SHORT NOTE ON GENERALIZED LUCAS SEQUENCES

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Abstract

In this note, we consider some generalizations of the Lucas sequence, which essentially extend sequences to triangular arrays. Some new and elegant results are derived.

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1 Introduction

The Lucas sequences are certain integer sequences that satisfy the recurrence relation [6]

$$x_n = px_{n-1} + qx_{n-2},$$

where p and q are fixed integers. The Fibonacci and Lucas numbers are two well-known examples which have extensive applications in algorithms, data structure and biology [5]. Inspired by the ideas in [1, 2, 3, 9], we study triangular array generalizations of the Lucas sequences defined by

$$f_n^{(k+1)} = a^{nk+k} + b^{nk+k}, \tag{1}$$

and

$$g_n^{(k+1)} = a^{n+k} + b^{n+k}, \tag{2}$$

for $n \geq 0$ and $k \geq 1$, where a and b are the roots of the characteristic equation

$$x^2 - px + q = 0.$$

It is evident that $g_{kn}^{(k+1)} = f_n^{(k+1)}$ and when $k = 1$,

$$f_n^{(2)} = a^{n+1} + b^{n+1} = g_n^{(2)} := u_{n+1} \tag{3}$$

are the fundamental Lucas numbers and generalized Lucas primordial sequence [5]. Then trivially, we have $f_n^{(k+1)} = u_{nk+k}$ and $g_n^{(k+1)} = u_{n+k}$. Some less obvious results are presented below.

2 Main results

To begin with, we present some recurrence relations for the sequences $\{f_n^{(k+1)}\}$ and $\{g_n^{(k+1)}\}$.

Proposition 1. *For $k, n \geq 1$, the sequence $\{f_n^{(k+1)}\}$ satisfies the second order recurrence relation*

$$f_{n+1}^{(k+1)} = u_k f_n^{(k+1)} - q^k f_{n-1}^{(k+1)}. \tag{4}$$

Proof. By using (1), (2) and (3), we have

$$\begin{aligned} u_k f_n^{(k+1)} - q^k f_{n-1}^{(k+1)} &= (a^k + b^k)(a^{nk+k} + b^{nk+k}) - (ab)^k (a^{nk} + b^{nk}) \\ &= a^{nk+2k} + b^{nk+2k} \\ &= f_{n+1}^{(k+1)} \end{aligned}$$

as required. □

Proposition 2. *For $k, n \geq 1$, the sequence $\{g_n^{(k+1)}\}$ satisfies the second order recurrence relation*

$$g_{n+1}^{(k+1)} = pg_n^{(k+1)} - qg_{n-1}^{(k+1)}. \tag{5}$$

Proof. Similarly, by using (1) and (2), we have

$$\begin{aligned} pg_n^{(k+1)} - qg_{n-1}^{(k+1)} &= (a+b)(a^{n+k} + b^{n+k}) - ab(a^{n-1+k} + b^{n-1+k}) \\ &= a^{n+1+k} + b^{n+1+k} \\ &= g_{n+1}^{(k+1)} \end{aligned}$$

as required. □

The generating functions of the sequences $\{f_n^{(k+1)}\}$ and $\{g_n^{(k+1)}\}$ are provided in the following result. The generating function methods are of special interest in the study of integer sequences [4]. More applications may be found in e.g. [7, 8].

Proposition 3. *For $k \geq 1$, we have*

$$\sum_{n=0}^{\infty} f_n^{(k+1)} x^n = \frac{u_k - 2q^k x}{1 - u_k x + q^k x^2}, \tag{6}$$

and

$$\sum_{n=0}^{\infty} g_n^{(k+1)} x^n = \frac{u_k - qu_{k-1} x}{1 - px + qx^2}. \tag{7}$$

Proof. Let $f(x) = \sum_{n=0}^{\infty} f_n^{(k+1)} x^n$. From (4), it follows that

$$\begin{aligned} (1 - u_k x + q^k x^2) f(x) &= f_0^{(k+1)} + (f_1^{(k+1)} - f_0^{(k+1)} u_k) x \\ &= a^k + b^k + (a^{2k} + b^{2k} - (a^k + b^k)^2) x \\ &= u_k - 2q^k x. \end{aligned}$$

Next, let $g(x) = \sum_{n=0}^{\infty} g_n^{(k+1)} x^n$. By virtue of (5), we obtain

$$\begin{aligned}
(1 - px + qx^2)g(x) &= g_0^{(k+1)} + (g_1^{(k+1)} - g_0^{(k+1)} p)x \\
&= a^k + b^k + (a^{k-1} + b^{k-1} - (a^k + b^k)(a + b))x \\
&= u_k - qu_{k-1}x.
\end{aligned}$$

□

Finally, we derive an analogue of the famous Simson's identity [11] for the Lucas sequence. More results of this flavor can be found in [10].

Proposition 4. *Define a normalized sequence by*

$$\tilde{g}_n^{(k+1)} = \frac{g_n^{(k+1)}}{a^k - b^k},$$

then

$$\tilde{g}_{n-k}^{(k+1)} \tilde{g}_{n+k}^{(k+1)} - (\tilde{g}_n^{(k+1)})^2 = q^n. \tag{8}$$

Proof. By definition, it suffices to prove

$$g_{n-k}^{(k+1)} g_{n+k}^{(k+1)} - (g_n^{(k+1)})^2 = q^n (a^k - b^k)^2. \tag{9}$$

The right-hand side of (9) reduces to

$$\begin{aligned}
(a^n + b^n)(a^{n+2k} + b^{n+2k}) - (a^{n+k} + b^{n+k})^2 &= (ab)^n (a^k - b^k)^2 \\
&= q^n (a^k - b^k)^2
\end{aligned}$$

as required. □

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