Analytic Functions for Clifford Algebras

Celebrating the 200^{th} anniversary of Cauchy Integration Theorem

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To the Memory of Augustin Louis Cauchy

Abstract. In this article, the Cauchy theory is applied and extended to n dimensional functions in (Clifford) algebras.
I already touched on this in [3] for path integration of classical fields, which might not be evident. So, I publish the details here as a celebration of Cauchy's outstanding lecture "Sur les intégrales définies", held on the 22nd of August 1814, so 200 years ago, which has been re/published in 1825, and is publicly accessible online in [2].

1. Introduction

At the end of the 19th century, Henri Poincaré discovered as was then to be known as Poincaré's Lemma; it states that on star-shaped open regions closed differential forms are necessarily exact (see [1]). This triggered the beginning of algebraic geometry, which became one of the most important branches of mathematics, especially in France.
It is often overlooked that Poincaré's intention at that time was not the abstract development of a theory of cohomologies, but he wanted to unravel the curious nature of electromagnetic fields. With the help of this lemma, he could show that (in 3 dimensions) an electromagnetic field of zero divergence is the curl of a vector potential $\mathbf{A}$, which in turn is fundamental to derive gauge invariance and the Lorentz representation of Maxwell's equations.

In 1895 Volterra showed that Poincaré's Lemma extends as the equivalence of closed and exact differential forms, and Elie Cartan independently rediscovered this a decade or two later. As can be seen from [1], for 1-forms that means that an n-tuple of continuously differential function $(f_1, \dots, f_n)$ on an open, star-shaped region $U \subset \mathbb{R}^n$ is integrable (i.e. defines an exact 1-form) in that region, if and only if its derivative (the Jacobi matrix) is symmetric.

Obviously, that is not what Cauchy understood as what an integrable function should look like: Cauchy examined a pair of differentable real-valued functions (u, v) on $\mathbb{R}^2$, and then he came to a remarkable solution: He defined $u(x+iy) := u(x, y), v(x+iy) := v(x, y)$, and then he defined a complex-valued function f on $U \subset \mathbb{C}$ as $f : U \ni x + iy \mapsto u(x + iy) + iv(x + iy) \in \mathbb{C}$. He then proved that f is integrable in U if it is complex differentiable, that complex differentiability is equivalent to analyticity, and more over, this is equivalent to the Jacobi matrix of $(u(x, y), v(x, y))$ to be anti-symmetric in its off-diagonal elements, namely to follow the Cauchy-Riemann equations $\partial u/\partial x = \partial v/\partial y$, and $\partial u/\partial y = -\partial v/\partial x$!

This not only antedates Poincaré's differential geometric results 80 years later, even then, Cauchy's work appears to be ahead of that time:
Let's examine the reason for the seemingly controversial results of the conditions of integrability:
The answer is that Poincaré is integrating within Euclidean geometry, whereas Cauchy is integrating in the complex plane. There are two reasons in favour of Cauchy's technique: Firstly, one cannot divide by a vector of two or more dimensions, but one can divide by complex numbers. It is this clever substitution $(x, y) \mapsto x + iy$ that allowed Cauchy to rigorously define (complex) differentiability. Secondly, by doing so, Cauchy instantaneously carried out the path integration in a vector space $\mathbb{C}$ with its intrinsic hyperbolic metric, and not in the Euclidean metric:

Whereas Poincaré used $(a_1, a_2) \cdot (b_1, b_2) := a_1 b_1 + a_2 b_2$ as inner product, for Cauchy it is $(a_1, ia_2) \cdot (b_1, ib_2) := a_1 b_1 - a_2 b_2$.
That explains, why Cauchy's results lead to the unsymmetric Jacobi matrix, whereas Poincaré's Jacobi matrix is to be symmetric. So, Cauchy was also the first one to carry integration out in hyperbolic vector spaces, something that Poincaré himself never thought of, even after he and H.A. Lorentz derived the covariant Maxwell equations, which A. Einstein and H. Minkowski then proved to be a consequence of space-time being hyperbolic, rather than Euclidean!

2. Preliminaries: Clifford Algebras

I want to deal with 2 or more dimensions of complex numbers. Then, according to Cauchy, I have to represent these vectors as numbers, in order to be able to divide by these. Because only then, I'll be able to go with his strong notion of differentiability.

The technique to use has been readily exposed by Hermann Graßman and William Kingdon Clifford:

Let X be an n-dimensional real vector space with $n \geq 1$, and let Q be a non-degenerate quadratic form on X. This means that one can find a linear basis $a_1, \ldots, a_n \in X$, w.r.t. which Q is defined through a symmetric, invertable $n \times n$-matrix A. Then there is an orthogonal transformation O on $\mathbb{R}^n$, i.e.: $O^{-1} = O^t$, where $U^t_{ij} := O_{jt}$ is the transpose of U, such that OAO^{-1} is a diagonal matrix with real eigenvalues $\lambda_1, \ldots \lambda_n \in \mathbb{R}^n$, and by scaling the basis elements a_j with a positive factor $|\lambda_j|^-1$, we arrive at: every non-degenerate quadratic form in n dimensions defines an orthonormal basis, and it falls into one of n possible categories: $(n, 0), (n-1, 1), \ldots, (0, n)$, where $(p, n-p)$ signifies that the first p eigenvalues are $+1$, and the $n-p$ others are -1. This is termed the *signature* of the quadratic form.

Proposition 2.1. *For each $n \in \mathbb{N}$ there is an $m \in \mathbb{N}$ and n $m \times m$-matrices $\alpha_1, \ldots, \alpha_n$, such that $\big(\alpha_1 + \cdots + \alpha_n\big)^2 = \alpha_1^2 + \cdots + \alpha_n^2$, where $\alpha_1^2 = \cdots = \alpha_n^2 = \mathbb{I}_m$ is the $m \times m$-unit matrix.*

Proof. The statement is trivial for $n = 1$. So, let $n > 1$. Then suitable matrices can be picked from the vector space of all endomorphisms on the $n \times n$-dimensional space of $n \times n$-matrices. □

Remark 2.2. Actually, it can be shown that $m = 2^{n/2}$ is the minimal m, if n is even, and then $m = 2^{(n+1)/2}$ will do for uneven n, but that is irrelevant for now.

Definition 2.3. For a non-degenerate quadratic form of signature $(p, n-p)$, the matrices $\alpha_1, \ldots, \alpha_p, i\alpha_{p+1}, \ldots, i\alpha_n$ are called the *generators* of the Clifford algebra $Cl_{p,n-p}(\mathbb{R})$, where $\alpha_1, \ldots, \alpha_n$ are as in the preceding proposition. $Cl_{p,n-p}(\mathbb{R})$ is defined as the (non-commutative) algebra of all real linear combinations of the α_k and all products of these. Let $X(n)$ be the vector space over $\mathbb{R}$ spanned by the α_k. $Cl_{p,n-p}(\mathbb{R})$ becomes a (finite dimensional) Banach space, when equipped with its natural supremum norm

$$\|\cdot\| : x \mapsto \sup_{\|\chi\| \leq 1} x(\chi),$$

and $X(n)$ then becomes a closed subspace of $Cl_{p,n-p}(\mathbb{R})$.

Remark 2.4. Although $Cl_{p,n-p}(\mathbb{R})$ is a vector space over the field of real numbers, it captures complex fields, since $\mathbb{C}^n$ isomorphically embeds into $Cl_{n,n}(\mathbb{R})$.

The important point now is that $\sum \lambda_k \alpha_k$ is an invertible matrix, if and only if $(\lambda_k)_{1 \leq k \leq n}$ is unequal zero. Therefore x is invertible for all $x \in X(n) \setminus \{0\}$, and right as well as left division are well-defined on $Cl_{p,n-p}(\mathbb{R})$ for every $x \in X(n) \setminus \{0\}$. This allows

Definition 2.5. A continuous mapping $f : X(n) \supset U \to Cl_{p,n-p}(\mathbb{R})$ of an open subset $U \subset X(n)$ is said to be *differentiable* in $x_0 \in U$ if and only if $\lim_{x \to x_0}(f(x) - f(x_0))(x - x_0)^{-1}$ and $\lim_{x \to x_0}(x - x_0)^{-1}(f(x) - f(x_0))$ both exist and are equal. For short, I'll write this limit as $f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$.

Remark 2.6. An analytic function, which vanishes along any one of its n axis, must be zero throughout.

Remark 2.7. Also, given two differentiable, $Cl_{p,n-p}(\mathbb{R})$-valued functions f, g on an open $U \subset X(n)$, the product $fg : z \mapsto f(z)g(z) \in Cl_{p,n-p}(\mathbb{R})$ is differentiable either, and $(fg)'(z) = d(f(z)g(z))/dz = f'(z)g(z) + f(z)g'(z)$.

To get rid of the factors i for the following, let me write $\gamma_k := \alpha_k$, whenever the metrics is positive for that component, and $\gamma_k := i\alpha_k$, when it is negative.

3. Proceeding with Cauchy's theorems

We are now in the position to harvest the fruit from Cauchy's work:

Definition 3.1. Let $U \subset X(n)$ be an open subset of $X(n)$ and $z_0 \in U$. A function $f : U \ni z \mapsto Cl_{p,n-p}(\mathbb{R})$ is called *analytic* in z_0, if and only if there is a neighbourhood $V(z_0) \subset U$ of z_0, such that $f(z) = \sum_{k\geq 0} a_k(z - z_0)^k$ for all $z \in V(z_0)$, where $a_0, a_1, \cdots \in \mathbb{R}$.

For a strictly positive, real valued $r > 0$, let us define the r-ball $B_r(0) := \{z = (z_1, \ldots, z_n) \in X_n : \sum_{1\leq k\leq n} \left|z_j\right|^2 \leq r^2\}$, and let $S_n(r)$ be its boundary.

Then the $B_\epsilon(0)$, $\epsilon > 0$, are base of zero neighbourhoods of $X(n)$, which means that for every open neighbourhood U of 0 there is some $\epsilon > 0$ such that $B_\epsilon(0) \subset U$.

Definition 3.2. Although with $z = \sum_k x_k\gamma_k$ with $x_k \in \mathbb{R}$ the derivative $f'(z)$ is well-defined, the partial derivatives $\partial f(z)/\partial z_k$ are *not*: Generally, when f is differentiable in z, right and left partial derivatives will be unequal! In order to deal with partial derivatives, we need to confine to partial derivatives to the right, which will be denoted by $\partial_r/\partial x_k$, $(1 \leq k \leq n)$.

A function $If : X_n \supset U \to Cl_{p,n-p}(\mathbb{R})$ will be called integral of $f : U \to Cl_{n,n-p}(\mathbb{C})$, if $(If)'(z) = f(z)$ for all $z \in U$. More generally, for $k \in \mathbb{N}$, the k-th integral of f is a function $I^{(k)}f : U \to Cl_{p,n_p}(\mathbb{R})$, such that $d^k f(z)/dz^k = f(z)$ for all $z \in U$.

Proposition 3.3. *Let $f : U \to Cl_{p,n-p}(\mathbb{R})$ be differentiable. Then, writing $z = z_1\gamma_1 + \cdots + z_n\gamma_n$ for $z \in U$,*

1. $\partial f(z)/\partial z_k = \partial f(z)/\partial z_l \gamma_l^{-1}\gamma_k$, $(1 \leq k \neq l \leq n)$.
2. *If f is twice differentiable, then* $\partial^2 f(z)/\partial z_k \partial z_l = -\partial^2 f(z)/\partial z_l \partial z_k$, $(1 \leq k \neq l \leq n)$.

Proof. We have $\partial f(z)/\partial z_k = f'(z)\partial z/\partial z_k = f'(z)\gamma_k$ for all k, from which the statement 1 follows. Taking a second partial derivative, delivers 2. $\square$

Remark 3.4. For $Cl_{1,1}(\mathbb{R})$, the above equations are the Cauchy-Riemann equations.

We want to extend the Cauchy integral theorem from 1 to $n \in \mathbb{N}$ dimensions. So, we need to have a notion of surface integration over the unit ball:

First off, the r-sphere is the sphere in Euclidean metrics, not in the hyperbolic metric of $Cl_{p,n-p}$ where $p > 1$. That is, integration must be w.r.t $dx = \sum_j \alpha_j dx_j$, but not w.r.t. $dz = \sum_j \gamma_j dx_j$, even when f is an analytic function of $z = \sum_j \gamma_j x_j = \sum_{j \leq p} \alpha_j x_j + \sum_{j>p} i\alpha_j x_j$. What needs to be done, is to embed the $Cl_{p,n-p}$-valued functions f on X_n into functions $\tilde{f}$ on $\mathbb{R}^n$ and integrate these by means of Euclidean differential forms:
As above, let $U \subset X_n$ be open and $f : U \ni z = \sum_j \gamma_j x_j \mapsto f(z) \in Cl_{p,n-p}$ be analytic. Let $\tilde{U} \subset \mathbb{R}^n$ be the set of all $(x_j)_{1 \leq j \leq n}$, such that $\sum_j \gamma_j x_j \in U$. Then $\tilde{f} : \tilde{U} \ni (x_1, \ldots, x_n) \mapsto f(\alpha_1 x_1, \ldots, \alpha_p x_p, i\alpha_{p+1} x_{p+1}, \ldots, i\alpha_n x_n) \in Cl_{p,n-p}$ is smooth, and for $1 \leq m \leq n$ the differential m-form $dx_{k_1} \wedge \cdots \wedge dx_{k_m}$ defines the orientated integral $\int_{\tilde{S}} \tilde{f}(x_1, \ldots, ix_m) dx_{k_1} \wedge \cdots \wedge dx_{k_m}$ of a m-dimensional, plane, compact region $\tilde{S} \subset \tilde{U}$, where the sign of the integration is chosen positive, if $(k_1, \ldots, k_m)$ is a cyclic subpartition of $(1, \ldots, n)$ and negative otherwise. We can now define: $\int_S f(\alpha_1 x_1, \ldots, i\alpha_n x_n) \alpha_{k_1} dx_{k_1} \cdots \alpha_{k_m} dx_{k_m} := \int_{\tilde{S}} \tilde{f}(x_1, \ldots, ix_n) dx_{k_1} \wedge \cdots \wedge dx_{k_m}$, where S is the set of all $\sum_j \gamma_j x_j \in X_n$ with $(x_1, \ldots, x_n) \in \tilde{S}$.

In particular, the n-dimensional volume differential becomes the ordered product $d^n x := \alpha_1 dx_1 \cdots \alpha_n dx_n$, and with $V(r) := \{y \in \mathbb{R}^n | y_1^2 + \cdots + y_n^2 = 1\}$, we have

$$\int_{B_r(0)} f(\sum_j \alpha_j x_j) d^n x = \int_{V(r)} \tilde{f}(x_1, \ldots, x_n) dx_1 \cdots dx_n.$$

In case that $n - p$ elements $\gamma_{p+1}, \ldots, \gamma_n$ have a negative signature, up to an additional factor i^{n-p} we get the same result:

$$\int_{B_r(0)} f(\alpha_1 x_1 + \cdots + \alpha_p x_p + i\alpha_{p+1} x_{p+1} + \cdots + i\alpha_n x_n) d^n x$$
$$= i^{n-p} \int_{V(r)} \tilde{f}(x_1, \ldots, x_n) dx_1 \cdots dx_n.$$

Analogously, given an analytic $f : U \to Cl_{p,n-p}$, $n > 1$, and $S_n(r) \subset U$ the (n-1)-dimensional sphere of an n-dimensional ball, then the surface integral of f over $S_n(r)$ is given by:

$$\int_{S_n(r)} f(z) d^{n-1} a = \int_{S_n(r)} f(z) \sum_{1 \leq k_1 \neq k2 \neq \cdots \neq k_{n-1} \leq n} \alpha_{k_1} dx_{k_1} \cdots \alpha_{k_{n-1}} dx_{n-1}.$$

Let's rewrite this into a simpler form:
We have: $r^2 = (\sum_j \alpha_j x_j)^2$, where $r \geq 0$ is the Euclidean radius, so that $dr = \frac{x}{r} dx$ and $dx = \frac{r}{x} dr$, where $x = \sum_j \alpha_j x_j$.

With this, let $F(r) := \int_{B_r(0)} f(\sum_j \alpha_j x_j)$ be the integral of a function $f : U \to Cl_{n,0}(\mathbb{R})$, where U is an open subset of X_n containing $B_r(0)$ such that $F(r)$ exists in $Cl_{n,0}$. Then the surface integral of f over $S_n(r)$ can be defined as: $\int_{S_n(r)} f(\sum_j \alpha_j x_j) d^{n-1} a := d(F(r)\frac{r}{x})/dx$. Again, if $f : U \to Cl_{p,n-p}(\mathbb{R})$, we get the same result with an additional factor i^{n-p}.

Armed with that, it is simple to prove

Lemma 3.5 (Cauchy Theorem). *For $n > 1$ let $f : U \to Cl_{p,n-p}(\mathbb{R})$ be analytic on an open subset $U \subset X_n$, $r > 0$, $B_r(z_0) \subset U$ be the closed r-ball around $z_0 \in U$, and let $S_r(z_0)$ be its $n-1$-dimensional boundary. Then the surface integral of f over $S_r(z_0)$ exists and vanishes, i.e.: $\int_{S_r(z_0)} f d^{2n-1}a = 0$.*

Proof. Because of local compactness of $\mathbb{R}^n$, it suffices to prove that statement for $r \to 0$. To prove existence for $p = n$, notice that f defines a smooth function $\tilde{f} : V(r) = \{x \in \mathbb{R}^n | \sum_j x_j^2 \leq r^2\} \ni y \mapsto f(y_1, \ldots, y_n) \in Cl_{n,0}$, so $F(r) = \int_{B_r(0)} \tilde{f}(y_1, \ldots, y_n) dy_1 \cdots dy_n$ exists and is differentiable in r. With an extra factor i^{n-p}, the existence then extends to f being an analytic function to $Cl_{p,n-p}$. The theorem then follows from $\left| int_{S_r(z_0)} f d^{2n-1}a \right| \leq S_n max_{z \in B_r(z_0)} \left| f(z) \right| r^{n-1}$, where S_n is the area of the unit sphere, which goes to 0 (even with degree r^{n-1}) as $r \to 0$. □

Proposition 3.6 (Cauchy Integral). *Let $f : U \to Cl_{p,n-p}(\mathbb{R})$ be analytic on an open subset $U \subset X_n$, $r > 0$, with $n > 1$ as in the above lemma. Then for $z_0 \in U$:*

$$\int_{S_r(z_0)} f(z)(z_0 - z)^{-1(n-1)} d^{n-1}a = i^{n-p} S_n f(z_0),$$

where again S_n denotes the area of the $n-1$-dimensional unit sphere.

Remark 3.7. Note that care must be taken as to whether the factor $(z - z_0)^{-n+1}$ goes to the left or the right of $f(z)$: if $f(z) = \gamma_j \lambda(z)$ for some j and some $\lambda(z) \in \mathbb{R}$, for instance, then, depending on $(z - z_0)$ and the dimension n, $(z - z_0)^n$ may anticommute with $f(z)$; only in case of f being real (or complex) valued, commutativity is ensured. In what follows, the factor $(z - z_0)^{-n+1}$ will be understood to be taken to the right of $f(z)$.

Proof. It suffices to prove the proposition for $z_0 = 0$.
We first contend that the statement holds for $f \equiv 1$: Let V_n be the volume of the n-dimensional unit ball $B_n(1)$. Then the volume of the r-ball $B_n(r)$ is given by $V_n r^n$. So, the function $\tilde{f} : \mathbb{R}^n \ni x \mapsto \frac{1}{|x|^n - 1} = 1/r^{n-1}$ has a volume integral over $B_n(r)$ given by rS_n, where S_n is the area of the unit sphere. So, the surface integral of $\tilde{f}$ over the sphere $S_r(0)$ of the r-ball $B_n(r)$ is S_n.
Now, if $n - 1$ is even, given $z = \sum_j \alpha_j x_j$, we have $r^{n-1} = z^{n-1}$; so for $f \equiv 1$ and even $n - 1 > 1$ the proposition already holds. So, let $n - 1$ be odd, i.e. $n \geq 2$ be even. Then $\frac{1}{z^{n-1}} = \frac{1}{r^{n-2}z}$, and because of $rdr = zdz$ with $z := (\sum_j \alpha_j x_j)$ and $dz := (\sum_j \alpha_j dx_j)$, $\frac{1}{r^{n-2}z} dz = \frac{1}{r^{n-1}} dr$. So, again, on the r-sphere $S_r(0)$, $\int_{S_r(0)} \frac{1}{z^{n-1}} da = \int_{S_r(0)} \frac{1}{r^{n-1}} da$, and it gives S_n, the surface area of the unit n-ball. This proves proposition for $f \equiv 1$. In general, if f is analytic in the origin, we have

$$\left| \int_{S_r(0)} (f(0)/r^{n-1} - f(z)/z^{n-1}) da \right| \leq \left| \int_{S_r(0)} r^{-n-2} (f(0)/r - f(z)/z) da \right|,$$

which converges to zero as $r \to 0$, because f is analytic in 0, and therefore $f(z) = f'(0)z + o(|z|)$. $\square$

The subsequent steps again are the same as in Cauchy's analysis:

Lemma 3.8. *Let* $1 : \mathbb{R}^n \to Cl_{p,n-p}(\mathbb{R})$ *be the constant function* $1(x) \equiv 1$. *Then* $\int_{S_r(0)} 1/z^k = 0$ *for* $k \geq n$.

Proof. As we can interchange the order of integration due to continuous differentiability, the integration over the $n-2$ polar angles can be carried out first, resulting in a smooth function $f(z)/z^{k-n+1}$ (outside 0). This function can be integrated to a function F, and the integral is the difference of F at start and end point. Because start and end point coincide for last azymuthal integration (from 0 to 2π), the lemma holds. $\square$

As an immediate consequence:

Corollary 3.9. *If* $f : U \to Cl_{p,n-p}(\mathbb{R})$ *be analytic on an open subset* $U \subset X_n$, $r > 0$, *with* $n > 1$, $z_0 \in U$, *and* $B_r(z_0) \subset U$. *Then for* $k \in \mathbb{N} \cup \{0\}$*:*

$$\int_{S_r(z_0)} f(z)(z_0 - z)^{-1(n-1+k)} d^{n-1}a = \frac{i^{n-p} S_n}{k!} \frac{d^k f(z_0)}{d^k z}.$$

This can now be extended to analytic functions $f(z)/(z - z_0)^k$, where f is analytic in an environment of z_0, and further to meromorphic functions, in line with Cauchy's theory of complex numbers.

Perhaps more importantly, the above corollary implies that if f is continuously differentiable in a point $z \in Cl_{p,n-p}(\mathbb{R})$, then it is analytic in z_0: Because $\int_{S_r(x_0)} f(z)(z - z_0)^{n-1} da$ then exists for sufficiently small $r > 0$, and as that is an analytic function in z outside $\{z_0\}$, $f(z_0)$ is the uniform limit of that function to $\{z_0\}$. So, the power series expansion converges uniformly as $z \to z_0$, which implies analyticity (the proof by itself being a copy of Cauchy's theory).

4. Complex Clifford Algebra

In all above, I restricted on a real-valued algebra, in which the elements $z = \sum_k x_k \gamma_k$ are such that $x_k \in \mathbb{R}$, rather than complex numbers. Of course, I could have replaced the real numbers x_k by complex numbers z_k from the beginning, but then one would be tempted to integrate over $2n-1$ dimensions, splitting the real part from the imaginary part of only one of the n complex dimensions, which was misleading:

In the definition of $Cl_{p,n-p}(\mathbb{R})$ we made a strict distinction between the components of positive signature, $x_1\alpha_1, \ldots, x_p\alpha_p$ and the $n-p$ ones with negative signature, $x_{p+1} i\alpha_{p+1}, \ldots, x_n i\alpha_n$. But at the end, we saw that up to a factor i there is no difference between both signs of signature of each component. So, if we have the α_k phase rotated by an extra factor $e^{i\phi_k}$, each, i.e. substitute $\alpha_k \to e^{i\phi_k}\alpha_k$, then in the Cauchy integration, we would get the same result, up to an additional factor $e^{i\phi_1+\cdots+i\phi_n}$. So, by letting the x_k become complex, the distinction of positive/negative signature (or metrics) becomes irrelevant, we can base all computations on the positive $\alpha_1,\ldots,\alpha_n$, still retain the results Cauchy's theory (without the factor i^{n-p}), and $Cl_{p,n-p}(\mathbb{R})$ simplifies to $Cl_n(\mathbb{C})$!

5. Conclusion

Let me come back to H. Poincaré's motivation: He was wondering what the Lorentz condition

$$\partial_0 A_0(x_0,\ldots,x_3)+\cdots+\partial_3 A_3(x_0,\ldots,x_3)=0$$

meant to electrodynamics. The A_μ represent electromagnetic vector field j_0 on regions of $\mathbb{R}^4$, so are to be taken as real-valued (and smooth) functions. Therefore they can be extended to the complex by defining

$$j_\mu(z_0,\ldots,z_3):=j_\mu(x+iy):=j_\mu(x)-ij_\mu(y),(0\le\mu\le 3).$$

Poincaré knew that the Lorentz condition stated nothing but a conservation law of some relativistically invariant energy. And he knew that this law was relativistically invariant. But why then does it get lost, when the Maxwell equations are transformed by Lorentz transformation and has to be regained anew by what is called a gauge?

Let us now accept that space-time is not Euclidean, but to be described in Minkowski metrics, instead, so is of signature $(1,3)$. So, let's rewrite:

$$f_\mu(z_0\gamma_0+\cdots+z_3\gamma_3)\gamma_\mu:=j_\mu(z_0,\ldots,z_3),(0\le\mu\le 3).$$

Then the Lorentz condition becomes

$$\sum_{0\le\mu\le 3}\partial(f_\mu(z_0\gamma_0+\cdots+z_3\gamma_3)\gamma_\mu)/\partial(z_\mu\gamma_\mu)=0.$$

By the above, we know that $(f_\mu\gamma_\mu)_{0\le\mu\le 3}$ is integrable to a function $F:X_n\ni z=z_0\gamma_0+z_n\gamma_3\mapsto F(z)\in Cl_{1,3}(\mathbb{R})$ if and only if $\partial_\mu j_\nu=-\partial_\nu j_\mu$ for $0\le\mu\ne\nu\le 3$. That means that $dF(z)/dz=\sum_{0\le\mu\le 3}f_\mu(z)\gamma_\mu$. So, $\partial F(z)/\partial(z_\mu\gamma_\mu)=f_\mu(z)\gamma_\mu$, hence the Lorentz condition enforces:

$$\Box F(z):=(\partial_0^2-\cdots-\partial_3^2)F(z)=0.$$

Note that integrability here means integrability in the Clifford algebra, but not in the standard Euclidean space: it's $\gamma_\mu\partial_\mu f_\nu\gamma_\nu=\gamma_\nu\partial_\nu f_\mu\gamma_\mu$, and it's *not* $\partial_\mu f_\nu=\partial_\nu f_\mu$ for $0\le\mu\ne\nu\le 3$!

In other words: the mapping

$$\Psi:\sum_{0\le\mu\le 3}\gamma_\mu f_\mu(\gamma_0 z_0+\cdots+\gamma_3 z_3)\mapsto(f_0(z_0,\ldots,z_3),\ldots,f_3(z_0,\ldots,z_3))$$

maps spinor-functions that are integrable within the Clifford algebra into Euclidean vector fields with a generally nontrivial rotation, which are therefore not integrable within the Euclidean space - the electromagnetic fields just fall into this class: see [3] for details.

References

[1] H. Cartan, *Differential Forms* Herman Kershaw, 1971.

[2] A. L. Cauchy, *Memoires sur les intégrales définies*, `https://archive.org/details/mmoiresurlesin00cauc`, 1825.

[3] H. D. Hüttenbach, *The Action Function of Adiabatic Systems*, `http://vixra.org/abs/1404.0441`, 2014.

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