$$\begin{cases} \underline{\hat{U}}_i(x) = e^{-\underline{\hat{\gamma}}x}\underline{\hat{C}}_2 \\ \underline{\hat{I}}_i(x) = -\underline{\hat{\hat{Y}}}_c e^{-\underline{\hat{\gamma}}x}\underline{\hat{C}}_2 \end{cases} \tag{4.31}$$

Constants $\underline{\hat{C}}_1$ and $\underline{\hat{C}}_2$ have the expressions (4.25).

From relations (4.30) it can be noticed that the last two terms of the second relation represent the forward voltage wave (meaning the first relation):

$$\underline{\hat{I}}_d(x) = \underline{\hat{\hat{Y}}}_c \underline{\hat{\hat{U}}}_d \tag{4.32}$$

Analogously for relation (4.31):

$$\underline{\hat{I}}_i = -\underline{\hat{\hat{Y}}}_c \underline{\hat{\hat{U}}}_{i1} \tag{4.33}$$

Relations (4.32) and (4.33), show that in any section of the three phase electrical line, the forward component of the current is related to the forward component of the voltage through characteristic admittance operator. The same thing can be said about the backward components of the current and voltage:

$$\begin{cases} \underline{\hat{U}}_d = \underline{\hat{\hat{Z}}}_c \underline{\hat{I}}_d \\ \underline{\hat{U}}_i = -\underline{\hat{\hat{Z}}}_c \underline{\hat{I}}_i \end{cases} \tag{4.34}$$

The reflection coefficient $\underline{\hat{\hat{K}}}_r$ is defined as a mathematical expression, which links the forward wave of voltage to the backward wave voltage at the end of the line (***x = l***).

In order to determine its expression, let us consider the two voltage waves at the end of the line. From relations (4.30) and (4.31), we obtain:

$$\begin{cases} \underline{\hat{U}}_d = e^{-\underline{\hat{\hat{\gamma}}}l}\underline{\hat{C}}_1 = e^{-\underline{\hat{\hat{\gamma}}}l}\dfrac{\underline{\hat{U}}_1 + \underline{\hat{\hat{Z}}}_c\underline{\hat{I}}_1}{2} \\ \underline{\hat{U}}_i = e^{\underline{\hat{\hat{\gamma}}}l}\underline{\hat{C}}_2 = e^{\underline{\hat{\hat{\gamma}}}l}\dfrac{\underline{\hat{U}}_1 - \underline{\hat{\hat{Z}}}_c\underline{\hat{I}}_1}{2} \end{cases} \tag{4.35}$$

Because of the matrix character of the quantities, we cannot define the ratio of the two voltage waves. Not to encounter this problem, we will write that:

$$\underline{\hat{U}}_i = \underline{\hat{\hat{K}}}_r\underline{\hat{U}}_d \tag{4.36}$$

Where the reflection coefficient must be a matrix, because $\underline{\hat{U}}_i$ and $\underline{\hat{U}}_d$ are column matrixes themselves.
From relations (4.35) and (4.36) we obtain:

$$e^{\underline{\hat{\hat{\gamma}}}l}(\underline{\hat{U}}_1 - \underline{\hat{\hat{Z}}}_c\underline{\hat{I}}_1) = \underline{\hat{\hat{K}}}_r e^{-\underline{\hat{\hat{\gamma}}}l}(\underline{\hat{U}}_1 + \underline{\hat{\hat{Z}}}_c\underline{\hat{I}}_1) \tag{4.37}$$

For easing the computations and emphasizing the role of the three phase consumer's impedance, we will write relation (4.37), as a function of the quantities at the end of the line, meaning $\underline{\hat{U}}_2, \underline{\hat{I}}_2, \underline{\hat{Z}}_s$. The voltages at the end of the line are identical with the voltages at the consumer. In previous conditions, relation (4.37) becomes:

$$\underline{\hat{U}}_2 - \underline{\hat{Z}}_c\underline{\hat{I}}_2 = \underline{\hat{\hat{K}}}_r(\underline{\hat{U}}_2 + \underline{\hat{Z}}_c\underline{\hat{I}}_2) \tag{4.38}$$

We introduce condition $\underline{\hat{U}}_2 = \underline{\hat{U}}_{output} = \underline{\hat{\hat{Z}}}_s\underline{\hat{I}}_{output}$ in relation (4.38) and obtain: $\underline{\hat{\hat{Z}}}_s - \underline{\hat{\hat{Z}}}_c = \underline{\hat{\hat{K}}}_r(\underline{\hat{\hat{Z}}}_s + \underline{\hat{\hat{Z}}}_c)$

By multiplying on the right with the inverse matrix of the coefficient $\underline{\hat{\hat{K}}}_r$, we obtain in the end:

$$\hat{\hat{\underline{K}}}_r = (\hat{\hat{\underline{Z}}}_s - \hat{\hat{\underline{Z}}}_c)(\hat{\hat{\underline{Z}}}_s + \hat{\hat{\underline{Z}}}_c)^{-1} \qquad (4.39)$$

For transmission lines, the reflection coefficient is generally a complex quantity, its module being equal to the ratio between the absolute values of the amplitudes of the forward and backward waves at the end of the line.

For a three-phase line, the reflection coefficient is a matrix because each of the three voltages and currents are reflected at the end of the line. It is obvious that in the case of an asymmetric three-phase line with asymmetric load consumers, each phase will have its own reflection coefficient.

4.4. The input impedance

In paragraph 4.3, relation (4.27), we have set the dependency between the electrical quantities at the output and the electrical quantities at the input. We solve the system of equations (4.27) as function of input electrical quantities $\hat{\underline{U}}_{input}$ and $\hat{\underline{I}}_{input}$ and we obtain:

$$\begin{cases} \hat{\underline{U}}_{input} = (ch\hat{\underline{\gamma}}l)\hat{\underline{U}}_{output} + (sh\hat{\underline{\gamma}}l)\hat{\hat{\underline{Z}}}_c \hat{\underline{I}}_{output} \\ \hat{\underline{I}}_{input} = (sh\hat{\underline{\gamma}}l)\hat{\hat{\underline{Y}}}_c \hat{\underline{U}}_{output} + (ch\hat{\underline{\gamma}}l)\hat{\underline{I}}_{output} \end{cases} \qquad (4.40)$$

If in the case of the electrical quadripole, the input impedance is defined as the ratio between the input voltage and current, in complex representation, in the case of the three-phase electrical multipole, both voltage and current are column matrixes which can not be divided algebraically.

We have reached the same problem as in the case of the reflection coefficient. In this situation, we define the input impedance for a three-phase electrical line, as a proportional constant between the matrix of the input voltage and the one of the input current. It is obvious that this constant is a matrix and has the dimension of impedance:

$$\underline{\hat{U}}_{input} = \underline{\hat{\hat{Z}}}_{input} \cdot \underline{\hat{I}}_{input} \tag{4.41}$$

We replace in the above relation $\underline{\hat{U}}_{input}$ and $\underline{\hat{I}}_{input}$ with the expressions from relations (4.40) and obtain:

$$(ch\underline{\hat{\gamma}}l)\underline{\hat{\hat{Z}}}_{s}\underline{\hat{I}}_{output} + (sh\underline{\hat{\gamma}}l)\underline{\hat{\hat{Z}}}_{c}\underline{\hat{I}}_{output} = \underline{\hat{\hat{Z}}}_{input}\left[(sh\underline{\hat{\gamma}}l)\underline{\hat{\hat{Y}}}_{c}\underline{\hat{\hat{Z}}}_{s}\underline{\hat{I}}_{output} + (ch\underline{\hat{\gamma}}l)\underline{\hat{I}}_{output}\right]$$

It can be noticed that the input impedance $\underline{\hat{\hat{Z}}}_{input}$ can be obtained directly from the above relation by multiplying on the right with the inverse of its coefficient:

$$\underline{\hat{\hat{Z}}}_{input} = ((ch\underline{\hat{\gamma}}l)\underline{\hat{\hat{Z}}}_{s} + (sh\underline{\hat{\gamma}}l)\underline{\hat{\hat{Z}}}_{c})\left[(sh\underline{\hat{\gamma}}l)\underline{\hat{\hat{Y}}}_{c}\underline{\hat{\hat{Z}}}_{s} + (ch\underline{\hat{\gamma}}l)\right]^{-1} \tag{4.42}$$

The open impedance $\underline{\hat{\hat{Z}}}_{input,0}$ and the short impedance $\underline{\hat{\hat{Z}}}_{input,k}$ of the three-phase can be obtained from relation (4.42) for the particular cases $\underline{\hat{\hat{Z}}}_{S} \to \infty$ and $\underline{\hat{\hat{Z}}}_{S} \to \mathbf{0}$. The expressions of these impedances are:

$$\underline{\hat{\hat{Z}}}_{input,0} = (ch\underline{\hat{\gamma}}l)\left[(sh\underline{\hat{\gamma}}l)\underline{\hat{\hat{Y}}}_{c}\right]^{-1} \tag{4.43}$$

$$\underline{\hat{\hat{Z}}}_{input,k} = \left[(sh\underline{\hat{\gamma}}l)\underline{\hat{\hat{Z}}}_{c}\right](ch\underline{\hat{\gamma}}l)^{-1} \tag{4.44}$$

Taking into account expressions (4.43) and (4.44), the following expressions are obtained:

$$\underline{\hat{\hat{Z}}}_{input,0}\underline{\hat{\hat{Z}}}_{input,k} = (ch\underline{\hat{\gamma}}l)\underline{\hat{\hat{Z}}}_{c}^{2}(ch\underline{\hat{\gamma}}l)^{-1} \tag{4.45}$$

$$\underline{\hat{\hat{Z}}}_{input,k}\underline{\hat{\hat{Z}}}_{input,0} = (sh\underline{\hat{\gamma}}l)\underline{\hat{\hat{Z}}}_{c}^{2}(sh\underline{\hat{\gamma}}l)^{-1} \tag{4.46}$$

From expressions (4.45) and (4.46), it can be noticed that in general operators $\hat{\hat{\underline{Z}}}_{input,0}$ and $\hat{\hat{\underline{Z}}}_{input,k}$ are not switchable. The necessary and sufficient condition to be switchable is that $\hat{\hat{\underline{Z}}}_{c}^{2}$ to be switchable with both $(ch\hat{\underline{\gamma}}l)$ and $(sh\hat{\underline{\gamma}}l)$.

In this situation there is obtained:

$$\hat{\hat{\underline{Z}}}_{input,0}\hat{\hat{\underline{Z}}}_{input,k} = \hat{\hat{\underline{Z}}}_{input,k}\hat{\hat{\underline{Z}}}_{input,0} = \hat{\hat{\underline{Z}}}_{c}^{2} \tag{4.47}$$

Relation (4.47) is known in the theory of the electrical quadripole.
Returning to relation (4.43) and (4.44) we compute the expressions:

$$\hat{\hat{\underline{Z}}}_{input,k}\hat{\hat{\underline{Z}}}_{input,0}^{-1} \text{ and } \hat{\hat{\underline{Z}}}_{input,0}^{-1}\hat{\hat{\underline{Z}}}_{input,k}$$

For the first relation we obtain:

$$\hat{\hat{\underline{Z}}}_{input,k}\hat{\hat{\underline{Z}}}_{input,0}^{-1} = (sh\hat{\underline{\gamma}}l)\hat{\hat{\underline{Z}}}_{c}(ch\hat{\underline{\gamma}}l)^{-1}(sh\hat{\underline{\gamma}}l)\hat{\hat{\underline{Y}}}_{c}(ch\hat{\underline{\gamma}}l)^{-1} \tag{4.48}$$

And for the second:

$$\hat{\hat{\underline{Z}}}_{input,0}^{-1}\hat{\hat{\underline{Z}}}_{input,k} = (sh\hat{\underline{\gamma}}l)\hat{\hat{\underline{Y}}}_{c}(ch\hat{\underline{\gamma}}l)^{-1}(sh\hat{\underline{\gamma}}l)\hat{\hat{\underline{Z}}}_{c}(ch\hat{\underline{\gamma}}l)^{-1} \tag{4.49}$$

It can be noticed that in this case as well, relations (4.48) and (4.49) are equal only when $\hat{\hat{\underline{Z}}}_{c}$ and $\hat{\hat{\underline{Y}}}_{c}$ switch the hyperbolic functions $ch\hat{\underline{\gamma}}l$ and $sh\hat{\underline{\gamma}}l$. There is obtained:

$$\hat{\hat{\underline{Z}}}_{input,k}\hat{\hat{\underline{Z}}}_{input,0}^{-1} = \hat{\hat{\underline{Z}}}_{input,0}^{-1}\hat{\hat{\underline{Z}}}_{input,k} = (sh\hat{\underline{\gamma}}l)^{2}\left[(ch\hat{\underline{\gamma}}l)^{-1}\right]^{2} \tag{4.50}$$

5. CONCLUSIONS

1. The three-phase electric multipole is a spatial electric circuit that is associated with a polyhedral shape. The external unit normal vector to the lateral surface is associated with the direction of path of the circuit.
2. The input-output voltages, respectively input-output currents form a column matrix with six elements, divided into two matrixes with three elements.
3. Between the matrix for input-output voltages and the matrix for input-output currents is a linear operational relation.
4. The reciprocity and symmetry conditions of three-phase electric multipole have a similar form to that from quadripole, where the impedance parameters are substituted by the associated impedance operator parameters.
5. The electric consumers: mono-phased, bi-phased, three-phased and the damages can be modulated by transversal impedances Z_{12}, Z_{23}, Z_{31}, Z_{10}, Z_{20}, Z_{30}.

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