Thin Spherical Matter Shell as Gravity Theory Filter

| Kevin Nolan Mar 06, 2014, edits Jul 18, 21, 29

Introduction

In GR (general relativity) a static thin solid and uniform spherical matter shell is the source of an external SM (Schwarzschild metric), interior flat MM (Minkowski metric), plus a shell wall metric not of interest here. In part 1 the above is shown to imply an illogical break in dependence on gravitational potential for the radial spatial metric component, exterior vs hollow interior regions, not shared by clocks. In part 2 further anomaly is found. In the gravitationally small regime, any infinitesimal element of shell mass is reasonably treated as an independent point source of SM. To within a tiny fractional error, linearly summing over the shell should but does not yield an interior metric consistent with the usual matching scheme of part 1. Conformally flat exterior metric as necessary cure is discussed in part 3. A tie-in to Mach's principle is discussed in part 4.

1: GR patching prescription = anomaly A

In the weak gravity limit, any viable theory of gravity must reduce to Newtonian gravity. For which interior to the wall of a rigid spherical shell of matter of mass *M*, radius *r = R* (for calculation simplicity taken to have a finite but arbitrarily small wall thickness *dR*), a Newtonian equipotential region with $\boldsymbol{g} = -\nabla\phi = 0$ applies:

$$\phi = -GM/r \to -GM/R\big|_{r\le R} = \text{constant for all}\, r \le R \quad (1\text{-}1)$$

In GR Birkhoff's theorem similarly demands an interior equipotential region [1]. To find the consequences of that, start with expression for standard SM line element in polar form:

$$ds^2 = c^2 d\tau^2 = \left(1 - r_s/r\right)c^2dt^2 - \left(1 - r_s/r\right)^{-1}dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \quad (1\text{-}2),\ \text{or}$$

$$ds^2 = c^2 d\tau^2 = g_{tt}c^2dt^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\varphi\varphi}d\varphi^2 \quad (1\text{-}3),\quad \text{where}$$

$$r_s = 2GM/c^2 = -2\phi r/c^2 \quad (1\text{-}4)$$

is the Schwarzschild radius for a spherically symmetric source of mass *M*. Holding respectively *r,t*, constant and equating the relevant terms in (1-3) yields the well-known relations

$$d\tau/dt = \sqrt{g_{tt}} \quad (1\text{-}5)$$ (coordinate determined clock-rate slows further down a potential well),

$$dr/ds = 1\big/\sqrt{-g_{rr}} = \sqrt{g_{tt}} \quad (1\text{-}6)$$ (radially oriented ruler coordinate length shrinks further down the well).

As inferred by a stationary observer at spatial infinity (where nominally $\phi = 0$), coordinate clock-rate and radial length scale are everywhere in the SM region equally affected by potential ϕ according to the factor

$$\sqrt{g_{tt}} = \sqrt{1 - r_s/r} = \sqrt{1 + 2\phi/c^2} = \sqrt{1 - 2GM/(rc^2)} \quad (1\text{-}7)$$

Yet transverse components $g_{\theta\theta}$, $g_{\varphi\varphi}$ are entirely unaffected. A ruler transversely oriented has an invariant coordinate length. While such coordinate determined spatial anisotropy is not directly reflected locally, it does have in-principle locally measurable consequences, and at any rate presents a consistency issue. Not remedied via recourse to the so-called Isotropic form of SM, as will be shown in part3. Standard SM forces that patching anisotropic exterior to flat and isotropic MM interior region can only be 'sensibly' satisfied one way. Given $g_{\theta\theta}$, $g_{\varphi\varphi}$ are *everywhere* unaffected by ϕ, then

$$\left|1\big/\sqrt{-g_{rr}}\right| \equiv \left|\sqrt{g_{tt}}\right| \to 1 (r < R) \quad (1\text{-}8).$$

Radial metric component losing all dependence on Newtonian potential ϕ in crossing from just exterior at $r = R^+$, to just interior at $r = R^-$ to the shell wall. (Vanishing of Newtonian $\boldsymbol{g} = -\nabla\phi \to 0\,(r < R)$ is in contrast completely consistent with **g**'s *invariant functional dependence on* ϕ *for all r*.) No-one questions consistently applying (1-1) into (1-5), (1-7) for redshifted clock-rate. Unlike radial ruler length, properly remaining depressed for $r < R$. That (1-8), demanded by an SM-to-MM force-fit, is not locally observable but 'merely' coordinate inferred, is beside the point. It is fundamentally illogical. Such anomaly has nothing to do with abruptness of shell transition – shell wall can be arbitrarily thick (for no gain in conceptual clarity). What matters is; logically identical coordinate behaviour of clocks and (radial pointing) rulers, everywhere exterior to shell, vs illogical disparity everywhere in hollow interior. But it gets better!

2: Direct determination of interior spatial metric = worse anomaly B

Any physically reasonable non-linear situation must reduce arbitrarily close to linearity, in some lower parameter limit, or over a sufficiently narrow range. In GR that situation corresponds to the so-called weak gravity regime typically characterised by $r_s/R \ll 1$. Apply to Fig.1 where for concreteness the thin shell is say a steel globe atlas (minus support stand), radius *R* = 10cm, mass *m* = *M* = 200gm. This yields $r_s/R \simeq 3.0\times10^{-26}$, weak gravity indeed! Arguing that a direct summation is invalid owing to not being a proper solution of the EFE's is in this context just silly. Any fractional error *in the final result* will reasonably be bounded to around the above value! As per intro then, treat each element of shell mass *dM* as an independent gravitating point source of SM - according to (1-3), (1-6), (1-7).

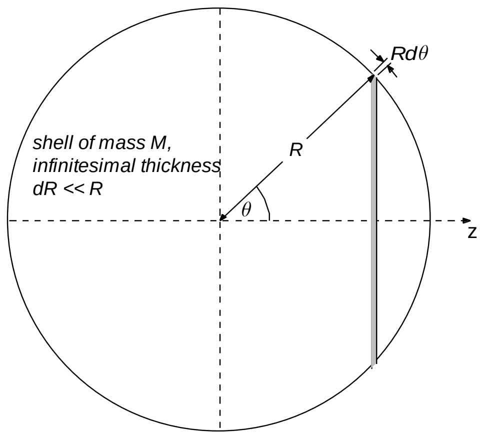

Fig. 1

Mathematically it is particularly easy to choose as target field point the shell centre *r* = 0 at which to evaluate a given spatial metric component, which as per Fig.1 will be taken as along axis **z**. Symmetry guaranteed Isotropy at *r* = 0 then demands the net result will equally apply along any other axis.

Using that in the gravity-free limit, $\sqrt{g_{tt}}\big|_{m\to0} = 1$, (1-6) can be recast as a fictitious mechanical 'strain' ε_{rr}

$$|\varepsilon_{rr}| = \left(\sqrt{g_{tt}}\big|_{m=0} - \sqrt{g_{tt}}\big|_{m=M}\right)\Big/\left(\sqrt{g_{tt}}\big|_{m=0}\right) = 1 - \sqrt{g_{tt}} \qquad \text{(2-1)}$$

No actual mechanical strain is implied - it's a useful representation of *coordinate determined* radial metric contraction. Of course both transverse 'strains' $\varepsilon_{\theta\theta}, \varepsilon_{\varphi\varphi} = 0$, virtually by definition. Expressing radial metric component as an 'at a point' purely uniaxial strain will be useful. First consider the case along axis **z** of Fig.1, just exterior to shell wall at $r = R^+$. Shell mass *M* can in that instance be treated as concentrated at the centre *r* = 0, so $|\varepsilon_{zz}| \equiv |\varepsilon_{rr}| = 1 - \sqrt{g_{tt}}$ is a maximum that from (1-7), (2-1) is given by

$$|\varepsilon_{zz}|_{r=R^+} = 1 - \sqrt{g_{tt}}\big|_{r=R^+} = 1 - \sqrt{1 - 2GM/(Rc^2)} \qquad \text{(2-2)}$$

That will be taken as a base value against which the computed interior metric 'strain' at *r* = 0, $|\varepsilon_{zz}|_{r=0}$, will be most easily expressed as a fraction $|\varepsilon^*_{zz}|_{r=0} = |\varepsilon_{zz}|_{r=0}\big/|\varepsilon_{zz}|_{r=R^+}$. That value *R* in (2-2) also applies to spatial

displacement of all shell mass elements from centre as target field point. Now *if* metric isotropy applied, angular position θ of any contributing mass element is immaterial and a simple scalar summing over all mass elements would yield $\left|\varepsilon^*_{zz}\right|_{r=0} = 1$, implying an isotropic compressive strain at *r* = 0 equal in amplitude to the radial strain of (2-2). In conflict with $\left|\varepsilon^*_{zz}\right|_{r=0} = 0$ enforced by the patching procedure in part 1. Here though SM anisotropy requires a slightly more involved integration procedure, which still won't deliver $\left|\varepsilon^*_{zz}\right|_{r=0} = 0$!
Consider the shown shell annular element, of radius $R\sin\theta$ thus area A_a and mass M_a given by

$$A_a = 2\pi(R\sin\theta)Rd\theta = 2\pi R^2\sin\theta d\theta \quad \text{(2-3)},$$

$$M_a = \sigma A_a = (M/(4\pi R^2))2\pi R^2\sin\theta d\theta = \tfrac{1}{2}M\sin\theta d\theta \quad \text{(2-4)},$$

where $\sigma = M/\left(4\pi R^2\right)$ is shell area mass density. All annulus sub-elements dM_a as point SM sources at distance $r = R$ from centre as target point, project at angle θ to axis **z** differential contributions $d\varepsilon_{rr}$ at *r* = 0. An initially s*calar* summing over the annular element, dividing by the maximal base value per eqn. (2-2), yields a nominal fractional contribution to $\left|\varepsilon^*_{zz}\right|_{r=0}$ that is just the mass ratio

$$M_a/M = \tfrac{1}{2}\sin\theta d\theta \quad \text{(2-5)} .$$

Summing in turn over all annuli would clearly, as mentioned earlier, give a unity value. Assuming an *isotropic* spatial metric that is. Each elemental $d\varepsilon_{rr}$ is in SM though not owing to one of a mutually orthogonal triple of components of an isotropic metric, but is a locally *uniaxial* strain acting along each relevant ***R*** axis. By axial symmetry only the resolved **z** axis components $d\varepsilon_{zz}$ survive summing over the annulus. Application of the pertinent mechanics resolved-stress/strain relations [2] yields a resolved $d\varepsilon_{zz}$ for each $d\varepsilon_{rr}$ according to

$$\left|d\varepsilon_{zz}\right| = \left|d\varepsilon_{rr}\right|\cos^2\theta \quad \text{(2-6)}$$

Combining (2-5), (2-6), angular integration over all annuli gives the total **z** axis *relative* strain at *r* = 0 as

$$\left|\varepsilon^*_{zz}\right|_{r=0} = \tfrac{1}{2}\int_0^\pi \sin\theta\cos^2\theta d\theta = \tfrac{1}{4}\int_0^\pi \sin 2\theta\cos\theta d\theta = \tfrac{1}{8}\int_0^\pi (\sin\theta + \sin 3\theta)d\theta \quad \text{(2-7)}$$

This readily evaluates 'by hand' to

$$\left|\varepsilon^*_{zz}\right|_{r=0} = \tfrac{1}{8}\left[-\cos\theta - \tfrac{1}{3}\cos 3\theta\right]_0^\pi = \tfrac{1}{8}\left(\left[1+\tfrac{1}{3}\right] - \left[-1-\tfrac{1}{3}\right]\right) = \tfrac{1}{3} \quad \text{(2-8)}$$

Not zero as demanded by the patching prescription of section 1. The fractional error in that result by 'naively' employing above linear integration should be no more than ≈ $r_s/R \simeq 3.0\times10^{-26}$ as remarked earlier. Evidently the anomaly stems from something else. Namely, the intrinsically inconsistent character of SM. A more involved treatment for arbitrary internal location would likely predict non-flatness. A pointless exercise, given the simplest case studied here has already undermined validity of SM.

3: Intrinsically isotropic aka conformally flat exterior metric as cure

3a: What won't work

It may be argued the above two issues are only apparent as evidenced by choosing rather than standard SM, the *notionally* physically equivalent ISM (isotropic SM) [3]:

$$ds^2 = c^2 d\tau^2 = \frac{(1 - r_s/(4r_1))^2}{(1 + r_s/(4r_1))^2} c^2 dt^2 - (1 + r_s/(4r_1))^4 \left(dr_1^2 + r_1^2 \left(d\theta^2 + sin^2\theta d\varphi^2\right)\right) \quad \text{(3-1)}$$

$$r_1 = \tfrac{1}{2} r - \tfrac{1}{4} r_s + \sqrt{\tfrac{1}{4} r(r - r_s)} \quad \text{(3-2)}$$

Which if applied in part 1 permits a shell transition without any break in ϕ dependency for any of the transformed metric components. And in part 2 yields a 'genuinely' scalar addition as per (2-5), eliminating the $\cos^2\theta$ factor in (2-6), leading to a unity value in (2-8). Thus an interior metric consistent with the 'new' patching result of part 1. While *formally* that would follow, is it really the case SM and ISM represent the same physical metric? For SM, proper volume between two closely spaced concentric spherical surfaces - *of given proper area difference* - and centred about a central point mass *M*, is greater by the factor $1/\sqrt{g_{tt}}$ than in the Euclidean gravity-free limit *M*=0. Owing to the proper radial spacing being greater than for the Euclidean case. Similarly between concentric great circles (or arbitrarily small sectors from such) – proper radial gap again greater by the factor $1/\sqrt{g_{tt}}$. So (in-principle!) locally physically determinable non-Euclidean geometry exists for SM. Which effects must vanish on transitioning to the flat MM shell interior.

ISM though is from (3-1) *by construction* locally Euclidean = conformally flat = zero Weyl curvature. If it is argued ISM truly is just a coordinate transformed but physically equivalent version of SM, such 'conformal flatness' is mock; a misleading artefact of using the transformed radial scale. In either case invalidates use of ISM to refute above anomalies found using SM. Which hinge on presence or not of genuine conformal flatness. Thus are thus not mere 'coordinate artefacts' but genuine and should be dealt with as such.

3b: What will work

A necessary *true* cure is an *intrinsically* isotropic aka genuinely conformally flat exterior metric. One maybe viable gravity theory consistent with that is YG (Yilmaz gravity) [4] (whereas say Nordstrom's 1912 theory, also conformally flat, is elsewise non-viable). The YG equivalent line element to that for SM is [5]

$$ds^2 = c^2 d\tau^2 = e^{-2\phi} c^2 dt^2 - e^{2\phi}\left(dr^2 + r^2\left(d\theta^2 + sin^2\theta d\varphi^2\right)\right) \quad \text{(3-3),}$$

with ϕ the Newtonian potential of (1-1). By inspection of (3-3), anomaly A then vanishes. In fact *all* 4 metric components keep pace in *all* regions, if temporal component is expressed as clock-rate. No jumps! Anomaly B is also absent. Each shell point-mass element is in YG the source of an isotropic thus effectively scalar spatial metric field. Summing therefore in identical manner to the anomaly-free temporal component, thus logically yielding unity in (2-8). No nasty shell-shock surprises using YG.

At least one site currently and perhaps maliciously references in a comparison table only the earlier pre 1974 scalar gravity versions of YG dealing just with static gravity [6]. Sort of like comparing 1911 GR with later rival theories in order to 'disprove' 1915 GR. According to proponents of YG it passes all the usual observational 'acid tests'. Yet interestingly GR and YG are opposites in a sense. In the vacuum region, SM hence GR has zero Ricci curvature and non-zero Weyl curvature, while the reverse is true of YG. As to whether 'gravity gravitates' as Yilmaz reasonably assumed and GR denies is still an open question. One study implies it does not [7], but certain underlying assumptions there may be suspect themselves. Regardless, the conformal flatness aspect of that theory ensures no 'event horizons', 'black holes' with attendant causally disconnected regions allowing for bizarre 'worm holes', 'CTC's' or similarly exotic solutions to EFE's. The current avantgarde interest in even more bizarre 'firewalls' just adds to the suspicion GR is sick at its core. YG may or may not be *the* classical gravity theory, but in passing the thin shell filter test, is here certified exotica free!

4: Discussion and conclusion

Conformal flatness naturally follows from correct application of a sensible version of Mach's principle. Where in the slow-motion regime matter is the source of a nominally $1/r$ scalar potential field acting on other matter to *isotropically* increase inertial mass. Directly implying a coordinate determined isotropic and equipartitioned slowing and shrinking. Inertially 'heavier' everything - fundamental particles, atoms, Planck-scale spacetime substructure, etc. Conformally flat spacetime curvature! Anisotropic SM is a nonsense seen from that perspective. 'Instantaneous' influences of distant matter is never invoked, any more than curved tracks in a bubble-chamber imply 'instantaneous' action on the newly formed particles by the surrounding current-carrying windings as source of applied B field. See post entries #27, 29 here for discussion of why the typical application of Mach's principle to rotating reference frames in just wrong-headed. In short, the usual argument that GR 'is not very Machian', while true, is so much the worse for GR, and not vice versa!

Anyways, SM pathology has been shown in two distinct but related ways. Requiring both a physically absurd selective break in functional dependence on Newtonian potential ϕ (part 1), and yielding self-contradictory results for interior spatial metric (part 2). Such anomalies *should* be GR's death sentence - given SM is touted as the unique solution to the EFE's for a static spherically symmetric mass distribution. Observational data *so far* supportive of GR *should* yield place to logic as final arbiter, when a theory claims the equivalent of 2+2=5.

In summary, SM and thus GR, predicts its own downfall in much simpler, classical ways than e.g. seemingly perennial 'singularity' or 'black hole information paradox' issues currently fashionable in quantum gravity circles. Not uncovered at the very outset of GR possibly owing to a shift in emphasis that sidelined Einstein's earlier gedanken experiment approach relying much on physical intuition. Whichever theory of gravity finally succeeds, the admittedly low-level arguments given here strongly imply it **must**, to avoid logical paradox, incorporate conformal flatness (part 3), at least in the classical regime.

References

[1] permalink (under 'Implications')

[2] link (slide 28, - 'strain' will follow the same resolving relation as for uniaxial stress considered there, as here there is no finite valued, material dependent Poisson's ratio to complicate matters.)

[3] permalink (sec 4)

[4] E.g. link (Note downloadable version is corrupted, and for legibility hit full-screen tab.)

[5] link (adapted from eqns (1), (2) there)

[6] prmalink (under Scalar Field theories, sec 7.1)

[7] link