Study of Fermat number

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Abstract

A number of $6n-1$ type is not odd perfect number, Fermat number is not also odd perfect number.

And, if Fermat number is composite number then Fermat number is factorized as below

$$\text{when } n \text{ is odd number}, 2^{2^n}+1=(2^{n+1}(3k+1)+1)(2^{n+1}(3m)+1)$$

$$\text{when } n \text{ is even number}, 2^{2^n}+1=\left(2^{n+1}\left(\frac{3k+1}{2}\right)+1\right)(2^{n+1}(3m)+1)$$

And, all Fermat number for $n \geq 5$ is composite number.

1. Introduction

We prove that if $N \equiv -1(mod\ 6)$ then N could not be odd perfect number and Fermat number could not be also odd perfect number by using the characteristics of $N=6n\pm 1$ type number.

But, we don't prove that an odd perfect number does not exist in call cases except Fermat number. And, when Fermat number is a composite number, we prove that Fermat number could be factorized by two factors of $2^{n+1}K+1$(From Euler) [3] type and we study to express K more specifically. And, we prove that all Fermat number for $n \geq 5$ is composite number by using mathematical induction.

2. Study of Fermat number

Definition 1. Unless otherwise stated, all of the numbers that are used in the contents of the following is a natural number.

Definition 2. $\therefore$ means therefore.

Definition 3. "→,➔" is an expression to simplify the distinction between the formula when we expand the numberical expression. For example, when we expand $a+1=0$ to obtain $a=-1$, we express $a+1=0$ ➔ $a=-1$.

Theorem 1. Fermat number and ddd perfect number

For an arbitrary natural number N, a number of $N \equiv -1(mod\ 6)$ type could not be odd perfect number.

If Fermat number is $F_n = 2^{2^n} + 1$ then $F_n \equiv -1(mod\ 6)$ and F_n is not odd perfect number.

Proof 1. For an arbitrary natural number N, in the case of $N \equiv -1(mod\ 6)$,

Let us $P_1, P_2, \dots, P_m$ be all divisor of N.

For an arbitrary $k, T_k = \frac{N}{P_k}$ is also divisor, so, $\sigma(N) = \sum_{k=1}^{m} P_k = \sum_{k=1}^{m} T_k$

Therefore, $2\sigma(N) = \sum_{k=1}^{m} P_k + \sum_{k=1}^{m} T_k = \sum_{k=1}^{m} (P_k + T_k)$

According to "theorem 1 in paper The formula of $\pi(N)$" [2] of myself,

if $N = PT$ then $P \equiv 1, T \equiv -1(mod\ 6)$ or $P \equiv -1, T \equiv 1(mod\ 6)$, so, $P_k + T_k \equiv 0(mod\ 6)$.

Therefore,

$$2\sigma(N) = \sum_{k=1}^{m} (P_k + T_k) \equiv 0(mod\ 6) \text{ ------------- } (1.1)$$

Because $\sigma(N) = 2N$ and $N \equiv -1(mod\ 6)$ according to the definition of odd perfect number [1]

$$2\sigma(N) = 2 \times 2N \equiv 2 \times -2 \equiv -4(mod\ 6) \text{ ------------ } (1.2)$$

Therefore, if $N \equiv -1(mod\ 6)$ then N is not odd perfect number because (1.1) and (1.2) is different.

Let us define $F_n = 2^{2^n} + 1$.

$F_n = 2^{2^n} + 1 \equiv -2 + 1 \equiv -1(mod\ 6)$ because 2^n is even number.

Therefore, Fermat number F_n is not odd perfect number according to the above result. ■

Theorem 2. Factor of Fermat composite number

When $F_n = 2^{2^n} + 1$ is a composite number and F_n is factorized by two factors, if we define $F_n = PT$ then $P = 2^{n+1}K + 1$(where K is positive integer)(From Euler) [3] and T is also $T = 2^{n+1}M + 1$. That is,

$$F_n = 2^{2^n} + 1 = PT = (2^{n+1}K + 1)(2^{n+1}M + 1)$$

And each factors P, T is same as following equations.

when n is odd number,

$$P = 2^{n+1}K + 1 = 2^{n+1}(3k + 1) + 1 \equiv -1(mod\ 6), k = 0,1,2,3, \dots$$

$$T = 2^{n+1}M + 1 = 2^{n+1}(3m) + 1 \equiv 1(mod\ 6), m = 1,2,3, \dots$$

(But, if k is odd then m is even, if k is even then m is odd)

when n is even number,

$$P = 2^{n+1}K + 1 = 2^{n+1}\left(\frac{3k + 1}{2}\right) + 1 \equiv -1(mod\ 6), k = 1,3,5, \dots$$

$$T = 2^{n+1}M + 1 = 2^{n+1}(3m) + 1 \equiv 1(mod\ 6), m = 1,2,3, \dots$$

(But, if $\frac{k-1}{2}$ is odd then m is odd, if $\frac{k-1}{2}$ is even then m is even)

For reference, if we make K, M to the sequence of $\left\{3k + 1, \frac{3l + 1}{2}, 3m\right\}$ type then

$$\left\{3k + 1, \frac{3l + 1}{2}, 3m\right\} = \{\{1,2,3\}, \{4,5,6\}, \{7,8,9\} \dots\}$$

Proof 2.

When $F_n = 2^{2^n} + 1$ is a composite number, let us define $F_n = PT, P = 2^{n+1}K + 1$ because $2^{n+1}K + 1 | F_n$(From Euler) [3]. According to "theorem 1 in paper The formula of $\pi(N)$" [2] of myself, when $2^{n+1}K + 1 \equiv -1(mod\ 6)$, if we define $P = 2^{n+1}K + 1 = 6p - 1, T = 6t + 1$,

$$F_n = P + 6tP = 2^{n+1}K + 1 + 6t(2^{n+1}K + 1) \to 2^{2^n} = 2^{n+1}K(6t + 1) + 6t = 2^{n+1}KT + 6t$$

If we divide by 2 both sides of the above equation,

$$2^{2^n - 1} = 2^n KT + 3t \text{ ------------- } (2.1)$$

Because t should be an even number in (2.1), if we define $t = 2u$ then

$$2^{2^n - 1} = 2^n KT + 6u$$

If we divide by 2 both sides of the above equation,

$$2^{2^n - 2} = 2^{n-1} KT + 3u \text{ ------------- } (2.2)$$

Because t should be an even number in (2.2), if we define $u = 2v$ then

$$2^{2^n - 2} = 2^{n-1} KT + 6v$$

If we divide by 2 both sides of the above equation,

$$2^{2^n - 3} = 2^{n-2} KT + 3v \text{ ------------- } (2.3)$$

Because (2.1),(2.2),(2.3) are same type, the process of the above is repeated.

For a certain integer k and for (2.1), if we repeat the above process then the following equation is satisfied.

$$2^{2^n - 1 - (n)} = 2^{n-(n)} KT + 3k \text{ ------------- } (2.4)$$

If we organize (2.4) then

$$2^{2^n - n - 1} = 2^0 K \frac{F_n}{2^{n+1}K + 1} + 3k \to \left(2^{2^n - n - 1} - 3k\right)(2^{n+1}K + 1) = KF_n \to$$

$$KF_n = 2^{2^n} K - 3k2^{n+1}K + 2^{2^n - n - 1} - 3k \to K\left(F_n - 2^{2^n} + 3k2^{n+1}\right) = 2^{2^n - n - 1} - 3k$$

$$\therefore K = \frac{2^{2^n - n - 1} - 3k}{F_n - 2^{2^n} + 3k2^{n+1}} = \frac{2^{2^n - n - 1} - 3k}{2^{2^n} + 1 - 2^{2^n} + 3k2^{n+1}} = \frac{2^{2^n - n - 1} - 3k}{2^{n+1}3k + 1}$$

And, $F_n = PT$, so

$$\therefore T = \frac{F_n}{2^{n+1}K + 1} = \frac{F_n}{2^{n+1}\frac{2^{2^n - n - 1} - 3k}{2^{n+1}3k + 1} + 1} = \frac{F_n(2^{n+1}3k + 1)}{2^{2^n} - 3k2^{n+1} + 2^{n+1}3k + 1} = 2^{n+1}3k + 1$$

Let us define $M = 3k$, because $T = 2^{n+1}3k + 1 = 2^{n+1}M + 1$,so, T is also $2^{n+1}K + 1$ type.

Because K is positive integer in $2^{n+1}K + 1$ type, so, $3k > 0$. Therefore $k > 0$

When $2^{n+1}K + 1 \equiv 1(mod\ 6)$, if we define $P = 2^{n+1}K + 1 = 6p + 1, T = 6t - 1$ then

$$F_n = -P + 6tP = -2^{n+1}K - 1 + 6t(2^{n+1}K + 1) \to 2^{2^n} + 2 = 2^{n+1}K(6t - 1) + 6t \to$$

$$2^{2^n} + 2 = 2^{n+1}KT + 6t$$

If we divide by 2 both sides of the above equation,

$$2^{2^n - 1} + 1 = 2^n KT + 3t \text{ ------------- } (2.5)$$

Because $2^n - 1$ is odd,

$$(2+1)(2^{2^n-2} - 2^{2^n-3} + \cdots + 2^4 - 2^3 + 2^2 - 2^1 + 2^0) = 2^n KT + 3t$$

Because K should be multiple of 3 to satisfy the above equation, if we define $K = 3a$ then

$$(2+1)(2^{2^n-2} - 2^{2^n-3} + \cdots + 2^4 - 2^3 + 2^2 - 2^1 + 2^0) = 2^n 3aT + 3t$$

if we divide by 3 both sides of the above equation then

$$2^{2^n-2} - 2^{2^n-3} + \cdots + 2^4 - 2^3 + 2^2 - 2^1 + 2^0 = 2^n aT + t \text{ ------------- } (2.6)$$

Because t should be odd number to satisfy (2.6), if we define $t = 2u + 1$ then

$$2^{2^n-2} - 2^{2^n-3} + \cdots + 2^4 - 2^3 + 2^2 - 2^1 + 2^0 = 2^n aT + 2u + 1 \rightarrow$$

$$2^{2^n-3} - 2^{2^n-4} + \cdots + 2^3 - 2^2 + 2^1 - 2^0 = 2^{n-1} aT + u \text{ ------------- } (2.7)$$

Because u should be odd number to satisfy (2.7), if we define $u = 2v + 1$ then

$$2^{2^n-3} - 2^{2^n-4} + \cdots + 2^3 - 2^2 + 2^1 - 2^0 = 2^{n-1} aT + 2v + 1 \rightarrow$$

$$2^{2^n-4} - 2^{2^n-5} + \cdots + 2^2 - 2^1 = 2^{n-2} aT + v \text{ ------------- } (2.8)$$

Because v should be odd number to satisfy (2.8), if we define $v = 2w$ then

$$2^{2^n-4} - 2^{2^n-5} + \cdots + 2^4 - 2^3 + 2^2 - 2^1 = 2^{n-2} aT + 2w \rightarrow$$

$$2^{2^n-5} - 2^{2^n-6} + \cdots + 2^3 - 2^2 + 2^1 - 2^0 = 2^{n-3} aT + w \text{ ------------- } (2.9)$$

Because (2.7),(2.9) are same type, the process from (2.7) to (2.9) is repeated.

For a certain integer k and for (2.7), when n is odd number, that is, let us define $n = 2m + 1$.

If we repeat the process of the above then the following equation is satisfied.

$$2^{2^{2m+1}-3-(2m)} - 2^{2^{2m+1}-4-(2m)} + \cdots + 2^3 - 2^2 + 2^1 - 2^0 = 2^{2m+1-1-(2m)} aT + k \text{ ------- } (2.10)$$

If we organize (2.10) then

$$2^{2^{2m+1}-2m-3} - 2^{2^{2m+1}-2m-4} + \cdots + 2^3 - 2^2 + 2^1 - 2^0 = 2^0 aT + k$$

If we multiply by 2 and add 1 to both sides of the above equation to organize the above equation for more simple then

$$2^{2^{2m+1}-2m-2} - 2^{2^{2m+1}-2m-3} + \cdots + 2^4 - 2^3 + 2^2 - 2^1 + 2^0 = 2^1 aT + 2k + 1$$

If we multiply by 3 to both sides of the above equation then

$$(2+1)\left(2^{2^{2m+1}-2m-2} - 2^{2^{2m+1}-2m-3} + \cdots + 2^4 - 2^3 + 2^2 - 2^1 + 2^0\right) = 3(2^1 aT + 2k + 1)$$

If we organize to reflect $K = 3a, n = 2m + 1$ in the above equation then

$$2^{2^n-n} + 1 = 2KT + 6k + 3 \text{ ------- } (2.11)$$

If we organize (2.11) then

$$2^{2^n-n} + 1 = 2K\frac{F_n}{2^{n+1}K+1} + 6k + 3 \rightarrow 2^{2^n-n} - 6k - 2 = 2K\frac{F_n}{2^{n+1}K+1} \rightarrow$$

$$2^{2^n-n-1} - 3k - 1 = K\frac{F_n}{2^{n+1}K+1} \rightarrow (2^{n+1}K+1)\left(2^{2^n-n-1} - 3k - 1\right) = KF_n \rightarrow$$

$$\therefore K = \frac{2^{2^n-n-1} - 3k - 1}{F_n - 2^{2^n} + 3k2^{n+1} + 2^{n+1}} = \frac{2^{2^n-n-1} - 3k - 1}{3k2^{n+1} + 2^{n+1} + 1} = \frac{2^{2^n-n-1} - (3k+1)}{2^{n+1}(3k+1) + 1}$$

And, $F_n = PT$,so,

$$\therefore T = \frac{F_n}{2^{n+1}K+1} = \frac{F_n}{2^{n+1}\frac{2^{2^n-n-1}-(3k+1)}{2^{n+1}(3k+1)+1}+1} = \frac{F_n\{2^{n+1}(3k+1)+1\}}{2^{2^n}-(3k+1)2^{n+1}+2^{n+1}(3k+1)+1}$$

$$= 2^{n+1}(3k+1)+1$$

Let us define $M = 3k+1$, because $T = 2^{n+1}(3k+1)+1 = 2^{n+1}M+1$,so, T is also $2^{n+1}K+1$ type.

Because K is positive integer in $2^{n+1}K+1$ type, so, $3k+1>0 \to k > -\frac{1}{3}$. Therefore $k \geq 0$

When $2^{n+1}K+1 \equiv 1(mod\ 6)$, n is even number, that is, let us define $n = 2m$.

For a certain integer k and for (2.7), if we repeat the process of the above then the following equation is satisfied.

$$2^{2^{2m}-3-(2m-2)} - 2^{2^{2m}-4-(2m-2)} + \cdots + 2^3 - 2^2 + 2^1 - 2^0 = 2^{2m-1-(2m-2)}aT + k \text{ ------- } (2.12)$$

If we organize (2.12) then

$$2^{2^{2m}-2m-1} - 2^{2^{2m}-2m-2} + \cdots + 2^3 - 2^2 + 2^1 - 2^0 = 2^1aT + k$$

If we multiply by 2 and add 1 to both sides of the above equation to organize the above equation for more simple then

$$2^{2^{2m}-2m} - 2^{2^{2m}-2m-1} + \cdots + 2^4 - 2^3 + 2^2 - 2^1 + 2^0 = 2^2aT + 2k + 1$$

If we multiply by 3 to both sides of the above equation then

$$(2+1)(2^{2^{2m}-2m} - 2^{2^{2m}-2m-1} + \cdots + 2^4 - 2^3 + 2^2 - 2^1 + 2^0) = 3(2^2aT + 2k + 1)$$

If we organize to reflect $K = 3a, n = 2m$ in the above equation then

$$2^{2^n-n+1} + 1 = 2^2KT + 6k + 3 \text{ ------- } (2.13)$$

If we organize (2.13) then

$$2^{2^n-n+1} + 1 = 2^2K\frac{F_n}{2^{n+1}K+1} + 6k + 3 \to 2^{2^n-n+1} - 6k - 2 = 2^2K\frac{F_n}{2^{n+1}K+1} \to$$

$$2^{2^n-n} - 3k - 1 = 2^1K\frac{F_n}{2^{n+1}K+1} \to (2^{n+1}K+1)\left(2^{2^n-n} - 3k - 1\right) = 2KF_n \to$$

$$K2^{2^n+1} - 2^{n+1}K3k - 2^{n+1}K + 2^{2^n-n} - 3k - 1 = 2KF_n \to$$

$$2^{2^n-n} - 3k - 1 = K\left(2F_n - 2^{2^n+1} + 2^{n+1}3k + 2^{n+1}\right)$$

$$\therefore K = \frac{2^{2^n-n}-3k-1}{2F_n - 2^{2^n+1} + 2^{n+1}3k + 2^{n+1}} = \frac{2^{2^n-n}-3k-1}{2^{n+1}3k + 2^{n+1} + 2} = \frac{1}{2}\frac{2^{2^n-n}-(3k+1)}{2^n(3k+1)+1}$$

And, $F_n = PT$,so,

$$\therefore T = \frac{F_n}{2^{n+1}K+1} = \frac{F_n}{2^{n+1}\frac{1}{2}\frac{2^{2^n-n}-(3k+1)}{2^n(3k+1)+1}+1} = \frac{F_n\{2^n(3k+1)+1\}}{2^{2^n}-(3k+1)2^n+2^n(3k+1)+1}$$

$$= 2^n(3k+1)+1 = 2^{n+1}\left(\frac{3k+1}{2}\right)+1$$

Let us define $M = \left(\frac{3k+1}{2}\right)$, because $T = 2^{n+1}\left(\frac{3k+1}{2}\right) + 1 = 2^{n+1}M + 1$, so,

T is also $2^{n+1}K + 1$ type.

Because K is positive integer in $2^{n+1}K + 1$ type, so, $\left(\frac{3k+1}{2}\right) > 0 \rightarrow k > -\frac{1}{3}$. Therefore $k \geq 0$

By summarizing the above contents, let us define $F_n = PT, P = 2^{n+1}K + 1, T = 2^{n+1}M + 1$ because T is also $2^{n+1}K + 1$ type in the contents of the above.

When n is odd number, if we define $P = 2^{n+1}K + 1 = 2^{n+1}(3k+1) + 1$ then
$P = 2^{n+1}(3k+1) + 1 \equiv -2(3k+1) + 1 \equiv -6k - 2 + 1 \equiv -1(mod\ 6)$.
$F_n = PT = P(2^{n+1}M + 1) \equiv -1(-2M + 1) \equiv 2M - 1(mod\ 6)$ and
$F_n \equiv -1 \equiv 2M - 1(mod\ 6) \rightarrow 2M \equiv 0(mod\ 6)$, because $F_n \equiv -1(mod\ 6)$ according to theorem 1
Therefore, because M is multiple of 3, if we define $T = 2^{n+1}(3m) + 1$ then
$T = 2^{n+1}(3m) + 1 \equiv -2(3m) + 1 \equiv 1(mod\ 6)$.
And, because it should be $3m > 0$, so, $m > 0$

When n is even number, if we define $P = 2^{n+1}K + 1 = 2^{n+1}\left(\frac{3k+1}{2}\right) + 1$ then

$P = 2^n(3k+1) + 1 \equiv -2(3k+1) + 1 \equiv -6k - 2 + 1 \equiv -1(mod\ 6)$.
And, $k \geq 0$, but it should be $2|3k+1$, so, $k = 1,3,5,\ldots$.
$F_n = PT = P(2^{n+1}M + 1) \equiv -1(-2M + 1) \equiv 2M - 1(mod\ 6)$ and
$F_n \equiv -1 \equiv 2M - 1(mod\ 6) \rightarrow 2M \equiv 0(mod\ 6)$.
Therefore, because M is multiple of 3, if we define $T = 2^{n+1}(3m) + 1$ then
$T = 2^{n+1}(3m) + 1 \equiv -2(3m) + 1 \equiv 1(mod\ 6)$.
And, because it should be $3m > 0$, so, $m > 0$

In addition, $F_n = 2^{2^n} + 1 = PT = (2^{n+1}K + 1)(2^{n+1}M + 1) = 2^{n+1}(2^{n+1}KM + K + M) + 1$
So, $2^{2^n} = 2^{n+1}(2^{n+1}KM + K + M) \rightarrow 2^{2^n - n - 1} = 2^{n+1}KM + K + M$
Therefore, $K + M$ should be even number.
When n is odd number, $K + M = 3k + 1 + 3m$, so, if k is odd number then m should be even number, if k is even number then m should be odd number.
When n is even number, because k should be odd number, if we define $k = 2a + 1$ then

$$K + M = \frac{3k+1}{2} + 3m = \frac{3(2a+1)+1}{2} + 3m = \frac{6a+4}{2} + 3m = 3a + 2 + 3m.$$

Because $a = \frac{k-1}{2}$, so, if $\frac{k-1}{2}$ is odd number then m is odd number,

if $\frac{k-1}{2}$ is even number then m is even number

By summarizing all of the above contents, if F_n is factorized by $F_n = PT$

when n is odd number,

$$P = 2^{n+1}K + 1 = 2^{n+1}(3k+1) + 1 \equiv -1(mod\ 6), k = 0,1,2,3,\ldots$$

$$T = 2^{n+1}M + 1 = 2^{n+1}(3m) + 1 \equiv 1(mod\ 6), m = 1,2,3,\ldots$$

(But, if k is odd then m is even, if k is even then m is odd)

when n is even number,

$$P = 2^{n+1}K + 1 = 2^{n+1}\left(\frac{3k+1}{2}\right) + 1 \equiv -1(mod\ 6), k = 1,3,5,\ldots$$

$$T = 2^{n+1}M + 1 = 2^{n+1}(3m) + 1 \equiv 1(mod\ 6), m = 1,2,3,\ldots$$

(But, if $\frac{k-1}{2}$ is odd then m is odd, if $\frac{k-1}{2}$ is even then m is even)

For reference, if we make K, M to the sequence of $\left\{3k+1, \frac{3l+1}{2}, 3m\right\}$ type then

$\left\{3k+1, \frac{3l+1}{2}, 3m\right\} = \{\{1,2,3\}, \{4,5,6\}, \{7,8,9\} \ldots\}$

because $k = 0,1,2,3,\ldots, p = 1,3,5,\ldots, m = 1,2,3,\ldots$

■

Theorem 3. Fermat composite number and $6n \pm 1$

If we define Fermat number as $F_n = 2^{2^n} + 1$ and $F_n = 6f_n - 1$.

$$f_n = 2^{2^n-2} - 2^{2^n-3} + 2^{2^n-4} - 2^{2^n-5} + \cdots + 2^4 - 2^3 + 2^2 - 2^1 + 2^0$$

And, if $F_n = PT$, that is, F_n is a composite number then the following equation is satisfied.

When n is odd number,

if $P = 2^{n+1}(3k+1) + 1 = 6p - 1, T = 2^{n+1}(3m) + 1 = 6t + 1$ then

$$p = 2^n k + (2^{n-1} - 2^{n-2} + \cdots + 2^4 - 2^3 + 2^2 - 2^1 + 2^0), t = 2^n m$$

When n is even number,

if $P = 2^{n+1}\left(\frac{3k+1}{2}\right) + 1 = 6p - 1, T = 2^{n+1}(3m) + 1 = 6t + 1$ then

$$p = 2^{n-1} k + (2^{n-2} - 2^{n-3} + \cdots + 2^4 - 2^3 + 2^2 - 2^1 + 2^0), t = 2^n m$$

Proof 3. Let us define $F_n = 2^{2^n} + 1 = 6f_n - 1$ because $F_n \equiv -1(mod\ 6)$ according to theorem 1.

$2^{2^n} + 1 = 6f_n - 1 \rightarrow 2^{2^n} + 2 = 6f_n \rightarrow 2^{2^n-1} + 1 = 3f_n$

Because $2^n - 1$ is an odd number

$2^{2^n-1} + 1 = (2+1)(2^{2^n-2} - 2^{2^n-3} + 2^{2^n-4} - 2^{2^n-5} + \cdots + 2^4 - 2^3 + 2^2 - 2^1 + 2^0) = 3f_n \rightarrow$

$2^{2^n-2} - 2^{2^n-3} + 2^{2^n-4} - 2^{2^n-5} + \cdots + 2^4 - 2^3 + 2^2 - 2^1 + 2^0 = f_n$

And, Let us define $F_n = PT$, F_n be a composite number..

When n is odd number, because $P = 2^{n+1}(3k+1) + 1 \equiv -1(mod\ 6)$ according to theorem 2,

if we define $P = 6p - 1 = 2^{n+1}(3k+1) + 1$ then

$$6p - 1 = 2^{n+1}(3k+1) + 1 \rightarrow 3p = 2^n(3k+1) + 1 \rightarrow 3p = 2^n 3k + 2^n + 1 \rightarrow$$

$$3p = 2^n 3k + (2+1)(2^{n-1} - 2^{n-2} + \cdots + 2^4 - 2^3 + 2^2 - 2^1 + 2^0) \rightarrow$$

$$p = 2^n k + (2^{n-1} - 2^{n-2} + \cdots + 2^4 - 2^3 + 2^2 - 2^1 + 2^0)$$

And, because $T = 2^{n+1}(3m) + 1 \equiv 1(mod\ 6)$ according to theorem 2,

if we define $T = 6t + 1 = 2^{n+1}(3m) + 1$ then

$$6t + 1 = 2^{n+1}(3m) + 1 \rightarrow 2t = 2^{n+1}(m) \rightarrow t = 2^n m$$

When n is even number, according to theorem 2

because $P = 2^{n+1}\left(\frac{3k+1}{2}\right) + 1 \equiv -1(mod\ 6)$, so,

if we define $P = 6p - 1 = 2^{n+1}\left(\frac{3k+1}{2}\right) + 1$ then

$$6p - 1 = 2^{n+1}\left(\frac{3k+1}{2}\right) + 1 \rightarrow 3p = 2^n\left(\frac{3k+1}{2}\right) + 1 \rightarrow 3p = 2^{n-1}3k + 2^{n-1} + 1 \rightarrow$$

$$3p = 2^{n-1}3k + (2+1)(2^{n-2} - 2^{n-3} + \cdots + 2^4 - 2^3 + 2^2 - 2^1 + 2^0) \rightarrow$$

$$p = 2^{n-1}k + (2^{n-2} - 2^{n-3} + \cdots + 2^4 - 2^3 + 2^2 - 2^1 + 2^0)$$

And, because $T = 2^{n+1}(3m) + 1 \equiv 1(mod\ 6)$ according to theorem 2

if we define $T = 6t + 1 = 2^{n+1}(3m) + 1$ then

$$6t + 1 = 2^{n+1}(3m) + 1 \rightarrow 2t = 2^{n+1}(m) \rightarrow t = 2^n m \blacksquare$$

Theorem 4. Fermat composite number and next Fermat composite number

If we define a Fermat number as $F_n = 2^{2^n} + 1$,

if we define the next Fermat number as $F_{n+1} = 2^{2^{n+1}} + 1$ then

$$F_{n+1} = F_n + 2^{2^n}\left(2^{2^n} - 1\right) = {F_n}^2 - 2^{2^n+1} = {F_n}^2 - 2F_n + 2 = F_n(F_n - 2) + 2$$

And, when F_n, F_{n+1} is all composite number,

If $F_n = (2^{n+1}K + 1)(2^{n+1}M + 1), F_{n+1} = (2^{n+2}U + 1)(2^{n+2}V + 1)$ then

$$2^{2^{n+1}-n-2} = 2^{n+2}UV + U + V = 2^n(2^{n+1}KM + K + M)^2$$

Proof 4.

Let us define a Fermat number as $F_n = 2^{2^n} + 1$ and

let us define the next Fermat number as $F_{n+1} = 2^{2^{n+1}} + 1$. And then

$$F_{n+1} - F_n = 2^{2^{n+1}} + 1 - \left(2^{2^n} + 1\right) = 2^{2^{n+1}} - 2^{2^n} = 2^{2^n}\left(2^{2^n} - 1\right) \rightarrow$$

$$F_{n+1} = F_n + 2^{2^n}\left(2^{2^n} - 1\right)$$

Because $2^{2^n} - 1 = 2^{2^n} + 1 - 2 = F_n - 2$, so

$$F_{n+1} = F_n + 2^{2^n}\left(2^{2^n} - 1\right) = F_n + 2^{2^n}(F_n - 2) = F_n + 2^{2^n}F_n - 2^{2^n+1} \rightarrow$$

$$F_{n+1} = F_n\left(1 + 2^{2^n}\right) - 2^{2^n+1} = F_nF_n - 2^{2^n+1} = {F_n}^2 - 2^{2^n+1}$$

Because $2^{2^n} = 2^{2^n} + 1 - 1 = F_n - 1$, so

$$F_{n+1} = {F_n}^2 - 2^{2^n+1} = {F_n}^2 - 2(F_n - 1) = {F_n}^2 - 2F_n + 2 = F_n(F_n - 2) + 2$$

And, when F_n, F_{n+1} is all composite number, according to theorem 2

If we define $F_n = (2^{n+1}K + 1)(2^{n+1}M + 1), F_{n+1} = (2^{n+2}U + 1)(2^{n+2}V + 1)$ then

$$F_n = (2^{n+1}K + 1)(2^{n+1}M + 1) = 2^{n+1}(2^{n+1}KM + K + M) + 1$$

$$F_{n+1} = (2^{n+2}U + 1)(2^{n+2}V + 1) = 2^{n+2}(2^{n+2}UV + U + V) + 1$$

If we define $Q = 2^{n+1}KM + K + M, W = 2^{n+2}UV + U + V$ then

$F_n = 2^{n+1}Q + 1, F_{n+1} = 2^{n+2}W + 1$이고 $F_{n+1} = F_n(F_n - 2) + 2$, so,

$2^{n+2}W + 1 = (2^{n+1}Q + 1)(2^{n+1}Q + 1 - 2) + 2 \rightarrow 2^{n+2}W + 1 = (2^{n+1}Q + 1)(2^{n+1}Q - 1) + 2 \rightarrow$

$2^{n+2}W + 1 = (2^{2n+2}Q^2 - 1) + 2 \rightarrow 2^{n+2}W + 1 = 2^{2n+2}Q^2 + 1 \rightarrow 2^{n+2}W = 2^{2n+2}Q^2 \rightarrow$

$$W = 2^nQ^2$$

And, $F_{n+1} = 2^{n+2}2^nQ^2 + 1 \rightarrow 2^{2^{n+1}} + 1 - 1 = 2^{n+2}2^nQ^2 \rightarrow 2^{2^{n+1}-n-2} = 2^nQ^2$

Therefore,

$$2^{2^{n+1}-n-2} = 2^{n+2}UV + U + V = 2^n(2^{n+1}KM + K + M)^2$$

■

Theorem 5. All $(n \geq 5)$ Fermat number is composite

All Fermat number for $n \geq 5$ is composite number.

Proof 5.

Let us define Fermat number as $F_n = 2^{2^n} + 1$ and $F_{n+1} = 2^{2^{n+1}} + 1$.

For $n < 5$, F_n is already proved to be prime number and

for $n \geq 5$, $F_5 = 641 \times 6700417 = (2^{5+1}(3 \times 3 + 1) + 1)(2^{5+1}(3 \times 34898) + 1)$

$$F_6 = 274177 \times 67280421310721 = (2^{n+1}(3 \times 714) + 1)\left(2^{6+1}\left(\frac{3 \times 350418860993 + 1}{2}\right) + 1\right)$$

is already proved to be composite number. So, we finish to prove the first of the mathematical induction. [3]

Let us suppose that F_n is composite number. That is, according to theorem 2

$$F_n = (2^{n+1}K + 1)(2^{n+1}M + 1) = 2^{n+1}(2^{n+1}KM + K + M) + 1$$

Let us define $Q = 2^{n+1}KM + K + M$, if we arrange the above equation then

$$F_n = 2^{n+1}(2^{n+1}KM + K + M) + 1 = 2^{n+1}Q + 1 \text{ ------------- } (5.1)$$

Now, Let us suppose that F_{n+1} is not composite number to show that F_{n+1} is also composite number. To prove this, let us define $2^{n+2}UV + U + V = 2^n Q^2$ for a certain natural number U and let us suppose that any certain natural number $V(V > 0)$ is not exist to satisfy this equation. Because V is not natural number , let us define $a \neq 0$, a, b is relative prime, $V = b/a > 0$ is an irreducible fraction. And, let us be $2^n Q^2 - U > 0 \rightarrow 2^n Q^2 > U$ for $V > 0$ because $V(2^{n+2}U + 1) = 2^n Q^2 - U$ in the above equation

By summarizing the above contents and applying theorem 4, let us suppose below equation.

$$2^{n+2}UV + U + V = 2^{n+2}U\frac{b}{a} + U + \frac{b}{a} = 2^n Q^2 = 2^{2^{n+1}-n-2}$$

$$\left(but, U < 2^n Q^2 = 2^{2^{n+1}-n-2}, V = \frac{b}{a} > 0, a \neq 0, a, b \text{ is relative prime}\right) \text{ ------------- } (5.2)$$

If we multiply 2^{n+2} and add 1 to the both sides of (5.2) and arrange then

$$2^{n+2}UV + U + V = 2^{2^{n+1}-n-2} \rightarrow 2^{n+2}(2^{n+2}UV + U + V) + 1 = 2^{n+2}2^{2^{n+1}-n-2} + 1 \rightarrow$$

$$2^{n+2}2^{n+2}UV + 2^{n+2}U + 2^{n+2}V + 1 = 2^{2^{n+1}} + 1 \rightarrow$$

$$2^{n+2}U(2^{n+2}V + 1) + (2^{n+2}V + 1) = 2^{2^{n+1}} + 1 \rightarrow$$

$$(2^{n+2}U + 1)(2^{n+2}V + 1) = 2^{2^{n+1}} + 1 = F_{n+1} \text{ ------------- } (5.3)$$

The above equation satisfies the assumption that F_{n+1} is not composite number, because V is not natural number but the left side of (5.3) is multiplication of two numbers. Therefore, (5.2) is appropriate to satisfy the assumption. And $V > 0$ is appropriate condition because if $V = 0$ then $(2^{n+2}U + 1)(2^{n+2}V + 1) = (2^{n+2}U + 1)(2^{n+2} \times 0 + 1) = 2^{n+2}U + 1$ in (5.3), so the assumption of that F_{n+1} is not composite number become meaninglessness.

If we multiply a to the both sides of (5.2) and arrange then

$$2^{n+2}Ub + aU + b = a2^{2^{n+1}-n-2} \rightarrow b(2^{n+2}U + 1) = a(2^{2^{n+1}-n-2} - U) \text{------------- (5.4)}$$

Let us define $2^{2^{n+1}-n-2} - U = bk$ for adequate natural number k because it should be $Gcd\left(b, (2^{2^{n+1}-n-2} - U)\right) \neq 1$ why a, b is relative prime and because $2^{2^{n+1}-n-2} - U > 0$ in the condition of (5.2). That is $2^{2^{n+1}-n-2} = U + bk$. If we apply this to (5.2) then

$$2^{n+2}U\frac{b}{a} + U + \frac{b}{a} = U + bk \rightarrow 2^{n+2}U\frac{b}{a} + \frac{b}{a} = bk \rightarrow because\ b \neq 0 \rightarrow 2^{n+2}U\frac{1}{a} + \frac{1}{a} = k \rightarrow$$

$$2^{n+2}U + 1 = ak \text{------------- (5.5)}$$

If we apply (5.5) to (5.3) then

$$F_{n+1} = (ak)(2^{n+2}V + 1) \rightarrow F_{n+1} = ak\left(2^{n+2}\frac{b}{a} + 1\right) \rightarrow F_{n+1} = k(2^{n+2}b + a) \text{------------- (5.6)}$$

The assumption is inconsistency because F_{n+1} is composite number as multiplication of k and $(2^{n+2}b + 1)$ in (5.6). So,the first assumption is wrong that F_{n+1} is not composite number.

So, F_{n+1} is composite number.

In the opposite, $2^{n+2}UV + U + V = 2^nQ^2 = 2^{2^{n+1}-n-2}$ of (5.2) for natural number V is concluded according to theorem 4 because F_n, F_{n+1} is composite number.

Therefore, all Fermat number for $n \geq 5$ is composite number according to mathematical induction.

■

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