A derivation of the negative binomial distribution

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Abstract. The negative binomial distribution is derived as a solution of a linear recurrence equation with an appropriate set of constraints.

The negative binomial distribution appears in many fields of science [1] and especially in particle physics where it is used to fit the observed multiplicity distributions of the charged particles produced in proton-antiproton collisions at high energies [2]. Various interpretations of the negative binomial distribution were given and used to derive its expression [1]. In the following the negative binomial distribution will be derived as a solution to a linear recurrence relation with an appropriate (suitable) choice of constraints.

The negative binomial distribution is given by [1]

$$P(n,k) = \binom{n+k-1}{k-1}\left(\frac{\frac{\overline{n}}{k}}{1+\frac{\overline{n}}{k}}\right)^n \frac{1}{\left(1+\frac{\overline{n}}{k}\right)^k} = \binom{n+k-1}{n}\left(\frac{\frac{\overline{n}}{k}}{1+\frac{\overline{n}}{k}}\right)^n \frac{1}{\left(1+\frac{\overline{n}}{k}\right)^k}$$

where $\overline{n}$ is the average of the distribution and k an integer parameter ($k \geq 1$) which determines the shape (width) of the distribution. For the case $k = 1$

$$P(n,1) = \left(\frac{\overline{n}}{1+\overline{n}}\right)^n \frac{1}{1+\overline{n}}$$

Now consider the linear recurrence relation

$$F_{n+1} = \lambda F_n, \quad n \geq 0$$

with $0 < \lambda < 1$. Its solution is given by

$$F_n = \lambda^n F_0$$

In order to determine the two constants λ and F_0 one then enforces on F_n the two constraints

$$\sum_{n=0}^{\infty} F_n = 1 \tag{1}$$

$$\sum_{n=0}^{\infty} nF_n = \overline{n} \tag{2}$$

which lead to (see appendix 1)

$$\lambda = \frac{\overline{n}}{1+\overline{n}}$$

$$F_0 = \frac{1}{1+\overline{n}}$$

and then

$$F_n = \left(\frac{\overline{n}}{1+\overline{n}}\right)^n \frac{1}{1+\overline{n}}$$

which is the expression of the negative binomial distribution for $k = 1$ given above.

For the case $k = 2$ the negative binomial distribution is given by

$$P(n,2) = (1+n)\left(\frac{\frac{\overline{n}}{2}}{1+\frac{\overline{n}}{2}}\right)^n \frac{1}{\left(1+\frac{\overline{n}}{2}\right)^2}$$

and one considers the linear recurrence relation

$$F_{n+2} + c_1 F_{n+1} + c_2 F_n = 0, \quad n \geq 0$$

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where c_1 and c_2 are real constants. Its characteristic equation given by

$$\lambda^2 + c_1\lambda + c_2 = 0$$

may have two roots λ_1 and λ_2 or a single root with multiplicity 2 denoted in the following λ. In the latter case the solution of the linear recurrence relation is given by

$$F_n = (a_0 + a_1 n)\lambda^n$$

where $0 < \lambda < 1$ and a_0 and a_1 are arbitrary constants.

Again one enforces the two constraints (1) and (2) on the solution above. But there are three constants (a_0, a_1 and λ) to be determined and one should add one more constraint to (1) and (2). Next we will see how to choose this additional constraint.

The constraint (1) leads to (appendix 2)

$$\frac{(a_1 - a_0)\lambda + a_0}{(1-\lambda)^2} = 1$$

Now by setting to zero the coefficient of λ in the numerator one gets an additional constraint

$$a_1 = a_0$$

and as a consequence

$$a_0 = (1-\lambda)^2$$

Therefore

$$F_n = a_0(n+1)\lambda^n$$

Applying the second constraint (2) to F_n leads to (appendix 2)

$$\lambda = \frac{\frac{\overline{n}}{2}}{1 + \frac{\overline{n}}{2}}$$

and

$$a_0 = \frac{1}{\left(1 + \frac{\overline{n}}{2}\right)^2}$$

Hence

$$F_n = (n+1)\left(\frac{\frac{\overline{n}}{2}}{1 + \frac{\overline{n}}{2}}\right)^n \frac{1}{\left(1 + \frac{\overline{n}}{2}\right)^2}$$

which is the negative binomial distribution for $k = 2$.

For the case $k = 3$ the negative binomial distribution is given by

$$P(n,3) = \frac{(n+2)(n+1)}{2}\left(\frac{\frac{\overline{n}}{3}}{1 + \frac{\overline{n}}{3}}\right)^n \frac{1}{\left(1 + \frac{\overline{n}}{3}\right)^3}$$

and one considers the linear recurrence relation

$$F_{n+3} + c_1 F_{n+2} + c_2 F_{n+1} + c_3 F_n = 0, \quad n \geq 0$$

where c_1, c_2 and c_3 are real constants. Its characteristic equation given by

$$\lambda^3 + c_1\lambda^2 + c_2\lambda + c_3 = 0$$

may have three roots λ_1, λ_2 and λ_3, two roots λ_1 and λ_2 one of them being of multiplicity 2 or a single root with multiplicity 3 denoted in the following λ. In the latter case the solution of the linear recurrence relation is given by

$$F_n = (a_0 + a_1 n + a_2 n^2)\lambda^n$$

where $0 < \lambda < 1$ and a_0, a_1 and a_2 are arbitrary constants.

There are four constants (a_0, a_1, a_2 and λ) to be determined and one should add two more constraints to (1) and (2). As in the case $k = 2$ above these constraints will be determined once the constraint (1) is developped.

The constraint (1) leads to (appendix 3)

$$\frac{a_0 + (-2a_0 + a_1 + a_2)\,\lambda + (a_0 - a_1 + a_2)\,\lambda^2}{(1-\lambda)^3} = 1$$

Now by setting in the numerator the coefficients of λ and λ^2 to zero one obtains two additional constraints

$$-2a_0 + a_1 + a_2 = 0$$

$$a_0 - a_1 + a_2 = 0$$

and as a consequence

$$a_0 = (1-\lambda)^3$$

For the solution of the above system of linear equations one finds $a_0 = 2a_2$ and $a_1 = 3a_2$. Therefore

$$\begin{aligned} F_n &= a_2(2 + 3n + n^2)\lambda^n \\ &= a_2(n+1)(n+2)\lambda^n \end{aligned}$$

Applying the second constraint (2) to F_n leads to (appendix 3)

$$\lambda = \frac{\frac{\overline{n}}{3}}{1 + \frac{\overline{n}}{3}}$$

and

$$a_0 = \frac{1}{\left(1 + \frac{\overline{n}}{3}\right)^3}$$

Now $a_2 = \frac{a_0}{2}$ and hence

$$F_n = \frac{(n+1)(n+2)}{2}\left(\frac{\frac{\overline{n}}{3}}{1 + \frac{\overline{n}}{3}}\right)^n \frac{1}{\left(1 + \frac{\overline{n}}{3}\right)^3}$$

which is the expression of the negative binomial distribution for $k = 3$.

We now turn to the general case and consider the following linear recurrence relation

$$F_{n+k} + c_1 F_{n+k-1} + c_2 F_{n+k-2} \cdots + c_2 F_{n+2} + c_{k-1} F_{n+1} + c_k F_n = 0, \quad n \geq 0$$

where $c_1, \cdots, c_k$ are real constants. Its characteristic equation given by

$$\lambda^k + c_1 \lambda^{k-1} + c_2 \lambda^{k-2} \cdots + c_2 \lambda^2 + c_{k-1}\lambda + c_k = 0$$

may have different roots and we are interested in the case of a unique root with multiplicity k denoted in the following λ. In the latter case the solution of the linear recurrence relation above is given by

$$F_n = \left(a_0 + a_1 n + a_2 n^2 + \cdots + a_{k-2} n^{k-2} + a_{k-1} n^{k-1}\right)\lambda^n$$

where $0 < \lambda < 1$ and $a_0, \cdots, a_{k-1}$ are arbitrary constants.

There are $k + 1$ constants ($a_0, \cdots, a_{k-1}$ and λ) to be determined while there are two constraints (1) and (2). One then needs $k - 1$ additional constraints and as in the special cases treated above these constraints will be determined from the constraint (1).

The constraint (1) leads to

$$\sum_{n=0}^{\infty} F_n = 1$$

$$\Rightarrow \sum_{n=0}^{\infty} \left(a_0 + a_1 n + a_2 n^2 + \cdots + a_{k-2} n^{k-2} + a_{k-1} n^{k-1}\right) \lambda^n = 1$$

$$\Rightarrow a_0 \sum_{n=0}^{\infty} \lambda^n + a_1 \sum_{n=0}^{\infty} n\lambda^n + a_2 \sum_{n=0}^{\infty} n^2 \lambda^n + \cdots + a_{k-2} \sum_{n=0}^{\infty} n^{k-2}\lambda^n + a_{k-1} \sum_{n=0}^{\infty} n^{k-1}\lambda^n = 1 \tag{3}$$

Now

$$n^j = \sum_{i=0}^{j} S(j,i)(n)_i \tag{4}$$

where $S(j,i)$ are the Stirling numbers of the second kind [3,4] and $(n)_i$ is the falling factorial $(n)_i = n(n-1)\cdots(n-i+1)$ with $(n)_0 = 1$. Reporting (4) into (3) leads to

$$a_0 \sum_{n=0}^{\infty} \lambda^n + a_1 \sum_{n=0}^{\infty} \left[\sum_{i=0}^{1} S(1,i)(n)_i\right] \lambda^n + a_2 \sum_{n=0}^{\infty} \left[\sum_{i=0}^{2} S(2,i)(n)_i\right] \lambda^n + a_3 \sum_{n=0}^{\infty} \left[\sum_{i=0}^{3} S(3,i)(n)_i\right] \lambda^n + \cdots$$
$$+ a_{k-2} \sum_{n=0}^{\infty} \left[\sum_{i=0}^{k-2} S(k-2,i)(n)_i\right] \lambda^n + a_{k-1} \sum_{n=0}^{\infty} \left[\sum_{i=0}^{k-1} S(k-1,i)(n)_i\right] \lambda^n = 1$$

And by swapping the sums $\sum_n$ and $\sum_i$ one is lead to

$$\begin{aligned} &a_0 \sum_{n=0}^{\infty} \lambda^n + a_1 \sum_{i=0}^{1} S(1,i) \left[\sum_{n=0}^{\infty} (n)_i \lambda^n\right] + a_2 \sum_{i=0}^{2} S(2,i) \left[\sum_{n=0}^{\infty} (n)_i \lambda^n\right] + a_3 \sum_{i=0}^{3} S(3,i) \left[\sum_{n=0}^{\infty} (n)_i \lambda^n\right] + \cdots \\ &+ a_{k-2} \sum_{i=0}^{k-2} S(k-2,i) \left[\sum_{n=0}^{\infty} (n)_i \lambda^n\right] + a_{k-1} \sum_{i=0}^{k-1} S(k-1,i) \left[\sum_{n=0}^{\infty} (n)_i \lambda^n\right] = 1 \end{aligned} \tag{5}$$

Now

$$\begin{aligned} \sum_{n=0}^{\infty} (n)_i \lambda^n &= \sum_{n=0}^{\infty} n(n-1)\cdots(n-i+1)\lambda^n = \lambda^i \sum_{n=0}^{\infty} n(n-1)\cdots(n-i+1)\lambda^{n-i} = \lambda^i \sum_{n=0}^{\infty} \left(\frac{d^i}{d\lambda^i}\lambda^n\right) \\ &= \lambda^i \frac{d^i}{d\lambda^i} \left(\sum_{n=0}^{\infty} \lambda^n\right) = \lambda^i \frac{d^i}{d\lambda^i} \left(\frac{1}{1-\lambda}\right) = \lambda^i i! \frac{1}{(1-\lambda)^{i+1}} = \frac{i!\,\lambda^i}{(1-\lambda)^{i+1}} \end{aligned} \tag{6}$$

Now one reports (6) into (5) to get

$$a_0 \frac{1}{1-\lambda} + a_1 \sum_{i=0}^{1} S(1,i) \frac{i!\,\lambda^i}{(1-\lambda)^{i+1}} + a_2 \sum_{i=0}^{2} S(2,i) \frac{i!\,\lambda^i}{(1-\lambda)^{i+1}} + a_3 \sum_{i=0}^{3} S(3,i) \frac{i!\,\lambda^i}{(1-\lambda)^{i+1}} + \cdots$$
$$+ a_{k-2} \sum_{i=0}^{k-2} S(k-2,i) \frac{i!\,\lambda^i}{(1-\lambda)^{i+1}} + a_{k-1} \sum_{i=0}^{k-1} S(k-1,i) \frac{i!\,\lambda^i}{(1-\lambda)^{i+1}} = 1$$

And by ordering following the powers of $\frac{1}{1-\lambda}$ one arrives to

$$\begin{aligned} &\left[a_0 + a_1 S(1,0) + a_2 S(2,0) + a_3 S(3,0) + \cdots + a_{k-2} S(k-2,0) + a_{k-1} S(k-1,0)\right] \frac{1}{1-\lambda} \\ &+ \left[a_1 S(1,1) + a_2 S(2,1) + a_3 S(3,1) + \cdots + a_{k-2} S(k-2,1) + a_{k-1} S(k-1,1)\right] \frac{\lambda}{(1-\lambda)^2} \\ &+ \left[a_2 S(2,2) + a_3 S(3,2) + \cdots + a_{k-2} S(k-2,2) + a_{k-1} S(k-1,2)\right] \frac{2!\,\lambda^2}{(1-\lambda)^3} \\ &+ \left[a_3 S(3,3) + \cdots + a_{k-2} S(k-2,3) + a_{k-1} S(k-1,3)\right] \frac{3!\,\lambda^3}{(1-\lambda)^4} \\ &\vdots \\ &+ \left[a_{k-3} S(k-3,k-3) + a_{k-2} S(k-2,k-3) + a_{k-1} S(k-1,k-3)\right] \frac{(k-3)!\,\lambda^{k-3}}{(1-\lambda)^{k-2}} \\ &+ \left[a_{k-2} S(k-2,k-2) + a_{k-1} S(k-1,k-2)\right] \frac{(k-2)!\,\lambda^{k-2}}{(1-\lambda)^{k-1}} \\ &+ a_{k-1} S(k-1,k-1) \frac{(k-1)!\,\lambda^{k-1}}{(1-\lambda)^k} \\ &= 1 \end{aligned}$$

Now $S(j,0)=0$ for $j \geq 1$ [3,4] and then the equation above reduces to

$$
\begin{aligned}
& a_0 \frac{1}{1-\lambda} \\
& \quad + \left[a_1 S(1,1) + a_2 S(2,1) + a_3 S(3,1) + \cdots + a_{k-2} S(k-2,1) + a_{k-1} S(k-1,1)\right] \frac{\lambda}{(1-\lambda)^2} \\
& \qquad + \left[a_2 S(2,2) + a_3 S(3,2) + \cdots + a_{k-2} S(k-2,2) + a_{k-1} S(k-1,2)\right] \frac{2!\,\lambda^2}{(1-\lambda)^3} \\
& \qquad\quad + \left[a_3 S(3,3) + \cdots + a_{k-2} S(k-2,3) + a_{k-1} S(k-1,3)\right] \frac{3!\,\lambda^3}{(1-\lambda)^4} \\
& \qquad\quad \vdots \\
& \qquad\quad + \left[a_{k-3} S(k-3,k-3) + a_{k-2} S(k-2,k-3) + a_{k-1} S(k-1,k-3)\right] \frac{(k-3)!\,\lambda^{k-3}}{(1-\lambda)^{k-2}} \\
& \qquad\qquad + \left[a_{k-2} S(k-2,k-2) + a_{k-1} S(k-1,k-2)\right] \frac{(k-2)!\,\lambda^{k-2}}{(1-\lambda)^{k-1}} \\
& \qquad\qquad\quad + a_{k-1} S(k-1,k-1) \frac{(k-1)!\,\lambda^{k-1}}{(1-\lambda)^{k}} \\
& \qquad\qquad\qquad = 1
\end{aligned}
$$

By setting to a common denominator in the above equation one gets

$$
\begin{aligned}
& a_0 \frac{(1-\lambda)^{k-1}}{(1-\lambda)^k} \\
& \quad + \left[a_1 S(1,1) + a_2 S(2,1) + a_3 S(3,1) + \cdots + a_{k-2} S(k-2,1) + a_{k-1} S(k-1,1)\right] \frac{\lambda\,(1-\lambda)^{k-2}}{(1-\lambda)^k} \\
& \qquad + \left[a_2 S(2,2) + a_3 S(3,2) + \cdots + a_{k-2} S(k-2,2) + a_{k-1} S(k-1,2)\right] \frac{2!\,\lambda^2\,(1-\lambda)^{k-3}}{(1-\lambda)^k} \\
& \qquad\quad + \left[a_3 S(3,3) + \cdots + a_{k-2} S(k-2,3) + a_{k-1} S(k-1,3)\right] \frac{3!\,\lambda^3\,(1-\lambda)^{k-4}}{(1-\lambda)^k} \\
& \qquad\quad \vdots \\
& \qquad\quad + \left[a_{k-3} S(k-3,k-3) + a_{k-2} S(k-2,k-3) + a_{k-1} S(k-1,k-3)\right] \frac{(k-3)!\,\lambda^{k-3}\,(1-\lambda)^2}{(1-\lambda)^k} \\
& \qquad\qquad + \left[a_{k-2} S(k-2,k-2) + a_{k-1} S(k-1,k-2)\right] \frac{(k-2)!\,\lambda^{k-2}\,(1-\lambda)}{(1-\lambda)^k} \\
& \qquad\qquad\quad + a_{k-1} S(k-1,k-1) \frac{(k-1)!\,\lambda^{k-1}}{(1-\lambda)^k} \\
& \qquad\qquad\qquad = 1
\end{aligned}
$$

Now the expansion of the powers of $(1-\lambda)$ in the numerators leads to

$$
\begin{aligned}
&a_0 \frac{\sum_{i=0}^{k-1} C_{k-1}^i (-\lambda)^i}{(1-\lambda)^k} \\
&\quad + \left[a_1 S(1,1) + a_2 S(2,1) + a_3 S(3,1) + \cdots + a_{k-2} S(k-2,1) + a_{k-1} S(k-1,1)\right] \frac{\lambda \sum_{i=0}^{k-2} C_{k-2}^i (-\lambda)^i}{(1-\lambda)^k} \\
&\qquad + \left[a_2 S(2,2) + a_3 S(3,2) + \cdots + a_{k-2} S(k-2,2) + a_{k-1} S(k-1,2)\right] \frac{2!\,\lambda^2 \sum_{i=0}^{k-3} C_{k-3}^i (-\lambda)^i}{(1-\lambda)^k} \\
&\qquad\quad + \left[a_3 S(3,3) + \cdots + a_{k-2} S(k-2,3) + a_{k-1} S(k-1,3)\right] \frac{3!\,\lambda^3 \sum_{i=0}^{k-4} C_{k-4}^i (-\lambda)^i}{(1-\lambda)^k} \\
&\qquad\quad \vdots \\
&\qquad\quad + \left[a_{k-3} S(k-3,k-3) + a_{k-2} S(k-2,k-3) + a_{k-1} S(k-1,k-3)\right] \frac{(k-3)!\,\lambda^{k-3}\,(1-\lambda)^2}{(1-\lambda)^k} \\
&\qquad\qquad + \left[a_{k-2} S(k-2,k-2) + a_{k-1} S(k-1,k-2)\right] \frac{(k-2)!\,\lambda^{k-2}\,(1-\lambda)}{(1-\lambda)^k} \\
&\qquad\qquad\quad + a_{k-1} S(k-1,k-1) \frac{(k-1)!\,\lambda^{k-1}}{(1-\lambda)^k} \\
&\qquad\qquad\qquad = 1
\end{aligned}
$$

The terms are then grouped following the powers of λ

$$
\begin{aligned}
&\frac{a_0}{(1-\lambda)^k} C_{k-1}^0 \\
&+\{-a_0 C_{k-1}^1 \\
&\quad +[a_1 S(1,1)+a_2 S(2,1)+a_3 S(3,1)+\cdots+a_{k-2} S(k-2,1)+a_{k-1} S(k-1,1)] C_{k-2}^0\} \frac{\lambda}{(1-\lambda)^k} \\
&+\{a_0 C_{k-1}^2 \\
&\quad -[a_1 S(1,1)+a_2 S(2,1)+a_3 S(3,1)+\cdots+a_{k-2} S(k-2,1)+a_{k-1} S(k-1,1)] C_{k-2}^1 \\
&\quad +2!\,[a_2 S(2,2)+a_3 S(3,2)+\cdots+a_{k-2} S(k-2,2)+a_{k-1} S(k-1,2)] C_{k-3}^0\} \frac{\lambda^2}{(1-\lambda)^k} \\
&+\{-a_0 C_{k-1}^3 \\
&\quad +[a_1 S(1,1)+a_2 S(2,1)+a_3 S(3,1)+\cdots+a_{k-2} S(k-2,1)+a_{k-1} S(k-1,1)] C_{k-2}^2 \\
&\quad -2!\,[a_2 S(2,2)+a_3 S(3,2)+\cdots+a_{k-2} S(k-2,2)+a_{k-1} S(k-1,2)] C_{k-3}^1 \\
&\quad +3!\,[a_3 S(3,3)+\cdots+a_{k-2} S(k-2,3)+a_{k-1} S(k-1,3)] C_{k-4}^0\} \frac{\lambda^3}{(1-\lambda)^k} \\
&+\{a_0 C_{k-1}^4 \\
&\quad -[a_1 S(1,1)+a_2 S(2,1)+a_3 S(3,1)+\cdots+a_{k-2} S(k-2,1)+a_{k-1} S(k-1,1)] C_{k-2}^3 \\
&\quad +2!\,[a_2 S(2,2)+a_3 S(3,2)+\cdots+a_{k-2} S(k-2,2)+a_{k-1} S(k-1,2)] C_{k-3}^2 \\
&\quad -3!\,[a_3 S(3,3)+a_4 S(4,3)+\cdots+a_{k-2} S(k-2,3)+a_{k-1} S(k-1,3)] C_{k-4}^1 \\
&\quad +4!\,[a_4 S(4,4)+a_5 S(5,4)+\cdots+a_{k-2} S(k-2,4)+a_{k-1} S(k-1,4)] C_{k-5}^0\} \frac{\lambda^4}{(1-\lambda)^k} \\
&+\cdots \\
&\vdots \\
&+\{(-1)^{k-2} a_0 C_{k-1}^{k-2} \\
&\quad +(-1)^{k-3}[a_1 S(1,1)+a_2 S(2,1)+a_3 S(3,1)+\cdots+a_{k-2} S(k-2,1)+a_{k-1} S(k-1,1)] C_{k-2}^{k-3} \\
&\quad +(-1)^{k-4} 2!\,[a_2 S(2,2)+a_3 S(3,2)+\cdots+a_{k-2} S(k-2,2)+a_{k-1} S(k-1,2)] C_{k-3}^{k-4} \\
&\quad +(-1)^{k-5} 3!\,[a_3 S(3,3)+a_4 S(4,3)+\cdots+a_{k-2} S(k-2,3)+a_{k-1} S(k-1,3)] C_{k-4}^{k-5} \\
&\quad +(-1)^{k-6} 4!\,[a_4 S(4,4)+a_5 S(5,4)+\cdots+a_{k-2} S(k-2,4)+a_{k-1} S(k-1,4)] C_{k-5}^{k-6} \\
&\quad \vdots \\
&\quad +(-1)^1 (k-3)!\,[a_{k-3} S(k-3,k-3)+a_{k-2} S(k-2,k-3)+a_{k-1} S(k-1,k-3)] C_2^1 \\
&\quad +(-1)^0 (k-2)!\,[a_{k-2} S(k-2,k-2)+a_{k-1} S(k-1,k-2)] C_1^0\} \frac{\lambda^{k-2}}{(1-\lambda)^k} \\
&+\{(-1)^{k-1} a_0 C_{k-1}^{k-1} \\
&\quad +(-1)^{k-2}[a_1 S(1,1)+a_2 S(2,1)+a_3 S(3,1)+\cdots+a_{k-2} S(k-2,1)+a_{k-1} S(k-1,1)] C_{k-2}^{k-2} \\
&\quad +(-1)^{k-3} 2!\,[a_2 S(2,2)+a_3 S(3,2)+\cdots+a_{k-2} S(k-2,2)+a_{k-1} S(k-1,2)] C_{k-3}^{k-3} \\
&\quad +(-1)^{k-4} 3!\,[a_3 S(3,3)+a_4 S(4,3)+\cdots+a_{k-2} S(k-2,3)+a_{k-1} S(k-1,3)] C_{k-4}^{k-4} \\
&\quad +(-1)^{k-5} 4!\,[a_4 S(4,4)+a_5 S(5,4)+\cdots+a_{k-2} S(k-2,4)+a_{k-1} S(k-1,4)] C_{k-5}^{k-5} \\
&\quad \vdots \\
&\quad +(-1)^2 (k-3)!\,[a_{k-3} S(k-3,k-3)+a_{k-2} S(k-2,k-3)+a_{k-1} S(k-1,k-3)] C_2^2 \\
&\quad +(-1)^1 (k-2)!\,[a_{k-2} S(k-2,k-2)+a_{k-1} S(k-1,k-2)] C_1^1 \\
&\quad +(-1)^0 (k-1)!\, a_{k-1} S(k-1,k-1) C_0^0\} \frac{\lambda^{k-1}}{(1-\lambda)^k} = 1
\end{aligned}
$$

To obtain the additional constraints one sets the coefficients of λ^m, with $m \geq 1$, to 0. Setting the coefficient of λ to 0 yields the first constraint

$$-a_0 C^1_{k-1} + [a_1 S(1,1) + a_2 S(2,1) + a_3 S(3,1) + \cdots + a_{k-2} S(k-2,1) + a_{k-1} S(k-1,1)]\, C^0_{k-2} = 0$$

which leads

$$a_1 S(1,1) + a_2 S(2,1) + a_3 S(3,1) + \cdots + a_{k-2} S(k-2,1) + a_{k-1} S(k-1,1) = a_0 C^1_{k-1} \quad (7)$$

where $C^0_{k-2} = 1$ was used.

With the coefficient of λ^2 set to 0 one gets the second constraint

$$\begin{aligned} a_0 C^2_{k-1} &- [a_1 S(1,1) + a_2 S(2,1) + a_3 S(3,1) + \cdots + a_{k-2} S(k-2,1) + a_{k-1} S(k-1,1)]\, C^1_{k-2} \\ &+ 2!\, [a_2 S(2,2) + a_3 S(3,2) + \cdots + a_{k-2} S(k-2,2) + a_{k-1} S(k-1,2)]\, C^0_{k-3} = 0 \end{aligned}$$

Using the first constraint (7) and $C^0_{k-3} = 1$ one gets

$$a_0 C^2_{k-1} - a_0 C^1_{k-1} C^1_{k-2} + 2!\, [a_2 S(2,2) + a_3 S(3,2) + \cdots + a_{k-2} S(k-2,2) + a_{k-1} S(k-1,2)] = 0$$

Now

$$C^1_{k-1} C^1_{k-2} = 2 C^2_{k-1}$$

and then

$$a_2 S(2,2) + a_3 S(3,2) + \cdots + a_{k-2} S(k-2,2) + a_{k-1} S(k-1,2) = a_0 \frac{C^2_{k-1}}{2!} \quad (8)$$

The third constraint is obtained by setting the coefficient of λ^3 to 0

$$\begin{aligned} -a_0 C^3_{k-1} &+ [a_1 S(1,1) + a_2 S(2,1) + a_3 S(3,1) + \cdots + a_{k-2} S(k-2,1) + a_{k-1} S(k-1,1)]\, C^2_{k-2} \\ &- 2!\, [a_2 S(2,2) + a_3 S(3,2) + \cdots + a_{k-2} S(k-2,2) + a_{k-1} S(k-1,2)]\, C^1_{k-3} \\ &+ 3!\, [a_3 S(3,3) + \cdots + a_{k-2} S(k-2,3) + a_{k-1} S(k-1,3)]\, C^0_{k-4} = 0 \end{aligned}$$

Using the first constraint (7), the second one (8) and $C^0_{k-4} = 1$ yields

$$-a_0 C^3_{k-1} + a_0 C^1_{k-1} C^2_{k-2} - 2!\, a_0 \frac{C^2_{k-1}}{2!} C^1_{k-3} + 3!\, [a_3 S(3,3) + \cdots + a_{k-2} S(k-2,3) + a_{k-1} S(k-1,3)] = 0$$

Now

$$C^1_{k-1} C^2_{k-2} = 3 C^3_{k-1}$$

$$C^2_{k-1} C^1_{k-3} = 3 C^3_{k-1}$$

and then

$$a_3 S(3,3) + \cdots + a_{k-2} S(k-2,3) + a_{k-1} S(k-1,3) = a_0 \frac{C^3_{k-1}}{3!} \quad (9)$$

Now we suppose that the constraint for some j is given by

$$a_j S(j,j) + a_{j+1} S(j+1,j) + \cdots + a_{k-2} S(k-2,j) + a_{k-1} S(k-1,j) = a_0 \frac{C^j_{k-1}}{j!} \quad (10)$$

and show that it holds for $j+1$ e.g.

$$a_{j+1} S(j+1,j+1) + a_{j+2} S(j+2,j+1) + \cdots + a_{k-2} S(k-2,j+1) + a_{k-1} S(k-1,j+1) = a_0 \frac{C^{j+1}_{k-1}}{(j+1)!} \quad (11)$$

Setting the coefficient of λ^{j+1} to 0 one gets

$$
\begin{aligned}
&(-1)^{j+1}a_0C_{k-1}^{j+1}\\
&+(-1)^j\left[a_1S(1,1)+a_2S(2,1)+\cdots+a_{k-2}S(k-2,1)+a_{k-1}S(k-1,1)\right]C_{k-2}^{j}\\
&+(-1)^{j-1}2\left[a_2S(2,2)+a_3S(3,2)+\cdots+a_{k-2}S(k-2,2)+a_{k-1}S(k-1,2)\right]C_{k-3}^{j-1}\\
&+(-1)^{j-2}3!\left[a_3S(3,3)+a_4S(4,3)+\cdots+a_{k-2}S(k-2,3)+a_{k-1}S(k-1,3)\right]C_{k-4}^{j-2}\\
&+(-1)^{j-3}4!\left[a_4S(4,4)+a_5S(5,4)+\cdots+a_{k-2}S(k-2,4)+a_{k-1}S(k-1,4)\right]C_{k-5}^{j-3}\\
&\vdots\\
&+(-1)^1j!\left[a_jS(j,j)+a_{j+1}S(j+1,j)+\cdots+a_{k-2}S(k-2,j)+a_{k-1}S(k-1,j)\right]C_{k-j-1}^{1}\\
&+(-1)^0(j+1)!\left[a_{j+1}S(j+1,j+1)+a_{j+2}S(j+2,j+1)+\cdots+a_{k-2}S(k-2,j+1)+a_{k-1}S(k-1,j+1)\right]C_{k-j-2}^{0}=0
\end{aligned}
$$

By reporting the expressions of the first j constraints given by eq. (10) one gets

$$
\begin{aligned}
&(-1)^{j+1}a_0C_{k-1}^{j+1}+(-1)^j a_0C_{k-1}^{1}C_{k-2}^{j}+(-1)^{j-1}a_0C_{k-1}^{2}C_{k-3}^{j-1}+(-1)^{j-2}a_0C_{k-1}^{3}C_{k-4}^{j-2}+(-1)^{j-3}a_0C_{k-1}^{4}C_{k-5}^{j-3}+\cdots+\\
&(-1)^1a_0C_{k-1}^{j}C_{k-j-1}^{1}+(j+1)!\left[a_{j+1}S(j+1,j+1)+a_{j+2}S(j+2,j+1)+\cdots+a_{k-2}S(k-2,j+1)+a_{k-1}S(k-1,j+1)\right]=0
\end{aligned}
$$

where $C_{k-j-2}^{0}=1$ was used.

Now

$$
\begin{aligned}
C_{k-1}^{i}C_{k-i-1}^{j-i+1}&=\frac{(k-1)!}{i!(k-i-1)!}\frac{(k-i-1)!}{(j-i+1)!(k-j-2)!}\\
&=\frac{(k-1)!}{i!(j-i+1)!(k-j-2)!}\\
&=\frac{(j+1)!}{i!(j-i+1)!}\frac{(k-1)!}{(j+1)!(k-j-2)!}\\
&=C_{j+1}^{i}C_{k-1}^{j+1}\\
&=C_{j+1}^{j-i+1}C_{k-1}^{j+1}
\end{aligned}
$$

Reporting this identity above yields

$$
\begin{aligned}
&(-1)^{j+1}a_0C_{k-1}^{j+1}+(-1)^j a_0C_{j+1}^{j}C_{k-1}^{j+1}+(-1)^{j-1}a_0C_{j+1}^{j-1}C_{k-1}^{j+1}+(-1)^{j-2}a_0C_{j+1}^{j-2}C_{k-1}^{j+1}+(-1)^{j-3}a_0C_{j+1}^{j-3}C_{k-1}^{j+1}+\cdots+\\
&(-1)^1a_0C_{j+1}^{1}C_{k-1}^{j+1}+(j+1)!\left[a_{j+1}S(j+1,j+1)+a_{j+2}S(j+2,j+1)+\cdots+a_{k-2}S(k-2,j+1)+a_{k-1}S(k-1,j+1)\right]=0
\end{aligned}
$$

Then factoring out C_{k-1}^{j+1} yields

$$
\begin{aligned}
&a_0C_{k-1}^{j+1}\left[(-1)^{j+1}+(-1)^jC_{j+1}^{j}+(-1)^{j-1}C_{j+1}^{j-1}+(-1)^{j-2}C_{j+1}^{j-2}+(-1)^{j-3}C_{j+1}^{j-3}+\cdots+(-1)^1C_{j+1}^{1}\right]\\
&\quad+(j+1)!\left[a_{j+1}S(j+1,j+1)+a_{j+2}S(j+2,j+1)+\cdots+a_{k-2}S(k-2,j+1)+a_{k-1}S(k-1,j+1)\right]=0
\end{aligned}
$$

Now by adding the term $(-1)^0C_{j+1}^{0}-(-1)^0C_{j+1}^{0}$ to the bracketed expression in the first line one gets

$$
\begin{aligned}
&a_0C_{k-1}^{j+1}\left[(-1)^{j+1}+(-1)^jC_{j+1}^{j}+(-1)^{j-1}C_{j+1}^{j-1}+(-1)^{j-2}C_{j+1}^{j-2}+(-1)^{j-3}C_{j+1}^{j-3}+\cdots+(-1)^1C_{j+1}^{1}+(-1)^0C_{j+1}^{0}-(-1)^0C_{j+1}^{0}\right]\\
&+(j+1)!\left[a_{j+1}S(j+1,j+1)+a_{j+2}S(j+2,j+1)+\cdots+a_{k-2}S(k-2,j+1)+a_{k-1}S(k-1,j+1)\right]=0
\end{aligned}
$$

Or in a compact form

$$
\begin{aligned}
&a_0C_{k-1}^{j+1}\left[-1+\sum_{i=0}^{j+1}(-1)^iC_{j+1}^{i}\right]\\
&\quad+(j+1)!\left[a_{j+1}S(j+1,j+1)+a_{j+2}S(j+2,j+1)+\cdots+a_{k-2}S(k-2,j+1)+a_{k-1}S(k-1,j+1)\right]=0
\end{aligned}
$$

which results in

$$
\begin{aligned}
&a_0C_{k-1}^{j+1}\left[-1+(1-1)^{j+1}\right]\\
&\quad+(j+1)!\left[a_{j+1}S(j+1,j+1)+a_{j+2}S(j+2,j+1)+\cdots+a_{k-2}S(k-2,j+1)+a_{k-1}S(k-1,j+1)\right]=0
\end{aligned}
$$

And finally one arrives to

$$a_{j+1}S(j+1,j+1)+a_{j+2}S(j+2,j+1)+\cdots+a_{k-2}S(k-2,j+1)+a_{k-1}S(k-1,j+1)=a_0\frac{C_{k-1}^{j+1}}{(j+1)!}$$

which is the desired expression (11).

Now by setting to 0 the coefficients of λ^m, with $m\geq 1$, one obtains the constraint equations and as a consequence

$$\frac{a_0}{(1-\lambda)^k}C_{k-1}^0=1$$

or

$$a_0=(1-\lambda)^k$$

where $C_{k-1}^0=1$ was used.

The constraints are then given by

$$\begin{aligned}
a_1S(1,1)+a_2S(2,1)+a_3S(3,1)+\cdots+a_{k-2}S(k-2,1)+a_{k-1}S(k-1,1)&=a_0C_{k-1}^1\\
a_2S(2,2)+a_3S(3,2)+\cdots+a_{k-2}S(k-2,2)+a_{k-1}S(k-1,2)&=a_0\frac{C_{k-1}^2}{2}\\
a_3S(3,3)+\cdots+a_{k-2}S(k-2,3)+a_{k-1}S(k-1,3)&=a_0\frac{C_{k-1}^3}{3!}\\
&\vdots\\
a_{k-2}S(k-2,k-2)+a_{k-1}S(k-1,k-2)&=a_0\frac{C_{k-1}^{k-2}}{(k-2)!}\\
a_{k-1}S(k-1,k-1)&=a_0\frac{C_{k-1}^{k-1}}{(k-1)!}
\end{aligned}$$

Or in a matrix form

$$S_k\begin{pmatrix}a_0\\a_1\\a_2\\a_3\\\vdots\\a_{k-2}\\a_{k-1}\end{pmatrix}=\begin{pmatrix}a_0\frac{C_{k-1}^0}{0!}\\a_0\frac{C_{k-1}^1}{1!}\\a_0\frac{C_{k-1}^2}{2!}\\a_0\frac{C_{k-1}^3}{3!}\\\vdots\\a_0\frac{C_{k-1}^{k-2}}{(k-2)!}\\a_0\frac{C_{k-1}^{k-1}}{(k-1)!}\end{pmatrix}$$

where

$$S_k=\begin{pmatrix}S(0,0)&0&0&0&\cdots&0&0\\0&S(1,1)&S(2,1)&S(3,1)&\cdots&S(k-2,1)&S(k-1,1)\\0&0&S(2,2)&S(3,2)&\cdots&S(k-2,2)&S(k-1,2)\\0&0&0&S(3,3)&\cdots&S(k-2,3)&S(k-1,3)\\\vdots&\vdots&\vdots&\ddots&\ddots&\vdots&\vdots\\0&0&0&0&0&S(k-2,k-2)&S(k-1,k-2)\\0&0&0&0&0&0&S(k-1,k-1)\end{pmatrix}$$

is the $k\times k$ Stirling matrix of the second kind. It's an upper triangular matrix; its inverse matrix is the upper triangular matrix s_k whose entries are the signed Stirling numbers of the first kind [5,6] and is called the Stirling matrix of the first kind. Hence

$$\begin{pmatrix}a_0\\a_1\\a_2\\a_3\\\vdots\\a_{k-2}\\a_{k-1}\end{pmatrix}=s_k\begin{pmatrix}a_0\frac{C_{k-1}^0}{0!}\\a_0\frac{C_{k-1}^1}{1!}\\a_0\frac{C_{k-1}^2}{2!}\\a_0\frac{C_{k-1}^3}{3!}\\\vdots\\a_0\frac{C_{k-1}^{k-2}}{(k-2)!}\\a_0\frac{C_{k-1}^{k-1}}{(k-1)!}\end{pmatrix}$$

where

$$s_k = \begin{pmatrix} s(0,0) & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & s(1,1) & s(2,1) & s(3,1) & \cdots & s(k-2,1) & s(k-1,1) \\ 0 & 0 & s(2,2) & s(3,2) & \cdots & s(k-2,2) & s(k-1,2) \\ 0 & 0 & 0 & s(3,3) & \cdots & s(k-2,3) & s(k-1,3) \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & s(k-2,k-2) & s(k-1,k-2) \\ 0 & 0 & 0 & 0 & 0 & 0 & s(k-1,k-1) \end{pmatrix}$$

Then

$$\begin{aligned} a_1 &= a_0 \left[s(1,1)\frac{C_{k-1}^1}{1!} + s(2,1)\frac{C_{k-1}^2}{2!} + s(3,1)\frac{C_{k-1}^3}{3!} + \cdots + s(k-2,1)\frac{C_{k-1}^{k-2}}{(k-2)!} + s(k-1,1)\frac{C_{k-1}^{k-1}}{(k-1)!} \right] \\ a_2 &= a_0 \left[s(2,2)\frac{C_{k-1}^2}{2!} + s(3,2)\frac{C_{k-1}^3}{3!} + \cdots + s(k-2,2)\frac{C_{k-1}^{k-2}}{(k-2)!} + s(k-1,2)\frac{C_{k-1}^{k-1}}{(k-1)!} \right] \\ a_3 &= a_0 \left[s(3,3)\frac{C_{k-1}^3}{3!} + s(4,3)\frac{C_{k-1}^4}{4!} + \cdots + s(k-2,3)\frac{C_{k-1}^{k-2}}{(k-2)!} + s(k-1,3)\frac{C_{k-1}^{k-1}}{(k-1)!} \right] \\ &\vdots \\ a_{k-2} &= a_0 \left[s(k-2,k-2)\frac{C_{k-1}^{k-2}}{(k-2)!} + s(k-1,k-2)\frac{C_{k-1}^{k-1}}{(k-1)!} \right] \\ a_{k-1} &= a_0 s(k-1,k-1)\frac{C_{k-1}^{k-1}}{(k-1)!} \end{aligned}$$

Or in a compact form

$$a_j = a_0 \sum_{i=j}^{k-1} s(i,j)\frac{C_{k-1}^i}{i!}, \quad 1 \le j \le k-1$$

For practical reasons in the next step it's convenient to express the a_j's as functions of a_{k-1}.

Now

$$a_{k-1} = a_0 s(k-1,k-1)\frac{C_{k-1}^{k-1}}{(k-1)!}$$

and since $s(k-1,k-1) = 1$ and $C_{k-1}^{k-1} = 1$

$$a_0 = (k-1)!\, a_{k-1}$$

Then

$$a_j = a_{k-1}(k-1)! \sum_{i=j}^{k-1} s(i,j)\frac{C_{k-1}^i}{i!}, \quad 0 \le j \le k-2$$

This sum is then evaluated and leads to (appendix 4)

$$a_j = a_{k-1}|s(k,j+1)|, \quad 0 \le j \le k-2$$

where $|s(i,j)|$ are the unsigned Stirling numbers of the first kind [5,6].

Then

$$\begin{aligned} F_n &= \left(a_0 + a_1 n + a_2 n^2 + \cdots + a_{k-2}n^{k-2} + a_{k-1}n^{k-1}\right)\lambda^n \\ &= a_{k-1}\left[|s(k,1)| + |s(k,2)|n + |s(k,3)|n^2 + \cdots + |s(k,k-1)|n^{k-2} + n^{k-1}\right]\lambda^n \\ &= a_{k-1}\sum_{i=0}^{k-1}|s(k,i+1)|n^i\lambda^n \\ &= a_{k-1}(n+1)(n+2)\cdots(n+k-1)\lambda^n \end{aligned}$$

where use was made of some properties of the unsigned Stirling numbers of the first kind [5,6], say

$$\sum_{i=0}^{k-1}|s(k,i+1)|n^i = (n+1)(n+2)\cdots(n+k-1)$$

and

$$|s(k,k)| = 1$$

Applying the constraint (2) leads to

$$\begin{aligned}
\sum_{n=0}^{\infty} nF_n &= a_{k-1} \sum_{n=0}^{\infty} n(n+1)(n+2)\cdots(n+k-1)\lambda^n = a_{k-1}\lambda \sum_{n=0}^{\infty} n(n+1)(n+2)\cdots(n+k-1)\lambda^{n-1} \\
&= a_{k-1}\lambda \sum_{n=0}^{\infty} \frac{d^k}{d\lambda^k}\lambda^{n+k-1} = a_{k-1}\lambda \frac{d^k}{d\lambda^k}\left(\sum_{n=0}^{\infty}\lambda^{n+k-1}\right) = a_{k-1}\lambda \frac{d^k}{d\lambda^k}\left(\lambda^{k-1}\sum_{n=0}^{\infty}\lambda^n\right) \\
&= a_{k-1}\lambda \frac{d^k}{d\lambda^k}\left(\frac{\lambda^{k-1}}{1-\lambda}\right) = a_{k-1}\lambda \frac{d^k}{d\lambda^k}\left(-\lambda^{k-2} - \lambda^{k-3} - \cdots - 1 + \frac{1}{1-\lambda}\right) \\
&= a_{k-1}\lambda \frac{k!}{(1-\lambda)^{k+1}} = \frac{a_0}{(k-1)!}\lambda \frac{k!}{(1-\lambda)^{k+1}} = \frac{(1-\lambda)^k}{(k-1)!}\lambda \frac{k!}{(1-\lambda)^{k+1}} = \frac{k\lambda}{1-\lambda} \\
&= \overline{n}
\end{aligned}$$

From which one deduces

$$\lambda = \frac{\frac{\overline{n}}{k}}{1+\frac{\overline{n}}{k}}$$

Therefore

$$\begin{aligned}
F_n &= a_{k-1}(n+1)(n+2)\cdots(n+k-1)\lambda^n \\
&= \frac{a_0}{(k-1)!}(n+1)(n+2)\cdots(n+k-1)\lambda^n \\
&= \frac{(1-\lambda)^k}{(k-1)!}(n+1)(n+2)\cdots(n+k-1)\lambda^n \\
&= (1-\lambda)^k \, \frac{(n+1)(n+2)\cdots(n+k-1)}{(k-1)!}\lambda^n \\
&= \frac{1}{\left(1+\frac{\overline{n}}{k}\right)^k} \frac{(n+1)(n+2)\cdots(n+k-1)}{(k-1)!}\left(\frac{\frac{\overline{n}}{k}}{1+\frac{\overline{n}}{k}}\right)^n \\
&= \frac{(n+1)(n+2)\cdots(n+k-1)}{(k-1)!}\left(\frac{\frac{\overline{n}}{k}}{1+\frac{\overline{n}}{k}}\right)^n \frac{1}{\left(1+\frac{\overline{n}}{k}\right)^k}
\end{aligned}$$

which is the expression for the negative binomial distribution.

Appendix 1.

Consider the linear recurrence relation

$$F_n = \lambda F_{n-1}$$

with $0 < \lambda < 1$. Its solution is given by

$$F_n = \lambda^n F_0$$

One then enforces on F_n the two constraints

$$\sum_{n=0}^{\infty} F_n = 1$$

$$\sum_{n=0}^{\infty} nF_n = \overline{n}$$

For the first constraint one has

$$\begin{aligned}
\sum_{n=0}^{\infty} F_n &= \sum_{n=0}^{\infty} \lambda^n F_0 = F_0 \sum_{n=0}^{\infty}\lambda^n = F_0 \frac{1}{1-\lambda} \\
&= 1
\end{aligned}$$

and then $F_0 = 1 - \lambda$.

For the second constraint one has

$$\sum_{n=0}^{\infty} nF_n = \sum_{n=0}^{\infty} n\lambda^n F_0 = F_0 \sum_{n=0}^{\infty} n\lambda^n$$
$$= \overline{n}$$

Now

$$\sum_{n=0}^{\infty} n\lambda^n = \lambda \sum_{n=0}^{\infty} n\lambda^{n-1} = \lambda \sum_{n=0}^{\infty} \frac{d}{d\lambda}\lambda^n = \lambda \frac{d}{d\lambda}\left(\sum_{n=0}^{\infty} \lambda^n\right) = \lambda \frac{d}{d\lambda}\left(\frac{1}{1-\lambda}\right) = \frac{\lambda}{(1-\lambda)^2}$$

Then

$$\sum_{n=0}^{\infty} nF_n = \overline{n}$$
$$\Rightarrow F_0 \frac{\lambda}{(1-\lambda)^2} = \overline{n}$$

From the first constraint one has $F_0 = 1 - \lambda$ and then

$$\frac{\lambda}{1-\lambda} = \overline{n}$$
$$\Rightarrow \lambda = \frac{\overline{n}}{1+\overline{n}}$$

Appendix 2.

The solution of the linear recurrence relation

$$F_{n+2} + c_1 F_{n+1} + c_2 F_n = 0, \quad n \geq 0$$

is given by

$$F_n = (a_0 + a_1 n)\lambda^n$$

in the case of a single root λ of multiplicity 2 (see text).

There are three constants a_0, a_1 and λ to be determined while there are two constraints (1) and (2). One then needs one more constraint.

The constraint (1) leads to

$$\sum_{n=0}^{\infty} F_n = \sum_{n=0}^{\infty}(a_0 + a_1 n)\lambda^n = a_0 \sum_{n=0}^{\infty} \lambda^n + a_1 \sum_{n=0}^{\infty} n\lambda^n = a_0 \sum_{n=0}^{\infty} \lambda^n + a_1\lambda \sum_{n=0}^{\infty} n\lambda^{n-1} = a_0 \sum_{n=0}^{\infty} \lambda^n + a_1\lambda \sum_{n=0}^{\infty} \frac{d}{d\lambda}\lambda^n$$
$$= a_0 \sum_{n=0}^{\infty} \lambda^n + a_1\lambda \frac{d}{d\lambda}\left(\sum_{n=0}^{\infty} \lambda^n\right) = a_0 \frac{1}{1-\lambda} + a_1\lambda \frac{d}{d\lambda}\left(\frac{1}{1-\lambda}\right) = \frac{a_0}{1-\lambda} + a_1\lambda \frac{1}{(1-\lambda)^2}$$
$$= \frac{a_0}{1-\lambda} + \frac{a_1\lambda}{(1-\lambda)^2} = \frac{a_0(1-\lambda)}{(1-\lambda)^2} + \frac{a_1\lambda}{(1-\lambda)^2} = \frac{(a_1 - a_0)\lambda + a_0}{(1-\lambda)^2}$$
$$= 1$$

Now setting in the numerator the coefficient of λ to zero one gets

$$a_1 = a_0$$

and consequently

$$a_0 = (1-\lambda)^2$$

Therefore

$$F_n = a_0(n+1)\lambda^n$$

The constraint (2) leads to

$$\sum_{n=0}^{\infty} nF_n = \sum_{n=0}^{\infty} a_0 n(n+1)\lambda^n = a_0 \sum_{n=0}^{\infty} n(n+1)\lambda^n = a_0\lambda \sum_{n=0}^{\infty} n(n+1)\lambda^{n-1} = a_0\lambda \sum_{n=0}^{\infty} \frac{d^2}{d\lambda^2}\lambda^{n+1} = a_0\lambda \frac{d^2}{d\lambda^2}\left(\sum_{n=0}^{\infty} \lambda^{n+1}\right)$$
$$= a_0\lambda \frac{d^2}{d\lambda^2}\left(\lambda \sum_{n=0}^{\infty} \lambda^n\right) = a_0\lambda \frac{d^2}{d\lambda^2}\left(\frac{\lambda}{1-\lambda}\right) = a_0\lambda \frac{2}{(1-\lambda)^3}$$
$$= \overline{n}$$

Now

$$a_0 = (1-\lambda)^2$$

thus

$$\frac{2\lambda}{1-\lambda} = \overline{n}$$

$$\Rightarrow \lambda = \frac{\frac{\overline{n}}{2}}{1+\frac{\overline{n}}{2}}$$

and

$$a_0 = (1-\lambda)^2 = \frac{1}{\left(1+\frac{\overline{n}}{2}\right)^2}$$

It follows that

$$F_n = (n+1)\left(\frac{\frac{\overline{n}}{2}}{1+\frac{\overline{n}}{2}}\right)^n \frac{1}{\left(1+\frac{\overline{n}}{2}\right)^2}$$

which is the negative binomial distribution for $k = 2$.

Appendix 3.

The solution of the linear recurrence relation

$$F_{n+3} + c_1 F_{n+2} + c_2 F_{n+1} + c_3 F_n = 0, \quad n \geq 0$$

is given by

$$F_n = (a_0 + a_1 n + a_2 n^2)\lambda^n$$

in the case of a single root λ of multiplicity 3 (see text).

There are four constants a_0, a_1, a_2 and λ to be determined while there are two constraints (1) and (2). One needs two more constraints which will be determined once the constraint (1) is worked out.

The constraint (1) leads to

$$\sum_{n=0}^{\infty} F_n = \sum_{n=0}^{\infty}(a_0 + a_1 n + a_2 n^2)\lambda^n = a_0 \sum_{n=0}^{\infty}\lambda^n + a_1 \sum_{n=0}^{\infty} n\lambda^n + a_2 \sum_{n=0}^{\infty} n^2\lambda^n$$
$$= 1$$

Now

$$n^2 = n(n-1) + n$$

and then

$$\begin{aligned}
\sum_{n=0}^{\infty} F_n &= a_0 \sum_{n=0}^{\infty}\lambda^n + a_1 \sum_{n=0}^{\infty} n\lambda^n + a_2 \sum_{n=0}^{\infty} n^2\lambda^n = a_0 \sum_{n=0}^{\infty}\lambda^n + a_1 \sum_{n=0}^{\infty} n\lambda^n + a_2 \sum_{n=0}^{\infty}[n(n-1)+n]\lambda^n \\
&= a_0 \sum_{n=0}^{\infty}\lambda^n + (a_1+a_2)\sum_{n=0}^{\infty} n\lambda^n + a_2 \sum_{n=0}^{\infty} n(n-1)\lambda^n = a_0 \sum_{n=0}^{\infty}\lambda^n + (a_1+a_2)\lambda\sum_{n=0}^{\infty} n\lambda^{n-1} + a_2\lambda^2 \sum_{n=0}^{\infty} n(n-1)\lambda^{n-2} \\
&= a_0 \sum_{n=0}^{\infty}\lambda^n + (a_1+a_2)\lambda\sum_{n=0}^{\infty}\left(\frac{d}{d\lambda}\lambda^n\right) + a_2\lambda^2 \sum_{n=0}^{\infty}\left(\frac{d^2}{d\lambda^2}\lambda^n\right) \\
&= a_0 \sum_{n=0}^{\infty}\lambda^n + (a_1+a_2)\lambda\frac{d}{d\lambda}\left(\sum_{n=0}^{\infty}\lambda^n\right) + a_2\lambda^2 \frac{d^2}{d\lambda^2}\left(\sum_{n=0}^{\infty}\lambda^n\right) \\
&= a_0\frac{1}{1-\lambda} + (a_1+a_2)\lambda\frac{d}{d\lambda}\left(\frac{1}{1-\lambda}\right) + a_2\lambda^2\frac{d^2}{d\lambda^2}\left(\frac{1}{1-\lambda}\right) = a_0\frac{1}{1-\lambda} + (a_1+a_2)\lambda\frac{1}{(1-\lambda)^2} + a_2\lambda^2\frac{2}{(1-\lambda)^3} \\
&= \frac{a_0(1-\lambda)^2 + (a_1+a_2)\lambda(1-\lambda) + 2a_2\lambda^2}{(1-\lambda)^3} = \frac{a_0 + \lambda(-2a_0 + a_1 + a_2) + \lambda^2(a_0 - a_1 + a_2)}{(1-\lambda)^3} \\
&= 1
\end{aligned}$$

Now by setting in the numerator the coefficients of λ and λ^2 to zero one obtains two additional constraints

$$-2a_0 + a_1 + a_2 = 0$$
$$a_0 - a_1 + a_2 = 0$$

and as a consequence

$$a_0 = (1-\lambda)^3$$

For the solution of the above system of linear equations one finds $a_0 = 2a_2$ and $a_1 = 3a_2$. Therefore

$$\begin{aligned} F_n &= a_2(2 + 3n + n^2)\lambda^n \\ &= a_2(n+1)(n+2)\lambda^n \end{aligned}$$

The constraint (2) leads to

$$\begin{aligned} \sum_{n=0}^{\infty} nF_n = \sum_{n=0}^{\infty} a_2 n(n+1)(n+2)\lambda^n &= a_2\lambda \sum_{n=0}^{\infty} n(n+1)(n+2)\lambda^{n-1} = a_2\lambda \sum_{n=0}^{\infty}\left(\frac{d^3}{d\lambda^3}\lambda^{n+2}\right) = a_2\lambda\frac{d^3}{d\lambda^3}\left(\sum_{n=0}^{\infty}\lambda^{n+2}\right) \\ &= a_2\lambda\frac{d^3}{d\lambda^3}\left(\lambda^2\sum_{n=0}^{\infty}\lambda^n\right) = a_2\lambda\frac{d^3}{d\lambda^3}\left(\frac{\lambda^2}{1-\lambda}\right) = a_2\lambda\frac{d^3}{d\lambda^3}\left(-\lambda - 1 + \frac{1}{1-\lambda}\right) = a_2\lambda\frac{6}{(1-\lambda)^4} \\ &= \overline{n} \end{aligned}$$

Now

$$\begin{aligned} a_2 &= \frac{a_0}{2} \\ &= \frac{(1-\lambda)^3}{2} \end{aligned}$$

Thus

$$\begin{aligned} \sum_{n=0}^{\infty} nF_n &= \frac{3\lambda}{1-\lambda} \\ &= \overline{n} \end{aligned}$$

from which one deduces

$$\lambda = \frac{\frac{\overline{n}}{3}}{1+\frac{\overline{n}}{3}}$$

and

$$\begin{aligned} a_0 &= (1-\lambda)^3 \\ &= \frac{1}{\left(1+\frac{\overline{n}}{3}\right)^3} \end{aligned}$$

Finally

$$\begin{aligned} F_n &= a_2(n+1)(n+2)\lambda^n \\ &= \frac{a_0}{2}(n+1)(n+2)\lambda^n \\ &= \frac{1}{2\left(1+\frac{\overline{n}}{3}\right)^3}(n+1)(n+2)\left(\frac{\frac{\overline{n}}{3}}{1+\frac{\overline{n}}{3}}\right)^n \\ &= \frac{(n+1)(n+2)}{2}\left(\frac{\frac{\overline{n}}{3}}{1+\frac{\overline{n}}{3}}\right)^n\frac{1}{\left(1+\frac{\overline{n}}{3}\right)^3} \end{aligned}$$

which is the expression of the negative binomial distribution for $k = 3$.

Appendix 4.

The expression for a_j will be evaluated by the Snake Oil method [7] which makes use of the generating functions. The expression for a_j is given by

$$a_j = a_{k-1}(k-1)!\sum_{i=j}^{k-1} s(i,j)\frac{C_{k-1}^i}{i!}$$

Now the binomial coefficient C_{k-1}^{i} vanishes for $i > k-1$ and then the expression for a_j may be written as

$$\begin{aligned} a_j &= a_{k-1}(k-1)! \sum_{i=j}^{k-1} s(i,j) \frac{C_{k-1}^{i}}{i!} \\ &= a_{k-1}(k-1)! \sum_{i=j}^{\infty} s(i,j) \frac{C_{k-1}^{i}}{i!} \\ &= a_{k-1}(k-1)! \sum_{i \geq j} s(i,j) \frac{C_{k-1}^{i}}{i!} \end{aligned}$$

To evaluate this expression with the Snake Oil method we set

$$\begin{aligned} f(k,j) &= \frac{a_j}{a_{k-1}} \\ &= (k-1)! \sum_{i \geq j} s(i,j) \frac{C_{k-1}^{i}}{i!} \end{aligned}$$

This expression is then multiplied by $\frac{x^{k-1}}{(k-1)!}$ and summed over k

$$\begin{aligned} \sum_{k \geq 1} f(k,j) \frac{x^{k-1}}{(k-1)!} &= \sum_{k \geq 1} \left[(k-1)! \sum_{i \geq j} s(i,j) \frac{C_{k-1}^{i}}{i!} \right] \frac{x^{k-1}}{(k-1)!} \\ &= \sum_{k \geq 1} \sum_{i \geq j} s(i,j) \frac{C_{k-1}^{i}}{i!} x^{k-1} \end{aligned}$$

The two summations on the RHS are then interverted so that

$$\sum_{k \geq 1} f(k,j) \frac{x^{k-1}}{(k-1)!} = \sum_{i \geq j} \frac{s(i,j)}{i!} \sum_{k \geq 1} C_{k-1}^{i} x^{k-1}$$

Now using the generating function technique [7]

$$\sum_{k \geq 1} C_{k-1}^{i} x^{k-1} = \frac{x^i}{(1-x)^{i+1}}$$

And then

$$\begin{aligned} \sum_{k \geq 1} f(k,j) \frac{x^{k-1}}{(k-1)!} &= \sum_{i \geq j} \frac{s(i,j)}{i!} \frac{x^i}{(1-x)^{i+1}} \\ &= \frac{1}{1-x} \sum_{i \geq j} \frac{s(i,j)}{i!} \frac{x^i}{(1-x)^i} \\ &= \frac{1}{1-x} \sum_{i \geq j} \frac{s(i,j)}{i!} u^i \end{aligned}$$

where $u = \frac{x}{1-x}$.
Using again the generating function technique [7]

$$\sum_{i \geq j} \frac{s(i,j)}{i!} u^i = \frac{[ln(1+u)]^j}{j!}$$

which is the expression for the generating function of the unsigned Stirling numbers of the first kind. It follows that

$$\sum_{k \geq 1} f(k,j) \frac{x^{k-1}}{(k-1)!} = \frac{1}{1-x} \frac{[ln(1+u)]^j}{j!}$$

Replacing u by its expression $u = \frac{x}{1-x}$ leads to

$$\begin{aligned} \sum_{k \geq 1} f(k,j) \frac{x^{k-1}}{(k-1)!} &= \frac{1}{1-x} \frac{\left[ln\left(1 + \frac{x}{1-x}\right) \right]^j}{j!} \\ &= \frac{1}{1-x} \frac{\left[ln\left(\frac{1}{1-x}\right) \right]^j}{j!} \\ &= \frac{d}{dx} \left\{ \frac{\left[ln\left(\frac{1}{1-x}\right) \right]^{j+1}}{(j+1)!} \right\} \end{aligned}$$

Now the expression between braces is the generating function for the unsigned Stirling numbers of the first kind e.g.

$$\frac{\left[ln\left(\frac{1}{1-x}\right)\right]^{j+1}}{(j+1)!} = \sum_{i\geq j+1} |s(i,j+1)|\frac{x^i}{i!}$$

It then follows that

$$\sum_{k\geq 1} f(k,j)\frac{x^{k-1}}{(k-1)!} = \frac{d}{dx}\left[\sum_{i\geq j+1} |s(i,j+1)|\frac{x^i}{i!}\right]$$
$$= \sum_{i\geq j+1} |s(i,j+1)|\frac{x^{i-1}}{(i-1)!}$$

Now by picking up the coefficient of $\frac{x^{k-1}}{(k-1)!}$ on the RHS one is lead to

$$f(k,j) = |s(k,j+1)|$$

and then

$$a_j = a_{k-1}|s(k,j+1)|$$

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