Nonuniform Dust, Oppenheimer-Snyder, and a Singular Detour to Nonsingular Physics

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Abstract

Oppenheimer and Snyder treated in "comoving coordinates" a finite-radius ball of self-gravitationally contracting dust whose energy density is initially static; this is incisively dealt with by use of Tolman's rarely-cited closed-form "comoving" metric solutions for all spherically-symmetric nonuniform dust distributions. Unaware of Tolman's general solutions, Oppenheimer and Snyder assumed that the uniform space-filling dust solution applies without modification to the interior of their dust ball, which is validated by Tolman's solutions. We also find that all nonuniform dust solutions which adhere to the Oppenheimer-Snyder initial conditions have a time-cycloid character that strikingly parallels Newtonian particle gravitational infall, and as well renders those solutions periodically singular. The highly intricate, and thus easily misapprehended, singular transformation of the Oppenheimer-Snyder dust-ball solution from "comoving" to "standard" coordinates is re-derived in detail; it reveals the completely nonsingular nature of the dust-ball metric in "standard" coordinates. Thus the periodically-singular quasi-Newtonian character of the "comoving" dust-ball metric is an artifact of the perceptibly unphysical "synthetic" nature of "comoving coordinates", whose definition requires the clocks of an infinite number of observers.

Introduction

When Oppenheimer and Snyder set out to solve the Einstein equation for the metric tensor of a self-gravitationally contracting ball of dust of initial radius a in "comoving coordinates" with initially static dust energy density, they apparently were unaware of Tolman's 1934 schematic closed-form "comoving" solutions for *any* spherically-symmetric *nonuniform* dust distribution [1]. Therefore Oppenheimer and Snyder were only able to deal analytically with *uniform space-filling dust* [2, 3], and made do with that limitation by *assuming* that this solution applies *without modification* to the *interior* region of their dust ball of radius a; the Tolman solutions in fact *validate* that assumption.

We now work out the detailed behavior of the Tolman solutions that correspond to given initial spherically-symmetric nonuniform dust energy densities plus the specific initial conditions for the "comoving" metric functions that were imposed by Oppenheimer and Snyder in the case of uniform space-filling dust. (Tolman's own article [1] presents solutions in terms of quadrature, which doesn't transparently reveal their behavior, nor are the presented solutions formally expressed in terms of their initial dust energy densities.)

Solving the comoving-metric Einstein equation for nonuniform dust

The spherically-symmetric "comoving coordinate system" metric is given by the line element [4],

$$ds^2 = (cdt)^2 - U(r,t)dr^2 - V(r,t)((d\theta)^2 + (\sin\theta d\phi)^2), \tag{1}$$

where $U(r,t)$ is dimensionless, but $V(r,t)$ has the dimension of length squared.

Pressure-free dust has the stress-energy tensor $T^{\mu\nu} = \rho U^\mu U^\nu$, where $\rho(r,t)$ is the dust energy density and U^μ is the dust's four-vector velocity field, which in "comoving coordinates" everywhere has the extremely simple constant form $U^\mu = (1,0,0,0)$ [5]. Therefore $T^{\mu\nu}$ has only the single nonzero component $T^{00} = \rho(r,t)$, and of the four components of the stress-energy tensor's *equation of continuity* $(T^\mu{}_\nu)_{;\mu} = 0$, *only the time component isn't an identity*, and that equation can be written in the form [6],

$$\partial(\rho V U^{\frac{1}{2}})/\partial t = 0, \tag{2a}$$

which implies that,

$$\rho(r,t) = \frac{\rho(r,t_0)V(r,t_0)(U(r,t_0))^{\frac{1}{2}}}{V(r,t)(U(r,t))^{\frac{1}{2}}}. \tag{2b}$$

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Of the ten components of the covariant symmetric second-rank tensor Einstein equation for the metric of this dynamical dust system in the "comoving coordinate system", there are only *four* components which are neither identities nor redundant on their face, namely [7],

$$-\tfrac{1}{V}+\tfrac{1}{U}\left(\tfrac{V''}{2V}-\tfrac{U'V'}{4UV}\right)-\tfrac{1}{c^2}\left(\tfrac{\ddot{V}}{2V}+\tfrac{\dot{U}\dot{V}}{4UV}\right)=-\tfrac{4\pi G\rho}{c^4}, \tag{3a}$$

$$\tfrac{1}{U}\left(\tfrac{V''}{V}-\tfrac{U'V'}{2UV}-\tfrac{V'^2}{2V^2}\right)-\tfrac{1}{c^2}\left(\tfrac{\ddot{U}}{2U}-\tfrac{\dot{U}^2}{4U^2}+\tfrac{\dot{U}\dot{V}}{2UV}\right)=-\tfrac{4\pi G\rho}{c^4}, \tag{3b}$$

$$\tfrac{1}{c^2}\left(\tfrac{\ddot{U}}{2U}+\tfrac{\ddot{V}}{V}-\tfrac{\dot{U}^2}{4U^2}-\tfrac{\dot{V}^2}{2V^2}\right)=-\tfrac{4\pi G\rho}{c^4}, \tag{3c}$$

and,

$$\tfrac{\dot{V}'}{V}-\tfrac{V'\dot{V}}{2V^2}-\tfrac{\dot{U}V'}{2UV}=0. \tag{3d}$$

We eliminate V'' by subtracting Eq. (3b) from twice Eq. (3a) to obtain,

$$-\tfrac{2}{V}+\tfrac{1}{U}\left(\tfrac{V'^2}{2V^2}\right)-\tfrac{1}{c^2}\left(\tfrac{\ddot{V}}{V}-\tfrac{\ddot{U}}{2U}+\tfrac{\dot{U}^2}{4U^2}\right)=-\tfrac{4\pi G\rho}{c^4}. \tag{4a}$$

We now eliminate both $\ddot{U}$ and $\dot{U}$ by subtracting Eq. (4a) from Eq. (3c) to obtain,

$$\tfrac{2}{V}-\tfrac{1}{U}\left(\tfrac{V'^2}{2V^2}\right)+\tfrac{1}{c^2}\left(\tfrac{2\ddot{V}}{V}-\tfrac{\dot{V}^2}{2V^2}\right)=0. \tag{4b}$$

We can now solve Eq. (4b) for U in terms of a formal fraction which depends on V and its partial derivatives. Since U is dimensionless, we make *both* the numerator and denominator of this formal fraction dimensionless,

$$U=\tfrac{((V')^2/V)}{4+(1/c^2)[4\ddot{V}-((\dot{V})^2/V)]}. \tag{4c}$$

Turning our attention now to Eq. (3d), we note that after it is multiplied through by $-2(V/V')$ it can be written,

$$(\dot{U}/U)-2(\dot{V}'/V')+(\dot{V}/V)=0, \tag{4d}$$

which is straightforwardly seen to be equivalent to,

$$\partial((UV)/(V')^2)/\partial t=0, \tag{4e}$$

and this in turn implies that,

$$U=C(r)((V')^2/V), \tag{4f}$$

where $C(r)$ *has no dependence on the time variable* t. We can now equate the right-hand side of Eq. (4f) to the right-hand side of Eq. (4c), and thus obtain the following second-order in time differential equation for V *alone*,

$$\ddot{V}-\tfrac{1}{4}((\dot{V})^2/V)=c^2((1/(4C(r)))-1). \tag{4g}$$

We can as well insert Eq. (4f) into Eq. (2b), and thus eliminate $U(r,t)$ from $\rho(r,t)$ to obtain,

$$\rho(r,t)=\tfrac{\rho(r,t_0)(V(r,t_0))^{\frac{1}{2}}V'(r,t_0)}{(V(r,t))^{\frac{1}{2}}V'(r,t)}. \tag{5}$$

We won't try to obtain the general solution of Eq. (4g), but seek its *particular* solution for the initial values of $\dot{V}(r,t)$ and $V(r,t)$ that were imposed by Oppenheimer and Snyder in the case of uniform space-filling dust. Those initial values are [8],

$$\dot{V}(r,t_0)=0, \tag{6a}$$

and,

$$V(r,t_0)=r^2. \tag{6b}$$

Insertion of Eq. (6b) into Eq. (5) yields,

$$\rho(r,t) = \rho(r,t_0)\left[\frac{2r^2}{(V(r,t))^{\frac{1}{2}}V'(r,t)}\right]. \tag{7}$$

The time derivative of Eq. (7), taken together with Eq. (6a), implies that $\dot{\rho}(r,t_0) = 0$. We thus see that the Oppenheimer-Snyder initial conditions imply that the dust energy density *is initially stationary*.

It is convenient at this point to eliminate $C(r)$ from Eqs. (4f) and (4g) in favor of $(1-(1/(4C(r)))$, which we shall denote as $K(r)$. Therefore Eq. (4f) now reads,

$$U = (((V')^2/V)/(4(1-K(r)))), \tag{8a}$$

which together with the initial condition $V(r,t_0) = r^2$ of Eq. (6b) implies that,

$$U(r,t_0) = (1-K(r))^{-1}, \tag{8b}$$

while the second-order in time differential equation for V that is given by Eq. (4g) now reads,

$$\ddot{V} - \tfrac{1}{4}((\dot{V})^2/V) = -c^2K(r). \tag{8c}$$

In order to *evaluate* $K(r)$ we need *further information that is contained in* Eqs. (3a) through (3d) *which is not already implicit in* Eqs. (8c) and (8a). Because $K(r)$ *is independent of the time variable* t, the needed information ought to follow from enforcing *any one* of the *three* Eqs. (3a), (3b) or (3c) *at the initial time* t_0 (note that Eq. (3d) *doesn't qualify because it is already implicit in* Eq. (8a)). Of the left-hand sides of Eqs. (3a) through (3c), that of Eq. (3a) *is the easiest to evaluate at the initial time* t_0 because *no knowledge of* $\dot{U}(r,t_0)$ *or* $\ddot{U}(r,t_0)$ *is needed* (the single occurrence of $\dot{U}(r,t_0)$ on the left-hand side of Eq. (3a) at time t_0 is multiplied by the factor $\dot{V}(r,t_0)$, which is equal to zero in accord with Eq. (6a)).

The further information *that is actually needed* for the evaluation of the left-hand side of Eq. (3a) at time t_0 consists of $V(r,t_0) = r^2$, in accord with Eq. (6b), from which $V'(r,t_0) = 2r$ and $V''(r,t_0) = 2$ as well follow; $U(r,t_0) = (1/(1-K(r)))$, in accord with Eq. (8b), from which $(U'(r,t_0)/U(r,t_0)) = (K'(r)/(1-K(r)))$ as well follows; and $\ddot{V}(r,t_0) = -c^2K(r)$, in accord with Eqs. (8c), (6a) and (6b). Therefore at time t_0 Eq. (3a) implies that,

$$-((rK'(r)+K(r))/(2r^2)) = -((4\pi G\rho(r,t_0))/c^4),$$

or,

$$(rK(r))' = ((8\pi Gr^2\rho(r,t_0))/c^4),$$

which determines $K(r)$ only up to an integration constant. But *the particular choice* of the integration constant that yields,

$$K(r) = ((8\pi G)/(c^4r))\int_0^r dr'(r')^2\rho(r',t_0), \tag{8d}$$

is the one which *accords* with the Birkhoff theorem result that a spherically-symmetric metric inside an empty-space central region is that of flat space. That result follows from Eq. (8d) because if $\rho(r,t_0) = 0$ for $0 \le r \le r_0$, where $r_0 > 0$ then *it is as well true* that $K(r) = 0$ for $0 \le r \le r_0$, and *for those values of* r it is clear from Eq. (8c) and its initial conditions which are given by Eqs. (6b) and (6a) that we have the simple solution $V(r,t) = r^2$, which together with $K(r) = 0$ and Eq. (8a) *also* implies that $U(r,t) = 1$. These results for $V(r,t)$ and $U(r,t)$ show that the "comoving coordinate system" metric given by Eq.(1) is that of flat space for $0 \le r \le r_0$ when $\rho(r,t_0)$ is equal to zero for those values of r, which is in accord with the Birkhoff theorem.

A physically suggestive way to present the $K(r)$ of Eq. (8d) is,

$$K(r) = ((2GM(r))/(c^2r)), \tag{8e}$$

where,

$$M(r) \stackrel{\text{def}}{=} (4\pi/c^2)\int_0^r dr'(r')^2\rho(r',t_0), \tag{8f}$$

is *the cumulative effective mass from the origin to* r of the spherically-symmetric initial energy distribution $\rho(r, t_0)$.

Having obtained the definitive Eq. (8d) general result for $K(r)$, we now turn to the final issue of solving Eq. (8c) when its initial conditions for $V(r, t)$ are given by Eqs. (6b) and (6a). The nonlinear first-derivative term $-\frac{1}{4}((\dot{V})^2/V)$ of Eq. (8c) is readily transformed away by changing the dependent variable from V to $W = V^{\frac{3}{4}}$, i.e., $V = W^{\frac{4}{3}}$, with the result,

$$\ddot{W} = -\tfrac{3}{4}c^2(K(r)/W^{\frac{1}{3}}),$$

whose initial conditions are given by $W(r, t_0) = r^{\frac{3}{2}}$ and $\dot{W}(r, t_0) = 0$. This second-order in time equation for W presents a distinctly Newtonian dynamical impression of purely radial motion in the presence of a (peculiar) central force. We therefore treat it accordingly, multiplying it through by the quintessential Newtonian dynamical integrating factor $2\dot{W}$ in order to carry out its dynamical first integration. Taking into account this equation's initial conditions, the result of its dynamical first integration is,

$$(\dot{W})^2 = -\left(\tfrac{3}{2}\right)^2 c^2 K(r)(W^{\frac{2}{3}} - r),$$

an equation form which virtually *begs* for its dependent variable to be changed to $\varrho = W^{\frac{2}{3}} = V^{\frac{1}{2}}$, i.e., $W = \varrho^{\frac{3}{2}}$. Since $\dot{W} = \frac{3}{2}\varrho^{\frac{1}{2}}\dot{\varrho}$, in terms of ϱ the above equation's form changes to,

$$(\dot{\varrho})^2 = -c^2 K(r)(1 - (r/\varrho)),$$

which is very close to the *time cycloid* equation form of Oppenheimer and Snyder [9]. It is, in fact, a minor matter to change the above equation to precisely that time cycloid form by the scaling change of the dependent variable to the dimensionless $R = (\varrho/r) = (V^{\frac{1}{2}}/r)$, i.e., $\varrho = rR$,

$$(\dot{R})^2 = (c^2/r^2)K(r)((1/R) - 1) = ((2GM(r))/r^3)((1/R) - 1), \tag{9a}$$

where the second equality follows from Eq. (8e). The initial condition,

$$R(r, t_0) = 1, \tag{9b}$$

is, of course, one of the initial conditions at $t = t_0$ of $((V(r, t))^{\frac{1}{2}}/r)$. The further "initial condition" $\dot{R}(r, t_0) = 0$ that corresponds to $\dot{V}(r, t_0) = 0$ follows from Eq. (9a) itself in conjunction with its Eq. (9b) initial condition.

A slightly simpler and more suggestive way to present Eq. (9a) is,

$$(\dot{R})^2 = (\omega(r))^2((1/R) - 1), \tag{9c}$$

where,

$$\omega(r) = (c/r)(K(r))^{\frac{1}{2}} = ((2GM(r))/r^3)^{\frac{1}{2}}. \tag{9d}$$

The time cycloid $R(r, t)$ of Eq. (9c) can't be expressed in terms of elementary functions, but *its inverse for its first half-period* (during which it falls from unity to zero strictly monotonically) *does* have an expression in terms of elementary functions,

$$\arccos((R(r,t))^{\frac{1}{2}}) + (R(r,t)(1 - R(r,t)))^{\frac{1}{2}} = \omega(r)(t - t_0) \text{ when } 0 \le (t - t_0) \le (\pi/(2\omega(r))). \tag{10a}$$

Differentiating both sides of the equality in Eq. (10a) with respect to time produces the the *specialized* time cycloid relation,

$$\dot{R}(r, t) = -\omega(r)((1/R(r, t)) - 1)^{\frac{1}{2}}, \tag{10b}$$

which is *consistent* with Eq. (9c) but *is itself only valid for the first half-period, namely*, $0 \le (t - t_0) < (\pi/(2\omega(r)))$. More generally, however, Eq. (9c) *itself* is obviously *consistent with*,

$$\dot{R}(r, t) = \pm[-\omega(r)((1/R(r, t)) - 1)^{\frac{1}{2}}], \tag{10c}$$

where *in fact* $\pm$ is the *plus* sign during the *odd* half-periods of the time cycloid $R(r,t)$ and is the *minus* sign during that time cycloid's *even* half-periods—*the time duration of a half-period of* $R(r,t)$ *is of course* $(\pi/(2\omega(r)))$. This *particular* choice of the $\pm$ sign ensures that $R(t,r)$ is not only *periodic in time* with time period $(\pi/\omega(r))$, but is also *continuous in time*, notwithstanding that its time derivative $\dot{R}(r,t)$ *diverges* at the *odd* half-period time points. Dividing both sides of Eq. (10c) by $\pm((1/R(r,t))-1)^{\frac{1}{2}}$ and then *integrating both sides of the result with respect to time* produces the *extension* of Eq. (10a) to,

$$\pm[\arccos((R(r,t))^{\frac{1}{2}}) + (R(r,t)(1-R(r,t)))^{\frac{1}{2}}] = \omega(r)(t-t_0), \tag{10d}$$

where $\pm$ is the *plus* sign during the *odd* half-periods of the time cycloid $R(r,t)$ and is the *minus* sign during that time cycloid's *even* half-periods—the time duration of a half-period of $R(r,t)$ is $(\pi/(2\omega(r)))$.

Another useful relationship follows from differentiating both sides of Eq. (10d) with respect to r, namely,

$$rR'(r,t) = \pm[-r\omega'(r)(t-t_0)((1/R(r,t))-1)^{\frac{1}{2}}] = ((r\omega'(r))/\omega(r))(t-t_0)\dot{R}(r,t), \tag{10e}$$

where the second equality follows from Eq. (10c). The presence of the factor of $(t-t_0)$ on the right-hand side of these two equalities shows that $rR'(r,t)$ *departs* from the *periodicity in time* (with full-period time duration $(\pi/\omega(r))$) which is manifested by the time cycloid $R(r,t)$ and its time derivative $\dot{R}(r,t)$.

This fact *isn't* relevant to the *particular* "comoving coordinate system" metric function $V(r,t)$ because $R(r,t) = ((V(r,t))^{\frac{1}{2}}/r)$ implies that,

$$V(r,t) = r^2(R(r,t))^2, \tag{11a}$$

which is clearly periodic in time if $R(r,t)$ is periodic in time.

However, because Eq. (11a) implies that,

$$V'(r,t) = 2rR(r,t)(R(r,t) + rR'(r,t)),$$

the departure from periodicity in time of $rR'(r,t)$ *is* relevant to the "comoving coordinate system" metric function $U(r,t)$, since it involves $(V')^2$ according to Eq. (8a) above, namely,

$$U = (((V')^2/V)/(4(1-K(r)))) = ((R+rR')^2/(1-K(r))), \tag{11b}$$

which shows that the departure from periodicity in time of rR' does infect the particular "comoving coordinate system" metric function U.

The density $\rho(r,t)$ as well departs from periodicity in time since it involves V' according to Eq. (7) above, namely,

$$\rho(r,t) = ((2r^2\rho(r,t_0))/(V^{\frac{1}{2}}V')) = (\rho(r,t_0)/(R^2(R+rR'))), \tag{11c}$$

which shows that the departure from periodicity in time of rR' infects $\rho(r,t)$.

With the *exception* of these special metric and density departures from periodicity in time, which bring to mind the perihelion *precession* of basically *periodic* planetary orbits, the "comoving coordinate system" metric tends to strongly reflect *merely Newtonian* gravitational physics.

For example, if we go to the limit that the initial effective mass density $(\rho(r,t_0)/c^2)$ describes a point mass at $r=0$, so that the cumulative mass $M(r)$ of Eq. (8f) is equal to some constant mass $M>0$ for all $r>0$, we see from Eq. (9a) that the basic "comoving coordinate system" metric time-cycloid equation becomes,

$$(\dot{R})^2 = ((2GM)/r^3)((1/R)-1), \tag{12}$$

for all $r>0$, with the initial condition $R(r,t_0)=1$.

Now compare the behavior of the metric-related dimensionsless entity R of Eq. (12) with the behavior of a Newtonian test particle which falls *from initial rest* toward a point mass $M>0$ located at the origin, from which that test particle *is initially separated by the radial distance* r_0. The Newtonian equation for that test particle's ensuing purely radial motion is,

$$\ddot{r} = -((GM)/r^2),$$

with the initial conditions $r(t_0) = r_0$ and $\dot{r}(t_0) = 0$. We multiply this equation of motion through by the usual Newtonian integrating factor $2\dot{r}$ to carry out its dynamical first integration, whose constant of integration of course accords with the two initial conditions given in the preceding sentence. The result is,

$$(\dot{r})^2 = (2GM)((1/r) - (1/r_0)),$$

which when differentiated with respect to time produces the original Newtonian equation of motion $\ddot{r} = -((GM)/r^2)$ noted above, and which at time t_0 as well accords with the two initial conditions. If we now make the simple scaling change to the new dimensionless variable $R = (r/r_0)$ whose initial condition is $R(t_0) = 1$, so that $r = r_0 R$, our above dynamical first integration result becomes,

$$(\dot{R})^2 = ((2GM)/(r_0)^3)((1/R) - 1),$$

which is identical in form to Eq. (12), and whose initial condition for R at time t_0 *is the same as well*.

Thus the "comoving coordinate system" evidences *no hint at all* of departures from Newtonian gravitational physics *that are due to gravitational time dilation*; it is *only* with respect to the existence of departures from *periodicity* (phenomena with a flavor reminiscent of planetary perihelion precession) that "comoving coordinate system" gravitational physics reveals clearly discernible *differences* from Newtonian gravitational physics.

Furthermore, the underlying *radius-dependent Newtonian time cycloids* that are *intrinsic* to the "comoving coordinate system" metric *manifestly cause that metric to periodically violate the Metric Signature Theorem* [10] *which follows directly from the Principle of Equivalence*. This occurs at the *odd* cycloid half-periods $t = t_0 + (2n+1)(\pi/(2\omega(r)))$, for $n = 0, 1, 2, \ldots$, where the cycloid has its periodic cusps and $R(r,t) = 0$, which implies that the diagonal-component metric function (and metric eigenvalue) $V(r,t) = 0$ (see Eqs. (11a) and (1)).

It is worth digressing a moment at this point to note that *the basic attributes* of "comoving" metrics predispose them *to unphysical behavior*. The fact that "comoving" metrics have g_{00} *fixed to unity* [11] is at loggerheads *both* with the weak-field static limit of g_{00} being $1 + 2\phi$, where ϕ is the Newtonian gravitational potential [12], *and* with $(g_{00})^{-\frac{1}{2}}$ being the static-limit gravitational time-dilation factor [13]. Furthermore, the fact that "comoving coordinates" are defined in terms *of the clocks of an infinite number of observers* [14] precludes their being, in the orthodox Einstein sense, *physical coordinate systems at all*.

The occurrence of time-periodic cycloid-cusp metric singularities that violate the Principle of Equivalence makes it intuitively apparent *that only singular space-time transformations* from the "comoving coordinate system" onto more "standard" coordinate systems (i.e., coordinate systems *capable* of accommodating gravitational time dilation because their g_{00} metric component *isn't* fixed to unity) of initially static, spherically-symmetric dust solutions *could possibly be compatible with these solutions' honoring the Principle of Equivalence in the latter coordinate systems*.

By way of buttressing the above-expressed intuition, we note that the fact that the metric eigenvalue $V(r,t)$ equals zero at the periodic cycloid cusps makes each member of our class of "comoving" metrics *non-invertible on a nonempty set of space-time points*. We shall now establish that a *nonsingular* space-time transformation $\bar{x}^\gamma(x^\sigma)$ of such a metric $g_{\kappa\lambda}(x^\sigma)$ *can't be invertible everywhere either*. The *resulting nonsingularly transformed metric* $g_{\alpha\beta}(\bar{x}^\gamma)$ is given by,

$$g_{\alpha\beta}(\bar{x}^\gamma) = (\partial x^\kappa/\partial\bar{x}^\alpha)(\partial x^\lambda/\partial\bar{x}^\beta)g_{\kappa\lambda}(x^\sigma),$$

where, because the transformation $\bar{x}^\gamma(x^\sigma)$ is *nonsingular*, its Jacobian matrix $\partial\bar{x}^\gamma/\partial x^\sigma$ and the inverse $\partial x^\tau/\partial\bar{x}^\delta$ thereof are *both* everywhere comprised *solely* of well-defined finite real components. From this it follows that we can *also* write our *original* metric $g_{\mu\nu}(x^\sigma)$ as a *nonsingular transformation* of the transformed metric $g_{\alpha\beta}(\bar{x}^\gamma)$, namely,

$$g_{\mu\nu}(x^\sigma) = (\partial\bar{x}^\alpha/\partial x^\mu)(\partial\bar{x}^\beta/\partial x^\nu)g_{\alpha\beta}(\bar{x}^\gamma).$$

From this relationship we can see that *if* the nonsingularly transformed metric $g_{\alpha\beta}(\bar{x}^\gamma)$ *really were* invertible everywhere, then given the invertibility everywhere of the Jacobian matrix $\partial\bar{x}^\gamma/\partial x^\sigma$ of the transformation $\bar{x}^\gamma(x^\sigma)$, our *original* metric $g_{\mu\nu}(x^\sigma)$ clearly would be invertible everywhere *as well*, which, of course, *it* *definitely is not*. We have thus showed that *any nonsingular transformation* of one of our class of metrics that isn't invertible everywhere *must itself not be invertible everywhere*. That *confirms* our intuition that *only singular space-time transformations* of our class of "comoving" metrics that aren't invertible everywhere *could possibly be invertible everywhere and honor the Metric Signature Theorem* [10] *and the Principle of Equivalence*.

As a speculative example of a such a usefully *singular* space-time transformation of one of our particular class of "comoving" metrics, we might suppose that *the gravitational time dilation* which the singularly transformed coordinate system manifests could be such that *all* times greater than or equal to the shortest of the time-cycloid half-periods of the "comoving" system *are singularly transformed to infinite time in the transformed system*; such a time-dilation *singularity* of the transformation *erases* in the transformed system all the time-periodic metric singularities of the "comoving" system which violate the Metric Signature Theorem and the Principle of Equivalence.

In the next section we work out in detail the singular Oppenheimer-Snyder transformation between the "comoving coordinate system" and the "standard" coordinate system" in the interior region of Oppenheimer and Snyder's dust ball—because of the Birkhoff theorem, the dust ball's *exterior-region* "comoving coordinate" metric is assumed to be transformed into a "standard" Schwarzschild metric. The transformation does indeed *singularly transform to infinite time* all the "comoving coordinate" times greater than or equal to the applicable "comoving" time-cycloid half-period (in its interior region the dust ball has *just one* "comoving" time-cycloid half-period).

The singular Oppenheimer-Snyder mapping worked out from scratch

Now that we have in hand from Eqs. (11a) and (11b) the full analytic details of the "comoving coordinate system" metric of Eq. (1) for spherically-symmetric radially-nonuniform dust that is initially started from rest, we need to transform this metric to one that *unlike* the unphysical "comoving coordinate system" metric *can exhibit gravitational time dilation because its g_{00} component is not fixed to unity*. Because of their intention to take technical advantage of the Birkhoff theorem, Oppenheimer and Snyder chose to map the spherically-symmetric "comoving coordinates" (t, r, θ, ϕ), in terms of which the invariant line element ds^2 is given by Eq. (1), into spherically-symmetric "standard coordinates" [15] $(\bar{t}, \bar{r}, \bar{\theta}, \bar{\phi})$, in terms of which the *same* invariant line element ds^2 is *as well* given by,

$$\begin{aligned} ds^2 &= B(\bar{r}, \bar{t})(cd\bar{t})^2 - A(\bar{r}, \bar{t})(d\bar{r})^2 - \bar{r}^2((d\bar{\theta})^2 + (\sin\bar{\theta}d\bar{\phi})^2) \\ &= (cdt)^2 - U(r,t)(dr)^2 - V(r,t)((d\theta)^2 + (\sin\theta d\phi)^2). \end{aligned} \tag{13a}$$

Inspection in Eq. (13a) *of the rightmost two terms* of the line element ds^2 in both its "standard" and its "comoving" form immediately reveals three very convenient mapping choices,

$$\bar{\theta} = \theta,\ \bar{\phi} = \phi \text{ and } \bar{r} = (V(r,t))^{\frac{1}{2}} = rR(r,t), \tag{13b}$$

where we have used the Eq. (11a) relation $V(r,t) = r^2(R(r,t))^2$. Next we obviously would like to obtain $\bar{t}$ as a function of r and t, just as has been done in Eq. (13b) for $\bar{r}$. Inspection of Eq. (13a), however, reveals that that task is completely entwined with the determination of B and A as functions of r and t; moreover *$\bar{t}$ itself doesn't occur in relations that can be extracted from* Eq. (13a), *only its partial derivatives* $(\partial\bar{t}/\partial t)$ *and* $(c(\partial\bar{t}/\partial r))$ *do*. We are thus faced with solving *both* simultaneous algebraic *and* first-order partial differential equations merely to obtain $\bar{t}(r,t)$! The path ahead thus appears long and arduous, *enough so that the ensuing development is almost never presented in complete detail*, which is all the *more* reason to do that *here*, particularly because the baffling intricacy of Oppenheimer and Snyder's singular result for $\bar{t}(r,t)$ is outright incomprehensible if the journey *isn't* painstakingly pursued to its conclusion.

Presenting now in greater detail that part of Eq. (13a) *which isn't eliminated* by the three mapping choices of Eq. (13b),

$$B[(\partial\bar{t}/\partial t)(cdt) + c(\partial\bar{t}/\partial r)dr]^2 - A[(1/c)(\partial\bar{r}/\partial t)(cdt) + (\partial\bar{r}/\partial r)dr]^2 = (cdt)^2 - U(r,t)(dr)^2. \tag{13c}$$

Since the three bilinear differential forms $(cdt)^2$, $(2c\,dt\,dr)$ and $(dr)^2$ are linearly independent, Eq. (13c) produces *the three simultaneous equations*,

$$B(\partial\bar{t}/\partial t)^2 - A((1/c)(\partial\bar{r}/\partial t))^2 = 1, \tag{14a}$$

$$B(\partial\bar{t}/\partial t)(c(\partial\bar{t}/\partial r)) - A((1/c)(\partial\bar{r}/\partial t))(\partial\bar{r}/\partial r) = 0, \tag{14b}$$

$$B(c(\partial\bar{t}/\partial r))^2 - A(\partial\bar{r}/\partial r)^2 = -U. \tag{14c}$$

We now eliminate A and B from Eqs. (14) in order to obtain the partial differential equation for $\bar{t}$. Solving Eq. (14b) for A yields,

$$A = \frac{B(\partial\bar{t}/\partial t)(c(\partial\bar{t}/\partial r))}{((1/c)(\partial\bar{r}/\partial t))(\partial\bar{r}/\partial r)}. \tag{15a}$$

We now insert this value of A into each one of Eqs. (14a) and (14c) and follow that by solving each one for $(1/B)$,

$$(1/B) = (\partial\bar{t}/\partial t)^2 - \left[\frac{((1/c)(\partial\bar{r}/\partial t))}{(\partial\bar{r}/\partial r)}\right](\partial\bar{t}/\partial t)(c(\partial\bar{t}/\partial r)), \tag{15b}$$

$$(1/B) = \left[\frac{(\partial\bar{r}/\partial r)}{U((1/c)(\partial\bar{r}/\partial t))}\right](\partial\bar{t}/\partial t)(c(\partial\bar{t}/\partial r)) - \left[\frac{1}{U}\right](c(\partial\bar{t}/\partial r))^2. \tag{15c}$$

Subtracting Eq. (15c) from Eq. (15b) reveals the vanishing of a homogeneous bilinear form in the two unknown partial derivatives $(\partial\bar{t}/\partial t)$ and $(c(\partial\bar{t}/\partial r))$,

$$(\partial\bar{t}/\partial t)^2 - \left(\left[\frac{((1/c)(\partial\bar{r}/\partial t))}{(\partial\bar{r}/\partial r)}\right] + \left[\frac{(\partial\bar{r}/\partial r)}{U((1/c)(\partial\bar{r}/\partial t))}\right]\right)(\partial\bar{t}/\partial t)(c(\partial\bar{t}/\partial r)) + \left[\frac{1}{U}\right](c(\partial\bar{t}/\partial r))^2 = 0. \tag{15d}$$

The homogeneous bilinear form in $(\partial\bar{t}/\partial t)$ and $(c(\partial\bar{t}/\partial r))$ on the left-hand side of Eq. (15d) *factors* into the product of two homogeneous *linear* forms in $(\partial\bar{t}/\partial t)$ and $(c(\partial\bar{t}/\partial r))$, namely,

$$\left((\partial\bar{t}/\partial t) - \left[\frac{((1/c)(\partial\bar{r}/\partial t))}{(\partial\bar{r}/\partial r)}\right](c(\partial\bar{t}/\partial r))\right)\left((\partial\bar{t}/\partial t) - \left[\frac{(\partial\bar{r}/\partial r)}{U((1/c)(\partial\bar{r}/\partial t))}\right](c(\partial\bar{t}/\partial r))\right) = 0. \tag{15e}$$

If the linear form in $(\partial\bar{t}/\partial t)$ and $(c(\partial\bar{t}/\partial r))$ in the *first* factor on the left-hand side of Eq. (15e) vanished, then both Eq. (15b) and Eq. (15c) would yield that $(1/B)$ vanishes. Since we seek a non-pathological result for the unknown "standard" metric function B, we must assume that the *second* factor on the left-hand side of Eq. (15e) vanishes, which yields the following homogeneous linear first-order partial differential equation for the unknown time part $\bar{t}(r,t)$ of the mapping from spherically-symmetric "comoving" space-time coordinates to spherically-symmetric "standard" space-time coordinates,

$$((1/c)(\partial\bar{r}/\partial t))(\partial\bar{t}/\partial t) = (1/U)(\partial\bar{r}/\partial r)(c(\partial\bar{t}/\partial r)). \tag{15f}$$

Since $\bar{r}(r,t) = rR(r,t)$ from Eq. (13b) and $U(r,t) = (R(r,t) + rR'(r,t)))^2/(1 - K(r))$ from Eq. (11b), the Eq. (15f) partial differential equation for $\bar{t}(r,t)$ is in detail,

$$(r/c)\dot{R}(r,t)(\partial\bar{t}/\partial t) = \left[\frac{R(r,t)(1-K(r))}{(R(r,t)+rR'(r,t))^2}\right](c(\partial\bar{t}/\partial r)). \tag{15g}$$

The only well-known way to obtain an *analytic* solution to a homogeneous linear partial differential equation such as Eq. (15g) is by separation of variables. But insofar as $R(r,t)$ depends on r as well as on t, it is apparent that Eq. (15g) doesn't lend itself to separation in the variables t and r. However, if the cumulative effective mass $M(r)$ defined by Eq. (8f) is proportional to r^3 for a range of values of r, then we see from Eqs. (9a) and (9b) that $R(r,t)$ will be independent of r over that range of values of r, and therefore Eq. (15g) *will* be analytically solvable by separation of variables for that range of values of r. That will be the case for a range of values of r of the form $0 < r \le a$ whenever $\rho(r,t_0)$ *is constant as a function of* r *over that range of values of* r. If *in addition* $\rho(r,t_0)$ *vanishes altogether for* $r > a$, *the Birkhoff theorem will apply to that empty-space region*, which provides a constraint on the values of the unknown "standard" metric functions A and B of Eq. (13a) *at the boundary radius* $r = a$ *of that empty-space region.*

This is, of course, precisely the dust ball of radius a which Oppenheimer and and Snyder treated; we see that we cannot really depart from their dust configuration if we want an analytic result for the mapping from "comoving" to "standard" coordinates.

It is convenient to characterize this dust configuration at the initial time t_0 by $M(a)$, which we denote as M. Then $\rho(r,t_0)$ has the value $((3Mc^2)/(4\pi a^3))$ for $0 \le r \le a$ and the value zero for $r > a$. From Eqs. (8f) and (9d) we further see that,

$$\omega = ((2GM)/a^3)^{\frac{1}{2}} \text{ for } 0 \le r \le a, \tag{16a}$$

and,

$$K(r) = (\omega/c)^2 r^2 \text{ for } 0 \le r \le a. \tag{16b}$$

Taking note of Eqs. (16a), (16b), (9a) and (9b) we also see that for $0 \le r \le a$, Eq. (15g) becomes,

$$(r/c)\dot{R}(t)(\partial\bar{t}/\partial t) = \left[\frac{(1-(\omega/c)^2 r^2)}{R(t)}\right](c(\partial\bar{t}/\partial r)), \tag{16c}$$

where,

$$(\dot{R}(t))^2 = \omega^2((1/R(t)) - 1). \tag{16d}$$

Making the variable-separation Ansatz $\bar{t}(r,t) = (1/\omega)\alpha(r)\beta(t)$ in Eq. (16c) then yields,

$$R(t)\dot{R}(t)(d(\ln(\beta(t)))/dt) = -p\omega^2 = (1 - (\omega/c)^2 r^2)(c^2/r)(d(\ln(\alpha(r)))/dr), \tag{16e}$$

where p is an arbitrary dimensionless constant. The separated equation for $\alpha(r)$ is straightforward to solve, and yields,

$$\alpha(r) = C_1(1 - (\omega/c)^2 r^2)^{p/2}, \tag{16f}$$

where C_1 is an arbitrary dimensionless constant, but to solve the separated equation for $\beta(t)$ it must be borne in mind that from Eq. (16d), $\dot{R} = \mp\omega((1-R)/R)^{\frac{1}{2}}$ and likewise, $dt = dR/\dot{R} = \mp(1/\omega)(R/(1-R))^{\frac{1}{2}}dR$. Using these relations in Eq. (16e), one readily obtains,

$$\beta(t) = C_2(1 - R(t))^p, \tag{16g}$$

where C_2 is an arbitrary dimensionless constant. From Eqs. (16f), (16g) and the variable-separation Ansatz above, we obtain,

$$\bar{t}(r,t) = (1/\omega)C[(1 - (\omega/c)^2 r^2)^{\frac{1}{2}}(1 - R(t))]^p, \tag{16h}$$

where C and p are arbitrary dimensionless constants. Since the Eq. (16c) partial differential equation is homogeneous and linear, *any linear combination of its solutions of the form of the solution given in* Eq. (16h) *is also a solution*. That fact leads us to expect that given *any arbitrary sufficiently smooth dimensionless function* $\phi(u)$ *of a single dimensionless variable* u, the form,

$$\bar{t}(r,t) = (1/\omega)\phi(u(r,t)), \text{ where } u(r,t) \stackrel{\text{def}}{=} [(1 - (\omega/c)^2 r^2)^{\frac{1}{2}}(1 - R(t))], \tag{16i}$$

will be a solution of the Eq. (16c) partial differential equation. That this expectation is actually true is readily verified by substitution of Eq. (16i) into Eq. (16c).

In the region $0 \le r \le a$ we now have the Eq. (16i) general form of $\bar{t}(r,t)$ in addition to our previous knowledge that $\bar{r}(r,t) = rR(t)$, $U(r,t) = ((R(t))^2/(1-(\omega/c)^2 r^2))$, and $\dot{R}(t) = \mp\omega((1-R(t))/R(t))^{\frac{1}{2}}$. This at long last permits us to use Eqs. (15b) and (15a) to obtain the unknown "standard"-form metric functions B and A. Requiring these "standard" B and A to adhere to the Birkhoff theorem at the empty-space boundary $r = a$ will then pin down $\bar{t}(r,t)$ *as a specific entity*, not merely the general form given by Eq. (16i). At that point *the singular nature of* $\bar{t}(r,t)$, the time part of the space-time mapping from the *unphysical* "comoving" coordinate system to the "standard" coordinate system, *will be manifest*.

First, however, we must have in hand the evaluated partial derivatives that are needed in Eqs. (15b) and (15a) to calculate the unknown "standard" metric functions B and A. By making use of Eq. (16i) for $\bar{t}(r,t)$ and the two relations $\bar{r}(r,t) = rR(t)$ and $\dot{R}(t) = \mp\omega((1-R(t))/R(t))^{\frac{1}{2}}$, we obtain the needed four partial derivatives,

$$(c(\partial\bar{t}/\partial r)) = -(\omega/c)r(1 - (\omega/c)^2 r^2)^{-\frac{1}{2}}(1 - R(t))\phi'(u(r,t)), \tag{17a}$$

$$(\partial\bar{t}/\partial t) = \pm(1 - (\omega/c)^2 r^2)^{\frac{1}{2}}((1 - R(t))/R(t))^{\frac{1}{2}}\phi'(u(r,t)), \tag{17b}$$

$$((1/c)(\partial\bar{r}/\partial t)) = \mp(\omega/c)r((1 - R(t))/R(t))^{\frac{1}{2}}, \tag{17c}$$

$$(\partial\bar{r}/\partial r) = R, \tag{17d}$$

where $u(r,t) = ((1-(\omega/c)^2r^2)^{\frac{1}{2}}(1-R(t)))$. Eqs. (17) together with Eq. (15b) for $(1/B)$ yield,

$$(1/B(r,t)) = ((1-R(t))/(R(t))^2)(R(t)-(\omega/c)^2r^2)(\phi'(u(r,t)))^2,$$

or,

$$B(r,t) = [(R(t))^2/((1-R(t))(R(t)-(\omega/c)^2r^2)(\phi'(u(r,t)))^2)]. \tag{18a}$$

Together with Eqs. (17) and (18a), Eq. (15a) for A then yields,

$$A(r,t) = (R(t)/(R(t)-(\omega/c)^2r^2)). \tag{18b}$$

Since Eq. (18b) for the particular "standard" metric function $A(r,t)$ *happens to have no dependence on the not yet determined dimensionless function* ϕ, we can straightaway *check* whether the behavior of $A(r,t)$ at the $r=a$ empty-space boundary accords with what would be expected from the Birkhoff theorem. First we absorb all dependence that $A(r,t)$ has on the "comoving" *time* t into the "standard" *spatial* radial coordinate $\bar{r} = rR(t)$ by everywhere *replacing* $R(t)$ by $(\bar{r}/r)$. As a result, $A(r,t)$ can be written,

$$A(r,\bar{r}) = (1/(1-(((\omega/c)^2r^3)/\bar{r}))), \tag{19a}$$

and its value at the $r=a$ empty-space boundary is,

$$A(r=a,\bar{r}) = (1/(1-(((\omega/c)^2a^3)/\bar{r}))) = (1/(1-((2GM)/(c^2\bar{r})))), \tag{19b}$$

where in the second equality of Eq. (19b) we have used the fact pointed out in Eq. (16a) that $\omega = ((2GM)/a^3)^{\frac{1}{2}}$ for $0 \le r \le a$. This second equality shows that the metric function A at the $r=a$ empty-space boundary indeed adheres to the form that it is expected to from the Birkhoff theorem.

From the "standard" metric function $B(r,t)$ of Eq. (18a) Oppenheimer and Snyder worked out the so far undetermined dimensionless $\phi'(u(r,t))$, where $u(r,t) = [(1-(\omega/c)^2r^2)^{\frac{1}{2}}(1-R(t))]$, by requiring that at the $r=a$ empty-space boundary $B(r=a,\bar{r})$ *also* adheres to the form that it is expected to from the Birkhoff theorem. With $\phi'(u(r,t))$ thus worked out, Oppenheimer and Snyder obtained $\bar{t}(r,t) = (1/\omega)\phi(u(r,t))$ by *integration* with respect to u of $\phi'(u)$. This time part $\bar{t}(r,t)$ of the mapping from "comoving" to "standard" coordinates turns out *to diverge for* $0 \le r \le a$ *at all sufficiently large "comoving" times* t; indeed its *divergence* occurs *even before* the "comoving" metric's time cycloid function $R(t)$ attains *the singular "comoving" metric value zero* at its *first* half-period time point $t = (t_0 + (\pi/(2\omega)))$. The *divergent* singular character of the $\bar{t}(r,t)$ time part of the *mapping* from "comoving" to "standard" coordinates *thus eliminates all the time-periodic singlarities that are inherent to the quasi-Newtonian "comoving" metric by properly accounting for gravitational redshift, which the unphysical "comoving" metric inherently cannot do*. The divergent singular nature of $\bar{t}(r,t)$ is obviously *a physically essential antidote to the quasi-Newtonian periodic singularities of the unphysical "comoving" metric*.

To work out $\phi'(u(r,t))$, we first eliminate, in analogy with Eqs. (19) above, from the $B(r,t)$ given by Eq. (18a) its dependence on the "comoving" time t by replacing all occurrences of $R(t)$ by $\bar{r}/r$,

$$B(r,\bar{r}) = [(\bar{r}/r)^2/((1-(\bar{r}/r))((\bar{r}/r)-(\omega/c)^2r^2)(\phi'((1-(\omega/c)^2r^2)^{\frac{1}{2}}(1-(\bar{r}/r))))^2)]. \tag{20a}$$

Adherence of $B(r,t)$ at the empty-space boundary $r=a$ to the form which is consistent with the Birkhoff theorem requires that,

$$B(r=a,\bar{r}) = (1-((2GM)/(c^2\bar{r}))) = [((\bar{r}/a)-(\omega/c)^2a^2)/(\bar{r}/a)], \tag{20b}$$

where the second equality follows from the Eq. (16a) result that $\omega = ((2GM)/a^3)^{\frac{1}{2}}$ for $0 \le r \le a$.

We insert Eq. (20a) into Eq. (20b), and solve the resulting equation for $\phi'((1-(\omega/c)^2a^2)^{\frac{1}{2}}(1-(\bar{r}/a)))$. It is convenient for the sake of more compact notation in subsequent steps to express that result as follows,

$$\phi'((1-(\omega/c)^2a^2)^{\frac{1}{2}}(1-(\bar{r}/a))) = F((\bar{r}/a)), \tag{20c}$$

where,
$$F(s) \stackrel{\text{def}}{=} [(s/(s-(\omega/c)^2a^2))(s/(1-s))^{\frac{1}{2}}]. \tag{20d}$$
It is now straightforward to obtain the function $\phi'(u)$ in terms of the function $F(s)$ that is defined by Eq. (20d). Taking u to be the particular argument of ϕ' in Eq. (20c), namely,

$$u = [(1-(\omega/c)^2a^2)^{\frac{1}{2}}(1-(\bar{r}/a))],$$

we readily see that $(\bar{r}/a)$ can be expressed in terms of u as follows,

$$(\bar{r}/a) = [1-((1-(\omega/c)^2a^2)^{-\frac{1}{2}}u)].$$

Insertion of this equality into both sides of Eq. (20c) yields,

$$\phi'(u) = F(1-((1-(\omega/c)^2a^2)^{-\frac{1}{2}}u)), \tag{20e}$$

where the function $F(s)$ is of course defined by Eq. (20d). To obtain $\bar{t}(r,t)$ from Eq. (16i) we require $\phi(u)$, which follows from Eq. (20e) as,

$$\phi(u) = \phi(u=0) + \int_0^u du' F(1-((1-(\omega/c)^2a^2)^{-\frac{1}{2}}u')). \tag{20f}$$

Eq. (20f) will obviously by simplified by changing its variable of integration from u' to,

$$s = [1-((1-(\omega/c)^2a^2)^{-\frac{1}{2}}u')],$$

as a result of which $du' = -(1-(\omega/c)^2a^2)^{\frac{1}{2}}ds$. Eq. (20f) then becomes,

$$\phi(u) = \phi(u=0) + (1-(\omega/c)^2a^2)^{\frac{1}{2}} \int_{\sigma(u)}^1 ds\, F(s), \tag{20g}$$

where,
$$\sigma(u) \stackrel{\text{def}}{=} [1-((1-(\omega/c)^2a^2)^{-\frac{1}{2}}u)]. \tag{20h}$$
We know from Eq. (16i) that $\bar{t}(r,t) = (1/\omega)\phi(u(r,t))$ where,

$$u(r,t) \stackrel{\text{def}}{=} [(1-(\omega/c)^2r^2)^{\frac{1}{2}}(1-R(t))]. \tag{20i}$$

Thus combining Eq. (16i) with Eq. (20g) yields,

$$\bar{t}(r,t) = \bar{t}_0 + (1/\omega)(1-(\omega/c)^2a^2)^{\frac{1}{2}} \int_{\sigma(u(r,t))}^1 ds\, F(s), \tag{20j}$$

where,
$$\bar{t}_0 \stackrel{\text{def}}{=} (1/\omega)\phi(u=0),$$
and because $\sigma(u(r,t_0)) = 1$ follows from $R(t_0) = 1$ and Eqs. (20i) and (20h), we see that,

$$\bar{t}(r,t_0) = \bar{t}_0 \text{ if } 0 \le r \le a. \tag{20k}$$

Explicit insertion into Eq. (20j) of the definition of $F(s)$ given by Eq. (20d) yields,

$$\bar{t}(r,t) = \bar{t}_0 + (1/\omega)(1-(\omega/c)^2a^2)^{\frac{1}{2}} \int_{S(r,t)}^1 ds[(s/(s-(\omega/c)^2a^2))(s/(1-s))^{\frac{1}{2}}], \tag{20l}$$

where,
$$S(r,t) \stackrel{\text{def}}{=} \sigma(u(r,t)) = [1-((1-(\omega/c)^2a^2)^{-\frac{1}{2}}(1-(\omega/c)^2r^2)^{\frac{1}{2}}(1-R(t)))], \tag{20m}$$
with the last equality of Eq. (20m) being the result of explicitly inserting Eq. (20i) into Eq. (20h).

The *crucial* property of the Eq. (20l) result for the time part $\bar{t}(r,t)$ of the mapping from the "comoving" coordinates to the "standard" coordinates *is that it diverges if*,

$$S(r,t) \le ((\omega/c)^2 a^2. \tag{20n}$$

Moreover, since from Eq. (20m),

$$R(t) - S(r,t) = [(1-(\omega/c)^2a^2)^{-\frac{1}{2}}(1-(\omega/c)^2r^2)^{\frac{1}{2}} - 1](1-R(t)) \ge 0 \text{ for } r \le a,$$

it is the case that,

$$R(t) \ge S(r,t). \tag{20o}$$

Therefore the time part $\bar{t}(r,t)$ of the mapping from the "comoving" to the "standard" coordinates definitely diverges at all "comoving" times t for which,

$$R(t) \le ((\omega/c)^2 a^2), \tag{20p}$$

which indeed shows that *none* the unphysical quasi-Newtonian periodic time cycloid singularities of the "comoving" metric exist in the "standard" metric, where gravitational redshift is permitted to act instead of being artificially suppressed. Moreover, in terms of $\bar{r} = rR(t)$, the "standard" coordinate system radial coordinate, Eq. (20p) tells us that $\bar{t}(r,t)$ *diverges* if,

$$\bar{r} \le ((\omega/c)^2a^2r) \le ((\omega/c)^2a^3) = ((2GM)/c^2), \tag{20q}$$

where we have again used the $\omega = ((2GM)/a^3)^{\frac{1}{2}}$ relation of Eq. (16a). Therefore in the "standard" coordinates, *the divergence of* $\bar{t}(r,t)$ *prevents the shrinking system from ever attaining its Schwarzschild radius* $((2GM)/c^2)$ *and forming an event horizon* [16]. In a nutshell, it is the *divergent* singular nature of the time part $\bar{t}(r,t)$ of the mapping from the "comoving" to the "standard" coordinates that *walls off* the *unphysical* features of the "comoving" metric, *preventing them from exerting any physically untoward effects whatsoever on the "standard" metric.*

The integral in the Eq. (20l) expression for $\bar{t}(r,t)$ can (with effort) be evaluated analytically *in the region where it doesn't diverge*, namely when,

$$S(r,t) > ((\omega/c)^2a^2). \tag{21a}$$

The caveat that the analytic result for $\bar{t}(r,t)$ *applies only in the region described by* Eq. (21a) *must be kept strictly in mind* because the analytic result itself *automatically* provides *a completely inapplicable* and potentially extremely misleading *analytic continuation into the region where the underlying integral diverges*. That fact has indeed sometimes caused confusion in the past [17, 18].

We now sketch the main steps of the analytic evaluation of the Eq. (20l) expression for $\bar{t}(r,t)$ *where it doesn't diverge*. To reduce notation bulk, we rewrite Eq. (20l) in the streamlined form,

$$(\omega(\bar{t}_\alpha(S) - \bar{t}_0)) = (1-\alpha)^{\frac{1}{2}} \int_S^1 ds[(s/(s-\alpha)(s/(1-s))^{\frac{1}{2}}], \tag{21b}$$

where,

$$\alpha \stackrel{\text{def}}{=} ((\omega/c)^2a^2), \tag{21c}$$

and,

$$S \stackrel{\text{def}}{=} S(r,t) = [1-((1-(\omega/c)^2a^2)^{-\frac{1}{2}}(1-(\omega/c)^2r^2)^{\frac{1}{2}}(1-R(t)))]. \tag{21d}$$

Note that the $(\omega(\bar{t}_\alpha(S) - \bar{t}_0))$ of Eq. (21b) *diverges whenever* $S \le \alpha$ *and is convergent only for* $S > \alpha$. The first step of its evaluation *in its region of convergence* is to change the variable of integration from s to $v = ((1-s)/s)^{\frac{1}{2}}$, so that $s = (1/(1+v^2))$, $ds = -2dv(v/(1+v^2)^2)dv$ and $sds(s/(1-s))^{\frac{1}{2}} = -2dv(1/(1+v^2)^3)$. The upshot of this change of variable is,

$$(\omega(\bar{t}_\alpha(S) - \bar{t}_0)) = 2(1-\alpha)^{-\frac{1}{2}} \int_0^{((1-S)/S)^{\frac{1}{2}}} dv[(1/(1+v^2)^2)(1/(1-(\alpha/(1-\alpha))v^2))]. \tag{21e}$$

The next step is the three-term partial fraction expansion of the integrand,

$$(\omega(\bar{t}_\alpha(S)-\bar{t}_0)) = 2(1-\alpha)^{\frac{1}{2}}\int_0^{((1-S)/S)^{\frac{1}{2}}} dv\{(1/(1+v^2)^2) + (\alpha/(1+v^2)) + ((\alpha^2/(1-\alpha))/(1-(\alpha/(1-\alpha))v^2))\}. \tag{21f}$$

In Eq. (21f) the third term *itself* requires a *further* elementary two-term partial fraction expansion,

$$(\omega(\bar{t}_\alpha(S)-\bar{t}_0)) = 2(1-\alpha)^{\frac{1}{2}}\int_0^{((1-S)/S)^{\frac{1}{2}}} dv\{(1/(1+v^2)^2) + (\alpha/(1+v^2)) + [(\alpha^2/(2(1-\alpha)))[(1/(1+(\alpha/(1-\alpha))^{\frac{1}{2}}v)) + (1/(1-(\alpha/(1-\alpha))^{\frac{1}{2}}v))]]\}. \tag{21g}$$

The last three terms of the above integrand are elementary to integrate, moreover it is readily verified that the first term yields,

$$2(1-\alpha)^{\frac{1}{2}}\int_0^{((1-S)/S)^{\frac{1}{2}}} dv(1/(1+v^2)^2) = [(1-\alpha)^{\frac{1}{2}}[(S(1-S))^{\frac{1}{2}} + \arctan(((1-S)/S)^{\frac{1}{2}})]]. \tag{21h}$$

The entire result, *which, however, only applies when* $S > \alpha$, is therefore,

$$\bar{t}_\alpha(S) = \bar{t}_0 + \{[(1/\omega)(1-\alpha)^{\frac{1}{2}}[(S(1-S))^{\frac{1}{2}} + ((1+2\alpha)\arctan(((1-S)/S)^{\frac{1}{2}}))]] + [(1/\omega)\alpha^{\frac{3}{2}}[\ln(1+(\alpha/(1-\alpha))^{\frac{1}{2}}((1-S)/S)^{\frac{1}{2}}) - \ln(1-(\alpha/(1-\alpha))^{\frac{1}{2}}((1-S)/S)^{\frac{1}{2}})]]\}. \tag{21i}$$

With regard to the range of applicability of Eq. (21i), namely $S > \alpha$, we *reiterate that* $\bar{t}_\alpha(S)$ *diverges for* $S \le \alpha$.

The ostensible "issue" of logarithms of negative argument [17, 18] *is merely a distraction by the completely inapplicable* (but automatically mathematically feasible) *analytic continuation of the valid convergent result for* $\bar{t}_\alpha(S)$ *when* $S > \alpha$ *into the region* $S < \alpha$ *where the actual integral expression of* Eq. (21b) *for* $(\omega(\bar{t}_\alpha(S)-\bar{t}_0))$ *clearly diverges*.

What is occurring here is "the vanquishing of singularity by singularity". In "comoving coordinates" the dust ball has an unphysical time-cycloid quasi-Newtonian metric which exhibits no trace of gravitational time dilation but manifests periodic singular violation of the Principle of Equivalence. This unphysical "comoving" metric does have a tortured space-time transformation relationship to valid gravitational physics, but as we showed near the end of the previous section that space-time transformation relationship *must necessarily be a singular one in order that it be able to banish the "comoving" metric's unphysical periodic time-cycloid singularities*. That is the *reason* why the Eq. (21b) expression which yields the time part $\bar{t}_\alpha(S)$ of the Oppenheimer-Snyder mapping *has divergence as its most prominent and physically relevant feature*.

References

[1] R. C. Tolman, Proc. Nat. Acad. Sci. U. S. A. **20**, 169–176 (1934).

[2] J. R. Oppenheimer and H. Snyder, Phys. Rev. **56**, 455 (1939).

[3] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (John Wiley & Sons, New York, 1972), Section 11.9, pp. 342–345.

[4] S. Weinberg, op. cit., Eq. (11.8.8), p. 340 and Eq. (11.9.1), p. 342.

[5] S. Weinberg, op. cit., Eqs. (11.9.2)–(11.9.3), p. 342.

[6] S. Weinberg, op. cit., Eq. (11.9.4), p. 342.

[7] S. Weinberg, op. cit., Eqs. (11.9.9)–(11.9.12), p. 343.

[8] S. Weinberg, op. cit., Eq. (11.9.13), p. 343 with Eqs. (11.9.22) and (11.9.17), p. 344.

[9] S. Weinberg, op. cit., Eq. (11.9.24), p. 344.

[10] S. Weinberg, op. cit., Section 3.6, pp. 85–86.

[11] S. Weinberg, op. cit., Eq. (11.8.1), p. 339 and Eq. (11.8.8), p. 340.

[12] S. Weinberg, op. cit., Eq. (3.4.5), p. 78.

[13] S. Weinberg, op. cit., Eq. (3.5.2), p. 79.

[14] S. Weinberg, op. cit., Section 11.8, second paragraph, p. 338.

[15] S. Weinberg, op. cit., Eq. (11.7.1), p. 336.

[16] S. Weinberg, op. cit., Eq. (11.9.40), p. 347.

[17] A. Mitra, arXiv:astro-ph/9910408 (1999); Found. Phys. Lett. **13**, 543 (2000).

[18] A. Mitra and K. K. Singh, International Journal of Modern Physics D **22**, No. 9 (2013) 1350054, DOI:10.1142/S0218271813500545.