The formula of number of twin prime

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Abstract

If $\pi_t(6n+1)$ is the number of twin prime of $6n+1$ or less then the formula of $\pi_t(6n+1)$ is described below.

$$\pi_t(6n+1) = n + 1 - \frac{2}{3}\sum_{k=1}^{n}\left(\frac{\pi\beta_t(6k)}{\pi\beta_t(6k)-1}\right) - \frac{2}{3\pi}\sum_{k=1}^{n}\sum_{m=1}^{\infty}\frac{\sin\left(\frac{2m\pi^2\beta_t(6k)}{\pi\beta_t(6k)-1}\right)}{m}$$

$$\text{where}, \beta_t(6k) = \{\tau(6k-1)-2\} + \{\tau(6k+1)-2\}, \ldots$$

1. Introduction

We build the formula to discriminate twin prime and we study to build the formula of the number of twin prime of N or less by using $\rho(N)$ defined in paper "The formula of $\pi(N)$" [1] of myself. And, we study to express the formula of the sequence of ordered pair of twin prime.

In addition, we express the formula of $\pi_t(6n+1)$ with $\pi(6n+1)$ by using apple box principle in theorem 6.

2. The formula of number of twin prime

Definition 1. We apply same definition of paper "The formula of $\pi(N)$" [1] of myself.

Definition 2. For arbitrary d

Let us define $\beta_t(6n) = \begin{cases} 0, & if \text{ all of } 6n-1, 6n+1 \text{ is prime, that is, twin prime} \\ d, & if \text{ one or more of } 6n-1, 6n+1 \text{ is composite number} \end{cases}$

Definition 3. Let us define

$$\rho_t(6n) = \begin{cases} 0, & if \text{ all of } 6n-1, 6n+1 \text{ is twin prime} \\ 1, & if \text{ one or more of } 6n-1, 6n+1 \text{ is composite number} \end{cases}$$

Definition 4. Let us define $\pi_t(N)$ as the number of twin prime of N or less.

Theorem 1. $\boldsymbol{\beta_t(6n)}$

$\beta_t(6n)$ could be used by the one of formula below.

$\beta_t(6n) = \beta(6n-1) + \beta(6n+1)$

$\beta_t(6n) = \rho(6n-1) + \rho(6n+1)$

$\beta_t(6n) = \{l(6n-1) - l(6(n-1)-1)\} + \{l(6n+1) - l(6(n-1)+1)\}$

$$= \left\{\left(\sum_{p=1}^{\left[\frac{n-1}{5}\right]}\left[\frac{n+p}{6p+1}\right] + \sum_{p=1}^{\left[\frac{n+1}{7}\right]}\left[\frac{n-p}{6p-1}\right]\right) - \left(\sum_{p=1}^{\left[\frac{(n-1)-1}{5}\right]}\left[\frac{(n-1)+p}{6p+1}\right] + \sum_{p=1}^{\left[\frac{(n-1)+1}{7}\right]}\left[\frac{(n-1)-p}{6p-1}\right]\right)\right\}$$

$$+ \left\{\left(\sum_{p=1}^{\left[\frac{n-1}{7}\right]}\left[\frac{n-p}{6p+1}\right] + \sum_{p=1}^{\left[\frac{n+1}{5}\right]}\left[\frac{n+p}{6p-1}\right]\right) - \left(\sum_{p=1}^{\left[\frac{(n-1)-1}{7}\right]}\left[\frac{(n-1)-p}{6p+1}\right] + \sum_{p=1}^{\left[\frac{(n-1)+1}{5}\right]}\left[\frac{(n-1)+p}{6p-1}\right]\right)\right\}$$

$\beta_t(6n) = \{r(6n-1) - r(6(n-1)-1)\} + \{r(6n+1) - r(6(n-1)+1)\}$

$$= \left\{\begin{array}{c}\left(\sum_{p=1}^{\left[\frac{\sqrt{6n}}{6}\right]}\left(\left[\frac{n+p}{6p+1}\right] - (p-1)\right) + \sum_{p=1}^{\left[\frac{\sqrt{6n}}{6}\right]}\left(\left[\frac{n-p}{6p-1}\right] - (p-1)\right)\right) \\ -\left(\sum_{p=1}^{\left[\frac{\sqrt{6(n-1)}}{6}\right]}\left(\left[\frac{(n-1)+p}{6p+1}\right] - (p-1)\right) + \sum_{p=1}^{\left[\frac{\sqrt{6(n-1)}}{6}\right]}\left(\left[\frac{(n-1)-p}{6p-1}\right] - (p-1)\right)\right)\end{array}\right\}$$

$$+ \left\{\begin{array}{c}\left(\sum_{p=1}^{\left[\frac{-1+\sqrt{6n+1}}{6}\right]}\left(\left[\frac{n-p}{6p+1}\right] - (p-1)\right) + \sum_{p=1}^{\left[\frac{1+\sqrt{6n+1}}{6}\right]}\left(\left[\frac{n+p}{6p-1}\right] - (p-1)\right)\right) \\ -\left(\sum_{p=1}^{\left[\frac{-1+\sqrt{6(n-1)+1}}{6}\right]}\left(\left[\frac{(n-1)-p}{6p+1}\right] - (p-1)\right) + \sum_{p=1}^{\left[\frac{1+\sqrt{6(n-1)+1}}{6}\right]}\left(\left[\frac{(n-1)+p}{6p-1}\right] - (p-1)\right)\right)\end{array}\right\}$$

$\beta_t(6n) = \{\tau(6n-1) - 2\} + \{\tau(6n+1) - 2\}$

$$= \sum_{p=1}^{6n-1}\left(\left[\frac{6n-1}{p}\right] - \left[\frac{6n-2}{p}\right]\right) + \sum_{p=1}^{6n+1}\left(\left[\frac{6n+1}{p}\right] - \left[\frac{6n}{p}\right]\right) - 4$$

$\beta_t(6n) = \{\sigma(6n-1) - (1 + 6n - 1)\} + \{\sigma(6n+1) - (1 + 6n + 1)\}$

Proof 1.

(prime = green, twin prime = red)

n	N=6n-1									N=6n+1								
	N	P=6p+1				P=6p-1				N	P=6p+1				P=6p-1			
		p=1	2	3	…	p=1	2	3	…		p=1	2	3	…	p=1	2	3	…
1	5									7								
2	11									13								
3	17									19								
4	23									25					5x5			
5	29									31								
6	35	7x5				5x7				37								
7	41									43								
8	47									49	7x7							
9	53									55					5x11	11x5		
10	59									61								
11	65		13x5			5x13				67								
12	71									73								
13	77	7x11					11x7			79								
14	83									85					5x17		17x5	
15	89									91	7x13	13x7						
16	95			19x5		5x19				97								
…																		

(Table 1.1) multiple, prime, twin prime of $N = 6n \pm 1$ type

We express the multiple of $N = 6n \pm 1$ type in the table 1.1. We display the row that the multiple does not exist to green and we display the twin prime that all of $6n - 1, 6n + 1$ is prime(green row) to red.

According to "definition 4 in paper The formula of $\pi(N)$" [1] of myself,
let us define if $6n-1$ is composite then $\beta(6n-1)=a$,
if $6n+1$ is composite then $\beta(6n+1)=b$.
If all of $6n-1, 6n+1$ is composite then

$$\beta(6n-1)+\beta(6n+1)=a+b, \rho(6n-1)+\rho(6n+1)=2$$

If one of $6n-1, 6n+1$ is composite then

$$\beta(6n-1)+\beta(6n+1)=a \text{ or } b, \rho(6n-1)+\rho(6n+1)=1$$

If all of $6n-1, 6n+1$ is prime,that is,twin prime then

$$\beta(6n-1)+\beta(6n+1)=0, \rho(6n-1)+\rho(6n+1)=0$$

So, $\beta(6n-1)+\beta(6n+1), \rho(6n-1)+\rho(6n+1)$ is satisfied with the definition of $\beta_t(6n)$.
Therefore, $\beta_t(6n)=\beta(6n-1)+\beta(6n+1), \beta_t(6n)=\rho(6n-1)+\rho(6n+1)$.

According to "definition 4, definition 6, theorem 2 in paper The formula of $\pi(N)$" [1] of myself,
if $6n-1$ is composite then $l(6n-1)-l(6(n-1)-1)>0, r(6n-1)-r(6(n-1)-1)>0$
if $6n-1$ is prime then $l(6n-1)-l(6(n-1)-1)=0, r(6n-1)-r(6(n-1)-1)=0$,
if $6n+1$ is composite then $l(6n+1)-l(6(n-1)+1)>0, r(6n+1)-r(6(n-1)+1)>0$
if $6n+1$ is prime then $l(6n+1)-l(6(n-1)+1)=0, r(6n+1)-r(6(n-1)+1)=0$

Therefore, if all of $6n-1, 6n+1$ is prime then

$$\{l(6n-1)-l(6(n-1)-1)\}+\{l(6n+1)-l(6(n-1)+1)\}=0,$$

$$\{r(6n-1)-r(6(n-1)-1)\}+\{r(6n+1)-r(6(n-1)+1)\}=0,$$

if one or more of $6n-1, 6n+1$ is composite then

$$\{l(6n-1)-l(6(n-1)-1)\}+\{l(6n+1)-l(6(n-1)+1)\}>0,$$

$$\{r(6n-1)-r(6(n-1)-1)\}+\{r(6n+1)-r(6(n-1)+1)\}>0$$

Therefore, all of $\{l(6n-1)-l(6(n-1)-1)\}+\{l(6n+1)-l(6(n-1)+1)\}$, $\{r(6n-1)-r(6(n-1)-1)\}+\{r(6n+1)-r(6(n-1)+1)\}$ is satisfied with the definition of $\beta_t(6n)$.
Therefore,
$\beta_t(6n)=\{l(6n-1)-l(6(n-1)-1)\}+\{l(6n+1)-l(6(n-1)+1)\}$

$$=\left\{\left(\sum_{p=1}^{\left[\frac{n-1}{5}\right]}\left[\frac{n+p}{6p+1}\right]+\sum_{p=1}^{\left[\frac{n+1}{7}\right]}\left[\frac{n-p}{6p-1}\right]\right)-\left(\sum_{p=1}^{\left[\frac{(n-1)-1}{5}\right]}\left[\frac{(n-1)+p}{6p+1}\right]+\sum_{p=1}^{\left[\frac{(n-1)+1}{7}\right]}\left[\frac{(n-1)-p}{6p-1}\right]\right)\right\}$$

$$+\left\{\left(\sum_{p=1}^{\left[\frac{n-1}{7}\right]}\left[\frac{n-p}{6p+1}\right]+\sum_{p=1}^{\left[\frac{n+1}{5}\right]}\left[\frac{n+p}{6p-1}\right]\right)-\left(\sum_{p=1}^{\left[\frac{(n-1)-1}{7}\right]}\left[\frac{(n-1)-p}{6p+1}\right]+\sum_{p=1}^{\left[\frac{(n-1)+1}{5}\right]}\left[\frac{(n-1)+p}{6p-1}\right]\right)\right\}$$

$$\beta_t(6n)=\{r(6n-1)-r(6(n-1)-1)\}+\{r(6n+1)-r(6(n-1)+1)\}$$

$$=\left\{\begin{array}{c}\left(\sum_{p=1}^{\left[\frac{\sqrt{6n}}{6}\right]}\left(\left[\frac{n+p}{6p+1}\right]-(p-1)\right)+\sum_{p=1}^{\left[\frac{\sqrt{6n}}{6}\right]}\left(\left[\frac{n-p}{6p-1}\right]-(p-1)\right)\right)\\ -\left(\sum_{p=1}^{\left[\frac{\sqrt{6(n-1)}}{6}\right]}\left(\left[\frac{(n-1)+p}{6p+1}\right]-(p-1)\right)+\sum_{p=1}^{\left[\frac{\sqrt{6(n-1)}}{6}\right]}\left(\left[\frac{(n-1)-p}{6p-1}\right]-(p-1)\right)\right)\end{array}\right\}$$

$$+\left\{\begin{array}{c}\left(\sum_{p=1}^{\left[\frac{-1+\sqrt{6n+1}}{6}\right]}\left(\left[\frac{n-p}{6p+1}\right]-(p-1)\right)+\sum_{p=1}^{\left[\frac{1+\sqrt{6n+1}}{6}\right]}\left(\left[\frac{n+p}{6p-1}\right]-(p-1)\right)\right)\\ -\left(\sum_{p=1}^{\left[\frac{-1+\sqrt{6(n-1)+1}}{6}\right]}\left(\left[\frac{(n-1)-p}{6p+1}\right]-(p-1)\right)+\sum_{p=1}^{\left[\frac{1+\sqrt{6(n-1)+1}}{6}\right]}\left(\left[\frac{(n-1)+p}{6p-1}\right]-(p-1)\right)\right)\end{array}\right\}$$

If all of $6n-1, 6n+1$ is prime then

$$\{\tau(6n-1)-2\}+\{\tau(6n+1)-2\}=0+0=0,$$

$$\{\sigma(6n-1)-(1+6n-1)\}+\{\sigma(6n+1)-(1+6n+1)\}=0+0=0$$

If one or more of $6n-1, 6n+1$ is composite then

$$\{\tau(6n-1)-2\}+\{\tau(6n+1)-2\}>0,$$

$$\{\sigma(6n-1)-(1+6n-1)\}+\{\sigma(6n+1)-(1+6n+1)\}>0$$

Therefore,

$\{\tau(6n-1)-2\}+\{\tau(6n+1)-2\}$, $\{\sigma(6n-1)-(1+6n-1)\}+\{\sigma(6n+1)-(1+6n+1)\}$ is also satisfied with the definition of $\beta_t(6n)$ and according to "theorem 2 in paper The formula of $\pi(N)$" [1] of myself.

$$\beta_t(6n)=\{\tau(6n-1)-2\}+\{\tau(6n+1)-2\}$$

$$=\sum_{p=1}^{6n-1}\left(\left[\frac{6n-1}{p}\right]-\left[\frac{6n-2}{p}\right]\right)+\sum_{p=1}^{6n+1}\left(\left[\frac{6n+1}{p}\right]-\left[\frac{6n}{p}\right]\right)-4$$

$$\beta_t(6n)=\{\sigma(6n-1)-(1+6n-1)\}+\{\sigma(6n+1)-(1+6n+1)\}$$

■

Theorem 2. $\boldsymbol{\rho_t(6n)}$

$$\rho_t(6n) = \left[\frac{\beta_t(6n)+1}{2}\right] = \left[\frac{\rho(6n-1)+\rho(6n+1)+1}{2}\right]$$

$$\rho_t(6n) = \left[\frac{\beta_t(6n)}{\beta_t(6n)-w}\right], 0 < w < \frac{1}{2}, w \in \overline{\mathbb{R}}, w = \frac{1}{e}, \frac{1}{\pi}, \frac{1}{N} (N > 2), \dots$$

If $6n-1, 6n+1$ is not twin prime then

$$\rho_t(6n) = \frac{\beta_t(6n)}{\beta_t(6n)-w} - \frac{1}{2} + \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{sin\left(\frac{2k\pi\beta_t(6n)}{\beta_t(6n)-w}\right)}{k}$$

If $6n-1, 6n+1$ is twin prime then

$$\rho_t(6n) = \left\{\frac{\beta_t(6n)}{\beta_t(6n)-w} - \frac{1}{2} + \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{sin\left(\frac{2k\pi\beta_t(6n)}{\beta_t(6n)-w}\right)}{k}\right\} + \frac{1}{2}$$

Especially, if $=\frac{1}{\pi}$,

when $6n-1, 6n+1$ is not twin prime ,

$$\rho_t(6n) = \frac{\pi\beta_t(6n)}{\pi\beta_t(6n)-1} - \frac{1}{2} + \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{sin\left(\frac{2k\pi^2\beta_t(6n)}{\pi\beta_t(6n)-1}\right)}{k}$$

when $6n-1, 6n+1$ is twin prime ,

$$\rho_t(6n) = \left\{\frac{\pi\beta_t(6n)}{\pi\beta_t(6n)-1} - \frac{1}{2} + \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{sin\left(\frac{2k\pi^2\beta_t(6n)}{\pi\beta_t(6n)-1}\right)}{k}\right\} + \frac{1}{2}$$

Proof 2.

Because $\beta_t(6n) = \rho(6n-1) + \rho(6n+1)$ according to theorem 1,

If all of $6n-1, 6n+1$ is composite then

$$\beta_t(6n) = \rho(6n-1) + \rho(6n+1) = 2 \rightarrow \left[\frac{\beta_t(6n)+1}{2}\right] = \left[\frac{2+1}{2}\right] = 1$$

If one of $6n-1, 6n+1$ is composite then

$$\beta_t(6n) = \rho(6n-1) + \rho(6n+1) = 1 \rightarrow \left[\frac{\beta_t(6n)+1}{2}\right] = \left[\frac{1+1}{2}\right] = 1$$

If all of $6n-1, 6n+1$ is prime,that is,twin prime then

$$\beta_t(6n) = \rho(6n-1) + \rho(6n+1) = 0 \rightarrow \left[\frac{\beta_t(6n)+1}{2}\right] = \left[\frac{0+1}{2}\right] = 0$$

So, $\left[\frac{\beta_t(6n)+1}{2}\right]$ is satisfied with the definition of $\rho_t(6n)$

Therefore, $\rho_t(6n) = \left[\frac{\beta_t(6n)+1}{2}\right] = \left[\frac{\rho(6n-1)+\rho(6n+1)+1}{2}\right]$

And, if we define $\rho_t(6n) = \left[\frac{\beta_t(6n)}{\beta_t(6n)-w}\right]$ like as "theorem 3 in paper The formula of $\pi(N)$"

[1]of myself then for $0 < w < \frac{1}{2}, w \in \mathbb{R}$

when all of $6n-1, 6n+1$ is prime , $\rho_t(6n) = \left[\frac{\beta_t(6n)}{\beta_t(6n)-w}\right] = \left[\frac{0}{0-w}\right] = 0$

when one of $6n-1, 6n+1$ is composite , $\rho_t(6n) = \left[\frac{\beta_t(6n)}{\beta_t(6n)-w}\right] = \left[\frac{1}{1-w}\right] = 1$

when all of $6n-1, 6n+1$ is composite, $\rho_t(6n) = \left[\frac{\beta_t(6n)}{\beta_t(6n)-w}\right] = \left[\frac{2}{2-w}\right] = 1$

$\rho_t(6n) = \left[\frac{\beta_t(6n)}{\beta_t(6n)-w}\right]$ and $w = \frac{1}{e}, \frac{1}{\pi}, \frac{1}{N}(N > 2), \ldots$ (detail proof is omitted)

And, when $6n-1, 6n+1$ is not twin prime, because $\beta_t(6n) > 0, 0 < w < \frac{1}{2}, w \in \mathbb{R}$

$1 < \frac{\beta_t(6n)}{\beta_t(6n)-w} < 2$, that is, $\frac{\beta_t(6n)}{\beta_t(6n)-w} \in \mathbb{R}$ and

for arbitrary $x \in \mathbb{R}$ $[x] = x - \frac{1}{2} + \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{sin(2k\pi x)}{k}$

[2],so,

$$\rho_t(6n)=\left[\frac{\beta_t(6n)}{\beta_t(6n)-w}\right]=\frac{\beta_t(6n)}{\beta_t(6n)-w}-\frac{1}{2}+\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{sin\left(\frac{2k\pi\beta_t(6n)}{\beta_t(6n)-w}\right)}{k}$$

When $6n-1,6n+1$ is twin prime, because $\beta_t(6n)=0$

$$\frac{\beta_t(6n)}{\beta_t(6n)-w}-\frac{1}{2}+\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{sin\left(\frac{2k\pi\beta_t(6n)}{\beta_t(6n)-w}\right)}{k}=\frac{0}{0-w}-\frac{1}{2}+\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{sin\left(2k\pi\frac{0}{0-w}\right)}{k}=-\frac{1}{2},$$

And, because $\rho_t(6n)=0$

$$\rho_t(6n)=0=-\frac{1}{2}+\frac{1}{2}=\left\{\frac{\beta_t(6n)}{\beta_t(6n)-w}-\frac{1}{2}+\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{sin\left(\frac{2k\pi\beta_t(6n)}{\beta_t(6n)-w}\right)}{k}\right\}+\frac{1}{2}$$

Especially, if $=\frac{1}{\pi}$,

when $6n-1,6n+1$ is not twin prime,

$$\rho_t(6n)=\frac{\beta_t(6n)}{\beta_t(6n)-\frac{1}{\pi}}-\frac{1}{2}+\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{sin\left(\frac{2k\pi\beta_t(6n)}{\beta_t(6n)-\frac{1}{\pi}}\right)}{k}$$

$$=\frac{\pi\beta_t(6n)}{\pi\beta_t(6n)-1}-\frac{1}{2}+\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{sin\left(\frac{2k\pi^2\beta_t(6n)}{\pi\beta_t(6n)-1}\right)}{k}$$

when $6n-1,6n+1$ is twin prime,

$$\rho_t(6n)=0=-\frac{1}{2}+\frac{1}{2}=\left\{\frac{\pi\beta_t(6n)}{\pi\beta_t(6n)-1}-\frac{1}{2}+\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{sin\left(\frac{2k\pi^2\beta_t(6n)}{\pi\beta_t(6n)-1}\right)}{k}\right\}+\frac{1}{2}$$

■

Theorem 3. $\boldsymbol{\pi_t(6n+1)}$

For $0 < w < \frac{1}{2}, w \in \overline{\mathbb{R}}, w = \frac{1}{e}, \frac{1}{\pi}, \frac{1}{N} (N > 2), \ldots$

$$\pi_t(6n+1) = n + 1 - \sum_{k=1}^{n} \rho_t(6k) = n + 1 - \frac{2}{3}\sum_{k=1}^{n}\left(\frac{\beta_t(6k)}{\beta_t(6k) - w}\right) - \frac{2}{3\pi}\sum_{k=1}^{n}\sum_{m=1}^{\infty}\frac{sin\left(\frac{2m\pi\beta_t(6k)}{\beta_t(6k)-w}\right)}{m}$$

$$= n + 1 - \frac{2}{3}\sum_{k=1}^{n}\left(\frac{\pi\beta_t(6k)}{\pi\beta_t(6k) - 1}\right) - \frac{2}{3\pi}\sum_{k=1}^{n}\sum_{m=1}^{\infty}\frac{sin\left(\frac{2m\pi^2\beta_t(6k)}{\pi\beta_t(6k)-1}\right)}{m}$$

Proof 3.

(3,5) is only that are not twin prime in $6k-1, 6k+1$ type of $6n+1$ or less,and

if $6k-1, 6k+1$ is twin prime then $1 - \rho_t(6n) = 1 - 0 = 1$,

if one or more of $6k-1, 6k+1$ is composite then $1 - \rho_t(6n) = 1 - 1 = 0$

So,

$$\pi_t(6n+1) = 1 + \sum_{k=1}^{n}\{1 - \rho_t(6k)\} = 1 + \sum_{k=1}^{n} 1 - \sum_{k=1}^{n}\rho_t(6k) = 1 + n - \sum_{k=1}^{n}\rho_t(6k)$$

$$= n + 1 - \sum_{k=1}^{n}\rho_t(6k) \text{ -------- } (3.1)$$

And, let us define $\mathbb{P}$ as a set which $6k-1, 6k+1$ is twin prime,

let us define $\mathbb{C}$ as a set which $6k-1, 6k+1$ is not twin prime, and let us define

$$A = \frac{\beta_t(6k)}{\beta_t(6k) - w} - \frac{1}{2} + \frac{1}{\pi}\sum_{m=1}^{\infty}\frac{sin\left(\frac{2m\pi\beta_t(6k)}{\beta_t(6k)-w}\right)}{m}$$

According to theorem 2, if $6k-1, 6k+1$ is not twin prime then $\rho_t(6k) = A$,

if $6k-1, 6k+1$ is twin prime then $\rho_t(6k) = A + \frac{1}{2}$,and

let us express $\sum_{\mathbb{Z}}^{n} u(k)$ with the sum of $u(k)$, only if $u(k) \in \mathbb{Z}$ in $1 \le k \le n$ for a certain $u(k)$, $\mathbb{Z}$

because $\mathbb{P} \cap \mathbb{C} = \emptyset$,

$$\sum_{k=1}^{n}\rho_t(6k) = \sum_{\mathbb{P}}^{n}\rho_t(6k) + \sum_{\mathbb{C}}^{n}\rho_t(6k) \text{ -------- } (3.2)$$

So, if we apply (3.2) to (3.1) then

$$\pi_t(6n+1) = n+1-\left(\sum_{\mathbb{P}}^{n}\rho_t(6k)+\sum_{\mathbb{C}}^{n}\rho_t(6k)\right) = n+1-\left(\sum_{\mathbb{P}}^{n}\left(A+\frac{1}{2}\right)+\sum_{\mathbb{C}}^{n}A\right)$$

$$= n+1-\left(\sum_{\mathbb{P}}^{n}\left(\frac{1}{2}\right)+\sum_{\mathbb{P}}^{n}A+\sum_{\mathbb{C}}^{n}A\right) \text{-------- (3.3)}$$

$$\sum_{\mathbb{P}}^{n}\left(\frac{1}{2}\right) = \frac{1}{2}\sum_{\mathbb{P}}^{n}1 = \frac{\pi_t(6n+1)-1}{2}, \sum_{\mathbb{P}}^{n}A+\sum_{\mathbb{C}}^{n}A = \sum_{k=1}^{n}A,$$

so,if we apply this to (3.3) then

$$\pi_t(6n+1) = n+1-\left(\frac{\pi_t(6n+1)-1}{2}+\sum_{k=1}^{n}A\right) = n+1-\frac{\pi_t(6n+1)-1}{2}-\sum_{k=1}^{n}A \text{-------- (3.4)}$$

If we substitute A to (3.4) and arrange then

$$\pi_t(6n+1)+\frac{\pi_t(6n+1)}{2} = n+1+\frac{1}{2}-\sum_{k=1}^{n}A \rightarrow \frac{3\pi_t(6n+1)}{2} = n+\frac{3}{2}-\sum_{k=1}^{n}A \rightarrow$$

$$\pi_t(6n+1) = \frac{2}{3}\left\{n+\frac{3}{2}-\sum_{k=1}^{n}\left(\frac{\beta_t(6k)}{\beta_t(6k)-w}-\frac{1}{2}+\frac{1}{\pi}\sum_{m=1}^{\infty}\frac{sin\left(\frac{2m\pi\beta_t(6k)}{\beta_t(6k)-w}\right)}{m}\right)\right\}$$

$$= \frac{2}{3}\left\{n+\frac{3}{2}-\sum_{k=1}^{n}\left(-\frac{1}{2}\right)-\sum_{k=1}^{n}\left(\frac{\beta_t(6k)}{\beta_t(6k)-w}\right)-\frac{1}{\pi}\sum_{k=1}^{n}\sum_{m=1}^{\infty}\frac{sin\left(\frac{2m\pi\beta_t(6k)}{\beta_t(6k)-w}\right)}{m}\right\}$$

$$= \frac{2}{3}\left\{\frac{3n}{2}+\frac{3}{2}-\sum_{k=1}^{n}\left(\frac{\beta_t(6k)}{\beta_t(6k)-w}\right)-\frac{1}{\pi}\sum_{k=1}^{n}\sum_{m=1}^{\infty}\frac{sin\left(\frac{2m\pi\beta_t(6k)}{\beta_t(6k)-w}\right)}{m}\right\}$$

$$= n+1-\frac{2}{3}\sum_{k=1}^{n}\left(\frac{\beta_t(6k)}{\beta_t(6k)-w}\right)-\frac{2}{3\pi}\sum_{k=1}^{n}\sum_{m=1}^{\infty}\frac{sin\left(\frac{2m\pi\beta_t(6k)}{\beta_t(6k)-w}\right)}{m} \text{-------- (3.5)}$$

If we substitute $w = \frac{1}{\pi}$ to (3.5) especially, then

$$\pi_t(6n+1) = n+1-\frac{2}{3}\sum_{k=1}^{n}\left(\frac{\beta_t(6k)}{\beta_t(6k)-\frac{1}{\pi}}\right)-\frac{2}{3\pi}\sum_{k=1}^{n}\sum_{m=1}^{\infty}\frac{sin\left(\frac{2m\pi\beta_t(6k)}{\beta_t(6k)-\frac{1}{\pi}}\right)}{m}$$

$$= n+1-\frac{2}{3}\sum_{k=1}^{n}\left(\frac{\pi\beta_t(6k)}{\pi\beta_t(6k)-1}\right)-\frac{2}{3\pi}\sum_{k=1}^{n}\sum_{m=1}^{\infty}\frac{sin\left(\frac{2m\pi^2\beta_t(6k)}{\pi\beta_t(6k)-1}\right)}{m}$$

■

Theorem 4. Next twin prime of $6n \pm 1$ type

If we define $P = 6p \pm 1$ as be the arbitrary twin prime of $6n \pm 1$ type, and if we define $X = 6x \pm 1$ as be the first twin prime of $6n \pm 1$ type after P.

$$x = p + 1 + \sum_{k=p+1}^{x-1} \rho_t(6k)$$

$$= p + 1 + \frac{1}{2}\sum_{k=p+1}^{x-1} \frac{\pi\beta_t(6k)+1}{\pi\beta_t(6k)-1} + \frac{1}{\pi}\sum_{k=p+1}^{x-1}\sum_{m=1}^{\infty}\frac{sin\left(\frac{2m\pi^2\beta_t(6k)}{\pi\beta_t(6k)-1}\right)}{m}$$

$$= p + 1 + \sum_{k=p+1}^{x} \rho_t(6k)$$

$$= p + \frac{3}{2} + \frac{1}{2}\sum_{k=p+1}^{x} \frac{\pi\beta_t(6k)+1}{\pi\beta_t(6k)-1} + \frac{1}{\pi}\sum_{k=p+1}^{x}\sum_{m=1}^{\infty}\frac{sin\left(\frac{2m\pi^2\beta_t(6k)}{\pi\beta_t(6k)-1}\right)}{m}$$

Proof 4.

Let us $P = 6p \pm 1$ as be the arbitrary twin prime of $6n \pm 1$ type, and let us define $X = 6x \pm 1$ as be the first twin prime of $6n \pm 1$ type after P.

$\rho_t(6k) = 1$ because $6k + 1$ or $6k - 1$ is a composite number in $p < k < x$ and

$\rho_t(6k) = 0$ because $6x \pm 1$ is a prime number. Therefore,

$$x = \sum_{k=1}^{x} 1 = \sum_{k=1}^{p} 1 + \sum_{k=p+1}^{x-1} 1 + \sum_{k=x}^{x} 1 = p + \sum_{k=p+1}^{x-1} \rho_t(6k) + 1$$

$$= p + 1 + \sum_{k=p+1}^{x-1} \rho_t(6k)$$

$$= \sum_{k=1}^{p} 1 + \sum_{k=p+1}^{x-1} 1 + \sum_{k=x}^{x} 1 + \sum_{k=x}^{x} 0 = \sum_{k=1}^{p} 1 + \sum_{k=x}^{x} 1 + \sum_{k=p+1}^{x-1} 1 + \sum_{k=x}^{x} 0$$

$$= p + 1 + \sum_{k=p+1}^{x-1} \rho_t(6k) + \sum_{k=x}^{x} \rho_t(6k) = p + 1 + \sum_{k=p+1}^{x} \rho_t(6k)$$

And,for $p < k < x$, $\rho_{twin}(6k) = \left[\frac{\beta_t(6k)}{\beta_t(6k)-w}\right]$, $1 < \frac{\beta_t(6k)}{\beta_t(6k)-w} < 2$,that is, $\frac{\beta_t(6k)}{\beta_t(6k)-w} \in \overline{\mathbb{R}}$,

so,according to theorem 2,if we arrange the above formula then

$$x = p + 1 + \sum_{k=p+1}^{x-1} \rho_t(6k)$$

$$= p + 1 + \sum_{k=p+1}^{x-1} \left\{ \frac{\pi\beta_t(6k)}{\pi\beta_t(6k) - 1} - \frac{1}{2} + \frac{1}{\pi} \sum_{m=1}^{\infty} \frac{sin\left(\frac{2m\pi^2\beta_t(6k)}{\pi\beta_t(6k) - 1}\right)}{m} \right\}$$

$$= p + 1 + \sum_{k=p+1}^{x-1} \left\{ \frac{1}{2}\left(\frac{\pi\beta_t(6k) + 1}{\pi\beta_t(6k) - 1}\right) + \frac{1}{\pi} \sum_{m=1}^{\infty} \frac{sin\left(\frac{2m\pi^2\beta_t(6k)}{\pi\beta_t(6k) - 1}\right)}{m} \right\}$$

$$= p + 1 + \frac{1}{2} \sum_{k=p+1}^{x-1} \frac{\pi\beta_t(6k) + 1}{\pi\beta_t(6k) - 1} + \frac{1}{\pi} \sum_{k=p+1}^{x-1} \sum_{m=1}^{\infty} \frac{sin\left(\frac{2m\pi^2\beta_t(6k)}{\pi\beta_t(6k) - 1}\right)}{m}$$

$$= p + 1 + \frac{1}{2} \sum_{k=p+1}^{x-1} \frac{\pi\beta_t(6k) + 1}{\pi\beta_t(6k) - 1} + \frac{1}{\pi} \sum_{k=p+1}^{x-1} \sum_{m=1}^{\infty} \frac{sin\left(\frac{2m\pi^2\beta_t(6k)}{\pi\beta_t(6k) - 1}\right)}{m}$$

$$= p + 1 + \sum_{k=p+1}^{x-1} \rho_t(6k) + \sum_{k=x}^{x} \rho_t(6k) = p + 1 + \sum_{k=p+1}^{x} \rho_t(6k)$$

$$= p + 1 + \frac{1}{2} \sum_{k=p+1}^{x} \frac{\pi\beta_t(6k) + 1}{\pi\beta_t(6k) - 1} + \frac{1}{\pi} \sum_{k=p+1}^{x} \sum_{m=1}^{\infty} \frac{sin\left(\frac{2m\pi^2\beta_t(6k)}{\pi\beta_t(6k) - 1}\right)}{m} + \sum_{k=x}^{x} \frac{1}{2}$$

$$= p + \frac{3}{2} + \frac{1}{2} \sum_{k=p+1}^{x} \frac{\pi\beta_t(6k) + 1}{\pi\beta_t(6k) - 1} + \frac{1}{\pi} \sum_{k=p+1}^{x} \sum_{m=1}^{\infty} \frac{sin\left(\frac{2m\pi^2\beta_t(6k)}{\pi\beta_t(6k) - 1}\right)}{m}$$

■

Theorem 5. Sequence of ordered pair of twin prime of $6n \pm 1$ type

If we define the sequence of ordered pair of twin prime of $6n \pm 1$ type as $\{(6p_1 - 1, 6p_1 + 1), (6p_2 - 1, 6p_2 + 1), \ldots\}$, that is, $\{(5,7), (11,13), (17,19), \ldots\}$ then the following formula is always true for all positive integer n

$$p_{n+1} = p_n + 1 + \sum_{k=p_n+1}^{p_{n+1}} \rho_t(6k)$$

Proof 5.

Let us define the sequence of ordered pair of twin prime of $6n \pm 1$ type as
$\{(6p_1 - 1, 6p_1 + 1), (6p_2 - 1, 6p_2 + 1), \ldots\}$,that is,$\{(5,7), (11,13), (17,19), \ldots\}$

If we define below (5.1) according to theorem 4 then

$$p_{n+1} = p_n + 1 + \sum_{k=p_n+1}^{p_{n+1}} \rho_t(6k) \text{------------- (5.1)}$$

When $n = 1$, the first twin prime is $(5,7)$ and $p_1 = 1$, the second twin prime is $(11,13)$ and $p_2 = 2$.

$$p_2 = 2 = p_1 + 1 + \sum_{k=p_1+1}^{p_2} \rho_t(6k) = 1 + 1 + \sum_{k=1+1}^{2} \rho_t(6k) = 1 + 1 + \rho_t(6 \times 2)$$

$$= 1 + 1 + 0 = 2$$

So, (5.1) is true when $n = 1$.

When $m = n$, if we suppose that (5.1) is true then

$$p_{m+1} = p_m + 1 + \sum_{k=p_m+1}^{p_{m+1}} \rho_t(6k) \text{------------- (5.2)}$$

Because $(6k - 1, 6k + 1)$ for k, $p_{m+1} < k < p_{m+2}$ is all composite,so $\rho_t(6k) = 1$ and because $(6p_{m+2} - 1, 6p_{m+2} + 1)$ is twin prime,so $\rho_t(6p_{m+2}) = 0$. Therefore,

$$\sum_{k=p_{m+1}+1}^{p_{m+2}} \rho_t(6k) = \sum_{k=p_{m+1}+1}^{p_{m+2}-1} \rho_t(6k) + \sum_{k=p_{m+2}}^{p_{m+2}} \rho_t(6k) = \sum_{k=p_{m+1}+1}^{p_{m+2}-1} 1 + \sum_{k=p_{m+2}}^{p_{m+2}} 0$$

$$= p_{m+2} - p_{m+1} - 1, \text{so,}$$

$$\sum_{k=p_{m+1}+1}^{p_{m+2}} \rho_t(6k) = p_{m+2} - p_{m+1} - 1 \text{------------- (5.3)}$$

If we add $\sum_{k=p_{m+1}+1}^{p_{m+2}} \rho_t(6k)$

to both sides of (5.2) then

$$p_{m+1} + \sum_{k=p_{m+1}+1}^{p_{m+2}} \rho_t(6k) = p_m + 1 + \sum_{k=p_m+1}^{p_{m+1}} \rho_t(6k) + \sum_{k=p_{m+1}+1}^{p_{m+2}} \rho_t(6k) \text{ ------------- } (5.4)$$

If we substitute (5.3) to left side of (5.4) then

$$p_{m+1} + p_{m+2} - p_{m+1} - 1 = p_{m+2} - 1$$

$$= p_m + 1 + \sum_{k=p_m+1}^{p_{m+1}} \rho_t(6k) + \sum_{k=p_{m+1}+1}^{p_{m+2}} \rho_t(6k) \text{ ------------- } (5.5)$$

Because $p_m = p_{m+1} - 1 - \sum_{k=p_m+1}^{p_{m+1}} \rho_t(6k)$

from (5.2), if we substitute this formula to p_m of (5.5) then

$$p_{m+2} - 1 = p_{m+1} - 1 - \sum_{k=p_m+1}^{p_{m+1}} \rho_t(6k) + 1 + \sum_{k=p_m+1}^{p_{m+1}} \rho_t(6k) + \sum_{k=p_{m+1}+1}^{p_{m+2}} \rho_t(6k)$$

Therefore,

$$p_{m+2} = p_{m+1} + 1 + \sum_{k=p_{m+1}+1}^{p_{m+2}} \rho_t(6k) \text{ ------------- } (5.6)$$

And, if we sustitue $m+1$ to m of (5.2) then

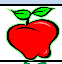
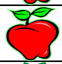
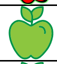
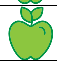
$$p_{m+1+1} = p_{m+1} + 1 + \sum_{k=p_{m+1}+1}^{p_{m+1+1}} \rho_t(6k) \rightarrow$$

$$p_{m+2} = p_{m+1} + 1 + \sum_{k=p_{m+1}+1}^{p_{m+2}} \rho_t(6k) \text{ ------------- } (5.7)$$

Therefore, (5.1) is always true for all positive integer n, because (5.6) is same as (5.7),

■

Theorem 6. Apple box principle

Space of Box	Apple Box A	Apple Box B
1 (R space)		
2 (M space)		
3 (G space)		
…	…	…
n	…	…

(Table 6.1)

Let us 2 apple box A, B be divided by space like Table 6.1 . Let us box be full filled and the kind of apples be only red and green. Let us the apple mixed red and green be not exist. That is, there's no red nor green apple.

Now, let us define n as the total number of space of apple box and let us define r as the total number of red apple and let us define g as the total number of green apple.

Let us define n_R as the number of space of all red apple like no 1 space,

let us define n_M as the number of space of one red apple like no 2 space,

let us define n_G as the number of space of all green apple like no 3 space.

The following equation is satisfied.

$$n_R = \frac{1}{2}(r - n_M) = r - n + n_G$$

$$g - n \leq n_G \leq \frac{g}{2} \ (if\ g > n)$$

Proof 6.

In the table 6.1, if r is the total number of red apple, g is the total number of green apple then

$$2n = r + g \text{------------- } (6.1)$$

If n is the total number of space, if R is the space that A, B is all red apple, n_R is the total number of R, if M is the space that one of A, B is red apple, n_M is the total number of M, if G is the space that A, B is all green apple, n_G is the total number of G then

$$n = n_R + n_G + n_M \text{------------- } (6.2)$$

Because the number of red apple in R is 2, in M is 1, in G is 0,

$$r = 2 \times n_R + 1 \times n_M + 0 \times n_G = 2n_R + n_M \rightarrow n_R = \frac{1}{2}(r - n_M)$$

Because the number of green apple in R is 0, in M is 1, in G is 2,

$$g = 0 \times n_R + 1 \times n_M + 2 \times n_G = 2n_G + n_M \text{------------- } (6.3)$$

If we apply (6.3),(6.2) to (6.1) then

$$2n = r + g = r + (2n_G + n_M) = r + 2n_G + (n - n_R - n_G) = r + n_G + n - n_R$$

Therefore

$$n_R = r - n + n_G$$

And, let us g the total number of green apple be larger than n the number of space,that is, $g > n$.
First of all, if after we fullfill green apple into A box, and we fill remain green apple into B box then

$$n_G = g - n$$

If we remove one from A box and we fill the one into B box then

$$n_G = g - n + 1$$

If we remove also another one from A box,that is,remove two, and we fill the one into B box then

$$n_G = g - n + 2$$

Continue using the above way, if we we remove a apple from A box and we fill into B box then

$$n_G = g - n + a \text{------------- (6.4)}$$

Therefore, n_G become bigger.

But, if the green apple filled in B box is same or bigger than the remain green apple in A box then n_G become smaller rather.
In (6.4) the remain green apple in A box is $n - a$, the green apple filled in B box is n_G,so,

$$n - a \leq n_G \to n - a \leq g - n + a \to 2n - g \leq 2a \to \left(n - \frac{g}{2}\right) \leq a$$

That is, n_G is minimum when we remove no one apple from A box,
n_G is maximum when we remove $(n - g/2)$ apple from A box.
Therefore,

$$g - n \leq n_G \leq g - n + n - \frac{g}{2} \to g - n \leq n_G \leq \frac{g}{2}$$

■

Theorem 7. $\boldsymbol{\pi_t(6n+1)}$ **expression by using** $\boldsymbol{\pi(6n+1)}$

If $\mathbb{M}$ is a set of that only one of $6k-1,6k+1$ is prime, $\mathbb{C}$ is a set of that $6k-1,6k+1$ is all composite then

$$\pi_t(6n+1)=\frac{1}{2}\left(\pi(6n+1)-\sum_{\mathbb{M}}^{n}\big(\rho(6k-1)+\rho(6k+1)\big)\right)$$

$$=\frac{1}{2}\left(\pi(6n+1)-2\sum_{k=1}^{n}\rho_t(6k)+\sum_{k=1}^{n}\big(\rho(6k-1)+\rho(6k+1)\big)\right)$$

$$=\frac{1}{2}\left(\pi(6n+1)-n+\sum_{\mathbb{C}}^{n}\rho_t(6k)\right)$$

$$=\frac{1}{2}\left(\pi(6n+1)-n+\frac{1}{2}\sum_{\mathbb{C}}^{n}\big(\rho(6k-1)+\rho(6k-1)\big)\right)$$

$$=\frac{\pi(6n+1)}{2}-\frac{1}{3}\sum_{\mathbb{M}}^{n}\left(\frac{\pi\beta(6k-1)}{\pi\beta(6k-1)-1}+\frac{\pi\beta(6k+1)}{\pi\beta(6k+1)-1}\right)$$

$$-\frac{1}{3\pi}\sum_{\mathbb{M}}^{n}\left(\sum_{m=1}^{\infty}\frac{sin\left(\frac{2m\pi^2\beta(6k-1)}{\pi\beta(6k-1)-1}\right)+sin\left(\frac{2m\pi^2\beta(6k+1)}{\pi\beta(6k+1)-1}\right)}{m}\right)$$

Proof 7.

Let us $\mathbb{M}$ be a set of that only one of $6k-1,6k+1$ is prime. 2,3 is only which are not prime of $6k\pm1$ type and (3,5) is only which are not twin prime of $6k\pm1$ type, so,if we regard prime as red apple and regard composite as green apple in theorem 6 then

$$\pi_t(6n+1)-1=\frac{1}{2}\left((\pi(6n+1)-2)-\sum_{\mathbb{M}}^{n}1\right)$$

According to theorem 3, because $\rho_t(6k)=1$ when $6k\pm1\in\mathbb{M}$

$$\pi_t(6n+1)-1=\frac{1}{2}\left((\pi(6n+1)-2)-\sum_{\mathbb{M}}^{n}\rho_t(6k)\right)\rightarrow$$

$$\pi_t(6n+1)=\frac{1}{2}\left(\pi(6n+1)-\sum_{\mathbb{M}}^{n}\rho_t(6k)\right)\text{------------- (7.1)}$$

When $6k\pm1\in\mathbb{M}$, because $\rho_t(6k)=1=1+0\ or\ 0+1=\rho(6k-1)+\rho(6k+1)$
if we apply this to (7.1) then

$$\pi_t(6n+1) = \frac{1}{2}\left(\pi(6n+1) - \sum_{\mathbb{M}}^{n}(\rho(6k-1)+\rho(6k+1))\right) \text{------------- (7.2)}$$

To brief the formulas, let us

$$\frac{\pi\beta_t(6k)}{\pi\beta_t(6k)-1} = b_t, \sum_{m=1}^{\infty}\frac{\sin\left(\frac{2m\pi^2\beta_t(6k)}{\pi\beta_t(6k)-1}\right)}{m} = s_t$$

According to theorem 2

$$\sum_{\mathbb{M}}^{n}\rho_t(6k) = \sum_{\mathbb{M}}^{n}\left(b_t - \frac{1}{2} + \frac{1}{\pi}s_t\right) = \sum_{\mathbb{M}}^{n}b_t - \frac{1}{2}\sum_{\mathbb{M}}^{n}1 + \frac{1}{\pi}\sum_{\mathbb{M}}^{n}s_t = \sum_{\mathbb{M}}^{n}b_t - \frac{1}{2}\sum_{\mathbb{M}}^{n}\rho_t(6k) + \frac{1}{\pi}\sum_{\mathbb{M}}^{n}s_t \rightarrow$$

$$\sum_{\mathbb{M}}^{n}\rho_t(6k) = \frac{2}{3}\sum_{\mathbb{M}}^{n}b_t + \frac{2}{3\pi}\sum_{\mathbb{M}}^{n}s_t \text{------------- (7.3)}$$

And, according to theorem 1, $\beta_t(6k) = \beta(6k-1)+\beta(6k+1)$ and
when $6k \pm 1 \in \mathbb{M}$, $\beta(6k-1) = 0 \ or \ \beta(6k+1) = 0$,so,

$$\frac{\pi\beta_t(6k)}{\pi\beta_t(6k)-1} = \frac{\pi\beta(6k-1)+\pi\beta(6k+1)}{\pi\beta(6k-1)+\pi\beta(6k+1)-1} = \frac{0+\pi\beta(6k+1)}{0+\pi\beta(6k+1)-1} \ or \ \frac{\pi\beta(6k-1)+0}{\pi\beta(6k-1)+0-1}$$

$$= \frac{0}{0-1} + \frac{\pi\beta(6k+1)}{\pi\beta(6k+1)-1} \ or \ \frac{\pi\beta(6k-1)}{\pi\beta(6k-1)-1} + \frac{0}{0-1}$$

$$= \frac{\pi\beta(6k-1)}{\pi\beta(6k-1)-1} + \frac{\pi\beta(6k+1)}{\pi\beta(6k+1)-1}$$

$$\sin\left(\frac{2m\pi^2\beta_t(6k)}{\pi\beta_t(6k)-1}\right) = \sin\left(\frac{2m\pi^2\beta(6k-1)}{\pi\beta(6k-1)-1}\right) + \sin\left(\frac{2m\pi^2\beta(6k+1)}{\pi\beta(6k+1)-1}\right)$$

Therefore,

$$b_t = \frac{\pi\beta(6k-1)}{\pi\beta(6k-1)-1} + \frac{\pi\beta(6k+1)}{\pi\beta(6k+1)-1}$$

$$s_t = \sum_{m=1}^{\infty}\frac{\sin\left(\frac{2m\pi^2\beta(6k-1)}{\pi\beta(6k-1)-1}\right) + \sin\left(\frac{2m\pi^2\beta(6k+1)}{\pi\beta(6k+1)-1}\right)}{m}$$

If we apply (7.2) ,(7.3) to (7.1) then

$$\pi_t(6n+1) = \frac{\pi(6n+1)}{2} - \frac{1}{3}\sum_{\mathbb{M}}^{n}\left(\frac{\pi\beta(6k-1)}{\pi\beta(6k-1)-1} + \frac{\pi\beta(6k+1)}{\pi\beta(6k+1)-1}\right)$$

$$-\frac{1}{3\pi}\sum_{\mathbb{M}}^{n}\left(\sum_{m=1}^{\infty}\frac{\sin\left(\frac{2m\pi^2\beta(6k-1)}{\pi\beta(6k-1)-1}\right) + \sin\left(\frac{2m\pi^2\beta(6k+1)}{\pi\beta(6k+1)-1}\right)}{m}\right) \text{------------- (7.4)}$$

And, if we apply theorem 3 to (7.1) then

$$n+1-\sum_{k=1}^{n}\rho_t(6k)=\frac{1}{2}\left(\pi(6n+1)-\sum_{\mathbb{M}}^{n}\rho_t(6k)\right)\rightarrow$$

$$\sum_{\mathbb{M}}^{n}\rho_t(6k)=\pi(6n+1)-2n-2+2\sum_{k=1}^{n}\rho_t(6k)$$

If we apply "theorem 4 of The formula of $\pi(N)$" [1] of myself to the above formula then

$$\sum_{\mathbb{M}}^{n}\rho_t(6k)=2n+2-\sum_{k=1}^{n}\big(\rho(6k-1)+\rho(6k+1)\big)-2n-2+2\sum_{k=1}^{n}\rho_t(6k)\rightarrow$$

$$\sum_{\mathbb{M}}^{n}\rho_t(6k)=2\sum_{k=1}^{n}\rho_t(6k)-\sum_{k=1}^{n}\big(\rho(6k-1)+\rho(6k+1)\big)$$

If we apply the above formula to (7.1) then

$$\pi_t(6n+1)=\frac{1}{2}\left(\pi(6n+1)-2\sum_{k=1}^{n}\rho_t(6k)+\sum_{k=1}^{n}\big(\rho(6k-1)+\rho(6k+1)\big)\right) \text{------------- (7.5)}$$

If we define $\mathbb{C}$ as a set of that $6k-1, 6k+1$ is all composite then $n_R=r-n+n_G$ according to theorem 6, so,

$$\pi_t(6n+1)-1=\frac{1}{2}\left((\pi(6n+1)-2)-n+\sum_{\mathbb{C}}^{n}1\right)\rightarrow$$

$$\pi_t(6n+1)=\frac{1}{2}\left(\pi(6n+1)-n+\sum_{\mathbb{C}}^{n}\rho_t(6k)\right) \text{------------- (7.6)}$$

When $6k-1\in\mathbb{C}, 6k+1\in\mathbb{C}$, $\rho(6k-1)+\rho(6k-1)=1+1=2$,so,

$$\rho_t(6k)=\frac{1}{2}\big(\rho(6k-1)+\rho(6k-1)\big) \text{------------- (7.7)}$$

If we apply (7.7) to (7.6) then

$$\pi_t(6n+1)=\frac{1}{2}\left(\pi(6n+1)-n+\frac{1}{2}\sum_{\mathbb{C}}^{n}\big(\rho(6k-1)+\rho(6k-1)\big)\right)$$

■

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