A Nonabelian Gauge Theory of Gravitation

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Abstract

The aim of the paper is to develop a gauge theory, which shall be on the one hand as similar as possible to the original ansatz of Einstein's theory of general relativity, and on the other hand in agreement with other gauge theories as, for instance, those of the electroweak or of the strong interaction. The result is a nonabelian gauge theory with the general linear group $GL(4, \mathbb{R})$ as its gauge group.

1. What is the problem?

There are three fundamental forces in nature. Each of them has its own theory by which it is described. These three theories are:

(a) the nonabelian gauge theory of strong interaction

(b) the nonabelian gauge theory of electro weak interaction

(c) the 'allgemeine Relativitätstheorie'

If the third theory is compared with the two other ones, there are considerable differences between them. The theory of general relativity is stemming from the year 1916 and hence an elder lady of almost one hundred years, while the other two theories are of recent date. On the one hand the theory of general relativity is not quantized and is treating space as a dynamical background, while on the other hand the two mentioned gauge theories are quantized, but are acting before a rigid background. These differences are a severe problem for the task to develop a quantum theory of gravitation, a program that shall not be discussed here (cf. f.i. [5] [6] [7]).

The aim of the present paper is to construct a gauge theory of gravitation, which is satisfying two conditions: On the one hand it shall be similar to the theory of general relativity as far as possible, and on the other hand fulfil the standards of modern gauge theories, also as far as possible.

In the next section it is investigated to what extent the theory of general relativity already has the shape of a gauge theory. As a result it can be shown that part of the theory of general relativity can be considered as a nonabelian gauge theory with the general linear group $GL(4,\mathbb{R})$ as its gauge group, But then difficulties arise. For the remaining part of the theory, Einstein's aim to save general invariance under arbitrary transformations of the coordinates, is not compatible with the essentials of modern gauge theories. A short review of electrodynamics shall illustrate the reason for this discrepancy in section 3. The rest of the paper is oriented towards modern gauge theories. The ansatz of section 4 is still more general and perhaps useful for astrophysical investigations, while the ansatz in section 5 is in correspondence with the gauge theories mentioned at the beginning and rather suited for studying gravitation in subatomic regions.

2. What part of the 'allgemeine Relativitätstheorie' has already the form of a gauge theory?

The element of general relativity most similar to a gauge theory is the formula

$$R^{\varrho}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\varrho}_{\sigma\nu} - \partial_\nu \Gamma^{\varrho}_{\sigma\mu} + \Gamma^{\varrho}_{\tau\mu}\Gamma^{\tau}_{\sigma\nu} - \Gamma^{\varrho}_{\tau\nu}\Gamma^{\tau}_{\sigma\mu} \tag{1}$$

Especially the last two terms are reminding of a commutator, like the last two terms in the formula

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + A_\mu A_\nu - A_\nu A_\mu \tag{2}$$

being valid in nonabelian gauge theories.

In order to investigate, whether this resemblance is only superficial or more profound, one should write equation (1) as

$$R^{\varrho}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\varrho}_{\sigma\nu} - \partial_\nu \Gamma^{\varrho}_{\sigma\mu} + K^{\varrho}_{\sigma\mu\nu} \tag{3}$$

with the quadratic part

$$K^{\varrho}_{\sigma\mu\nu} = \Gamma^{\varrho}_{\tau\mu}\Gamma^{\tau}_{\sigma\nu} - \Gamma^{\varrho}_{\tau\nu}\Gamma^{\tau}_{\sigma\mu} \tag{4}$$

being separated. By interchanging the two factors of the last term it will be achieved that the indices μ and ν are standing in the same seriation in both terms, and only the remaining indices ϱ, σ and τ may perhaps occur at different places.

$$K^{\varrho}_{\sigma\mu\nu} = \Gamma^{\varrho}_{\tau\mu}\Gamma^{\tau}_{\sigma\nu} - \Gamma^{\tau}_{\sigma\mu}\Gamma^{\varrho}_{\tau\nu} \tag{5}$$

If now one is trying to write this expression as

$$K^{\varrho}_{\sigma\mu\nu} = k^{\alpha\beta\varrho}_{\gamma\delta\sigma}\Gamma^{\alpha}_{\gamma\mu}\Gamma^{\beta}_{\delta\nu} \tag{6}$$

with certain constants k, then one has to choose these such that

$$k^{\varrho\tau\varrho}_{\tau\sigma\sigma} = +1 \qquad k^{\tau\varrho\varrho}_{\sigma\tau\sigma} = -1 \qquad k^{\alpha\beta\varrho}_{\gamma\delta\sigma} = 0 \quad \text{else} \tag{7}$$

The next calculation will reveal that the constants k are the same numbers as the structure constants h for the Liealgebra $gl(4,\mathbb{R})$ of the general linear group $GL(4,\mathbb{R})$.

The Liealgebra $gl(4,\mathbb{R})$ of $GL(4,\mathbb{R})$ can be spanned by the 16 generators

$$(s_\sigma^\varrho) = i\,(\delta_\sigma^\varrho) \qquad \text{row } \varrho \quad \text{column } \sigma \qquad 1 \le \varrho, \sigma \le 4 \tag{8}$$

with the Kronecker symbol δ_σ^ϱ. Determining the structure constants by calculating the commutators

$$[(s_\gamma^\alpha), \left(s_\delta^\beta\right)] = i \cdot h_{\gamma\delta\sigma}^{\alpha\beta\varrho} \cdot (s_\sigma^\varrho) \tag{9}$$

one first of all will find that

$$(s_\tau^\alpha) \cdot \left(s_\omega^\beta\right) = i \cdot \delta_\varrho^\alpha \cdot \delta_\tau^\beta \cdot \delta_\sigma^\omega \qquad \left(s_\omega^\beta\right) \cdot (s_\tau^\alpha) = i \cdot \delta_\omega^\alpha \cdot \delta_\varrho^\beta \cdot \delta_\sigma^\tau \tag{10}$$

This implies

$$h_{\tau\sigma\sigma}^{\varrho\tau\varrho} = +1 \qquad h_{\sigma\omega\sigma}^{\omega\varrho\varrho} = -1 \qquad h_{\gamma\delta\sigma}^{\alpha\beta\varrho} = 0 \quad \text{else} \tag{11}$$

Plugging this into the formula

$$R_{\sigma\mu\nu}^\varrho = \partial_\mu \Gamma_{\sigma\nu}^\varrho - \partial_\nu \Gamma_{\sigma\mu}^\varrho + h_{\gamma\delta\sigma}^{\alpha\beta\varrho} \Gamma_{\gamma\mu}^\alpha \Gamma_{\delta\nu}^\beta \tag{12}$$

for the tensor of curvature, and taking into acount that the summation over ω may be substituted by the summation over τ, one will find that (11) is identical with (7) and hence (12) with (1).

Result of this section:

A part of Einstein's general relativity can be written as a nonabelian gauge theory for the gauge group $GL(4,\mathbb{R})$.

3. A short review of electrodynamics

Why is it so difficult, if not to say impossible, to formulate the rest of the theory of general relativity as a gauge theory? The reason for this shall be illustrated by the following short review on the classical electrodynamics.

If one is beginning the reconstruction of this theory with the vector j^μ, which is comprising the charge and the current density, then the potential can be introduced as solution of the wave equation

$$\partial_\mu \partial^\mu A^\nu = e\, j^\nu \tag{13}$$

with suited boundary conditions. Additionally one has the Lorentz convention

$$\partial_\mu A^\mu = 0 \tag{14}$$

The field strengths are defined by the representation

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{15}$$

of the tensor $F_{\mu\nu}$ by potentials. The inhomogeneous Maxwell equations are inherent in the field equation

$$\partial_\mu F^{\mu\nu} = e\, j^\nu \tag{16}$$

while the homogeneous equations are an inference of (15).

Another part of electrodynamics is described by the energy momentum tensor. It is defined by

$$T^{\mu\nu} = -F^{\mu}_{\alpha} F^{\nu\alpha} + \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \tag{17}$$

For the free electromagnetic field it is conserved. But, if charges and currents are present, then one will get

$$\partial_{\mu} T^{\mu\nu} = F^{\mu\nu} j_{\mu} \tag{18}$$

In this case the energy momentum tensor describes the influence of the electromagnetic field on the charges and currents and the transfer of energy momentum.

Now the reasons for the already mentioned difficulties with the complete transformation of general relativity into a gauge theory will become a bit clearer.

Einstein's priority aim was the total invariance of his theory under arbitrary coordinate transformations. As mathematical tools for this task he had tensor calculus and differential geometry at his disposal. In order to achieve his aim Einstein has reduced the tensor curvature tensor to the Ricci tensor by the contraction

$$R^{\varrho}_{\sigma\mu\nu} \rightarrow R_{\sigma\nu} \tag{19}$$

and then described the interaction between matter and the gravitational field by an equation, which is connecting the Ricci tensor with the energy momentum tensor. But this procedure is an obstacle to the continuation of the intended construction of a nonabelian gauge theory. By the projection onto the Ricci tensor the nonabelian character of the theory is lost. The contraction of the index ϱ with the index μ cannot be carried out, because ϱ and σ belong together in a gauge theory and are substituted by a single index a.

The essential point is that Einstein is operating with the energy momentum tensor instead of deriving a field equation, which describes, as usual in quantum field theory, the interaction by a current density. This becomes especially clear by the fact that his description of electrodynamics is containing such an equation (formula (63) on page 813 of the original paper), which is lacking in his theory of gravitation.

For all these reasons the total transformation of the theory of general relativity into a gauge theory cannot be achieved. But instead the construction, as far as it already was done in section 2, shall be continued in the next two sections with the style of modern gauge theories as its guiding principle. The ansatz of section 4 is still more general, while the ansatz of section 5 is following the design of the already existing gauge theories of strong and electroweak interaction.

4. A general ansatz

There are two obstacles standing against the intention to continue the construction of a gauge theory:

(a) In equation (1) there is no coupling constant. But such a constant is needed for the quadratic term in nonabelian gauge theories.

(b) Instead of the two indices ϱ and σ there is only one variable a in a gauge theory.

In order to remove the first fault let be

$$\Gamma^{\varrho}_{\sigma\mu} = -gA^{\varrho}_{\sigma\mu} \qquad R^{\varrho}_{\sigma\mu\nu} = -gF^{\varrho}_{\sigma\mu\nu} \tag{20}$$

Under the condition that the coupling constant g doesn't vanish, one will have

$$F^{\varrho}_{\sigma\mu\nu} = \partial_\mu A^{\varrho}_{\sigma\nu} - \partial_\nu A^{\varrho}_{\sigma\mu} + gh^{\alpha\beta\varrho}_{\gamma\delta\sigma}\Gamma^{\alpha}_{\gamma\mu}\Gamma^{\beta}_{\delta\nu} \tag{21}$$

after having divided all parts of the equation through $-g$. By the further substitution

$$A^{\varrho}_{\sigma\mu} = A^a_\mu \qquad A^{\varrho}_{\tau\mu} = A^b_\mu \qquad A^{\tau}_{\sigma\mu} = A^c_\mu \qquad F^{\varrho}_{\sigma\mu\nu} = F^a_{\mu\nu} \qquad h^{\varrho\tau\varrho}_{\tau\sigma\sigma} = h^{abc} \tag{22}$$

the second deficiency will be removed, too. The result is

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - gh^{abc}A^b_\mu A^c_\nu \tag{23}$$

As a sake of simplicity and with the intention to write down some explicit results the rest of the investigation is reduced to one spatial dimension. In this case the Lie algebra $gl(2,\mathbb{R})$ has four generators

$$s^1 = i\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad s^2 = i\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad s^3 = i\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad s^4 = i\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{24}$$

The commutators

$$[s^a, s^b] = i \cdot h^{abc} \cdot s^c \tag{25}$$

are listed in the table

$[s^i, s_j]$	s^1	s^2	s^3	s^4
s^1	0	$s^3 - s^4$	$-s^1$	$+s^1$
s^2	$s^4 - s^3$	0	$+s^2$	$-s^2$
s^3	$+s^1$	$-s^2$	0	0
s^4	$-s^1$	$+s^2$	0	0

The structure constants are

$$\begin{array}{ll} h^{123} = -1 & h^{213} = +1 \\ h^{124} = +1 & h^{214} = -1 \\ h^{131} = +1 & h^{311} = -1 \\ h^{141} = -1 & h^{411} = +1 \\ h^{232} = -1 & h^{322} = +1 \\ h^{242} = +1 & h^{422} = -1 \end{array}$$

Now the interaction between matter and the gravitational field must be determined. For this purpose matter is described by a mass density ϱ and a suited current density $\vec{j}$, such that the vector j^μ satisfies the continuity relation

$$\partial_\mu j^\mu = 0 \qquad j^\mu = (\varrho, \frac{1}{c}\vec{j}) \tag{26}$$

This equation is taken as the starting point for the gauge formalism. Potentials are introduced by means of the covariant derivative

$$D_\mu = \partial_\mu + gA^a_\mu s^a \tag{27}$$

Variation of the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a,\mu\nu} + gA^a_\mu s^a j^\mu \qquad F^{a,\mu\nu} = \partial_\mu A^{a,\nu} - \partial_\nu A^{a,\mu} - gh^{abc}A^{b,\mu}A^{c,\nu} \tag{28}$$

with respect to the potentials and their derivatives will give the result

$$\partial_\mu F^{a,\mu\nu} = gh^{abc}A^b_\mu F^{c,\mu\nu} + gj^{a,\nu} \qquad j^{a,\nu} = j^\nu s^a \tag{29}$$

Additionally, as in electrodynamics, the Lorentz convention

$$\partial_\mu A^{a,\mu} = 0 \tag{30}$$

is taken as a gauge fixing condition.

5. A special ansatz

In this section the starting point of the gauge formalism is, as usual, the homogeneous Dirac equation

$$(i\gamma^\mu \partial_\mu - m)\psi = 0 \tag{31}$$

The covariant derivative is

$$D_\mu = \partial_\mu + igA^a_\mu s^a \tag{32}$$

Equation (32) then changes into

$$(i\gamma^\mu D_\mu - m)\psi = 0 \tag{33}$$

The tensor of the field strength is

$$F^{a,\mu\nu} = \partial_\mu A^{a,\nu} - \partial_\nu A^{a,\mu} - gh^{abc}A^{b,\mu}A^{c,\nu} \tag{34}$$

Variation of the Lagrangian density

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4}F^a_{\mu\nu}F^{a,\mu\nu} \tag{35}$$

with respect to the three fields inherent in it will give a system

$$
\begin{aligned}
i\gamma^\mu \partial_\mu \psi(x) - m\psi(x) &= g\gamma^\mu \psi(x) A^a_\mu s^a \\
-i\partial_\mu \bar{\psi}(x)\gamma^\mu - m\bar{\psi}(x) &= g\bar{\psi}(x)\gamma^\mu A^a_\mu s^a \\
\partial_\mu F^{a,\mu\nu} &= gh^{abc} A^b_\mu F^{c,\mu\nu} + g\bar{\psi}(x)\gamma^\nu s^a \psi(x)
\end{aligned}
$$

of three coupled equations.

A final remark concerning the special case of vanishing current density may be added.

If in the equation

$$\partial_\mu F^{a,\mu\nu} = gh^{abc} A^b_\mu F^{c,\mu\nu} + gj^{a,\nu} \tag{36}$$

the coupling constant g is set to zero, then (36) will become the free wave equation

$$\Box A^a_\nu = 0 \qquad \Box = \partial_\mu \partial^\mu \tag{37}$$

which is only the four-fold copy of the corresponding equation of electrodynamics. But if one leaves g different from zero and confines the gravitational field to the region without charges and currents, then one has $j^a_\nu = 0$. In this case equation (36) will become

$$\partial_\mu F^{a,\mu\nu} = gh^{abc} A^b_\mu F^{c,\mu\nu} \qquad F^{a,\mu\nu} = \partial^\mu A^{a,\nu} - \partial^\nu A^{a,\mu} - gh^{abc} A^{b,\mu} A^{c,\nu} \tag{38}$$

An additional term occurs, which is due to the nonabelian character of the general linear group $GL(4, \mathbb{R})$ and hence lacking in electrodynamics. By inserting $F^{a,\mu\nu}$ into the left hand side of the first equation in (38), separating the non linear term, and shifting it to the right hand side, one will get

$$\Box A^a_\nu = gh^{abc}(\partial^\mu(A^{b,\mu} A^{c,\nu}) + A^b_\mu F^{c,\mu\nu}) \qquad F^{c,\mu\nu} = \partial^\mu A^{c,\nu} - \partial^\nu A^{c,\mu} - gh^{cde} A^{d,\mu} A^{e,\nu} \tag{39}$$

This is an inhomogeneous field equation for the potentials A^a_μ with an interaction term on the right hand side. The interaction being inherent in it can be understood as self interaction of the gravitational field with itself. From this fact further questions arise: What does the application of perturbation theory bring about? Can the existence of black holes be explained by equation (39)?

But all this must remain a topic for future investigations.

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