Derivation of the reflection integral equation of the zeta function by the quaternionic analysis

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Abstract

We derive the reflection integral equation of the zeta function by the quaternionic analysis.

Many researchers have attempted proof of Riemann hypothesis, but they have not been successful. The proof of this Riemann hypothesis has been an important mathematical issue. In this paper, we attempt to derive the reflection integral equation from the quaternionic analysis as preparation proving Riemann hypothesis.

We obtain a generating function of the inverse Mellin-transform. We obtain new generating function by multiplying the generating function with exponents and reversing the sign. We derive the reflection integral equation from inverse Z-transform of the generating function.

CONTENTS

1 Introduction

1.1 Issue

Many researchers have attempted to prove the Riemann hypothesis, but they have not been successful. The proof of this Riemann hypothesis has been an important mathematical issue. In this paper, we attempt to derive the reflection integral equation from the quaternionic analysis as preparation proving Riemann hypothesis.

1.2 Importance of the issue

Proof of the Riemann hypothesis is one of the most important unsolved problems in mathematics.

For this reason, many mathematicians have tried the proof of Riemann hypothesis. However, those trials were not successful. One of the methods proving Riemann hypothesis is interpreting the zeros of the zeta function as the eigenvalues of a certain operator. However, the operator was not found until now. The reflection integral equation is considered as the one of the operators. For this reason, derivation of the reflection integral equation is an important issue.

1.3 Research trends so far

Leonhard Euler introduced the zeta function in 1737. Bernhard Riemann expanded the argument of the function to the complex number in 1859.

David Hilbert and George Polya[2] suggested that the zeros of the function were probably eigenvalues of a certain operator around 1914. This conjecture is called "Hilbert-Polya conjecture.
Zeev Rudnick and Peter Sarnak[3] are studying the distribution of zeros by random matrix theory in 1996. Shigenobu Kurokawa is studying the field with one element[4] around 1996. Alain Connes[5] showed the relation between noncommutative geometry and the Riemann hypothesis in 1998.
Christopher Deninger[6] is studying the eigenvalue interpretation of the zeros in 1998.

1.4 New derivation method of this paper

We obtain a generating function of the inverse Mellin-transform. We obtain new generating function by multiplying the generating function with exponents and reversing the sign. We derive the reflection integral equation from inverse Z-transform of the generating function.

(Reflection integral equation)

$$\zeta(1-s) = \oint_{S^3} \frac{dt^3}{2\pi^2} \frac{B(s,t+2)}{(t+1)t} \zeta(t) \qquad (1.1)$$

2 Confirmations of known results

In this chapter, we confirm known results.

2.1 Complex number

Euler used the complex number in about 1748.

$$i^2 = -1 \qquad (2.1)$$

We express the complex number as follows.

$$s = \tau + ix \in \mathbb{C} \tag{2.2}$$

$$\tau, x \in \mathbb{R} \tag{2.3}$$

The complex conjugate is shown below.

$$\bar{s} = \tau - ix \in \mathbb{C} \tag{2.4}$$

The function is shown below.

$$f(s) \in \mathbb{C} \tag{2.5}$$

The absolute square is shown below.

$$|s|^2 = s\bar{s} \tag{2.6}$$

In this paper, we have the following symbols.

$$\mathrm{Re}(s) = \frac{1}{2}(s + \bar{s}) = \tau \tag{2.7}$$

$$\mathrm{Im}(s) = \frac{1}{2}(s - \bar{s}) = ix \tag{2.8}$$

2.2 Quaternion

William Rowan Hamilton[7] published the quaternion in 1843.

$$i^2 = j^2 = k^2 = ijk = -1 \tag{2.9}$$

We express the quaternion as follows.

$$s = \tau + ix + jy + kz \in \mathbb{H} \tag{2.10}$$

$$\tau, x, y, z \in \mathbb{R} \tag{2.11}$$

Conjugation of quaternions is shown below.

$$\bar{s} = \tau - ix - jy - kz \in \mathbb{H} \tag{2.12}$$

The function is shown below.

$$f(s) \in \mathbb{H} \tag{2.13}$$

The absolute square is shown below.

$$|s|^2 = \mathrm{s}\bar{s} \tag{2.14}$$

In this paper, we have the following symbols.

$$\mathrm{Re}(s) = \frac{1}{2}(s + \bar{s}) = \tau \tag{2.15}$$

$$\mathrm{Im}(s) = \frac{1}{2}(s - \bar{s}) = ix + jy + kz \tag{2.16}$$

2.3 Complex analysis

Augustin-Louis Cauchy [8] introduced the following equation for complex analysis in 1814. Riemann[9] used this equation for complex analysis in 1851.
(Cauchy - Riemann differential equation)

$$\frac{\partial f}{\partial \tau} + i\frac{\partial f}{\partial x} = 0 \tag{2.17}$$

We write the above equation as follows shortly.

$$\frac{\partial f}{\partial \bar{s}} = 0 \tag{2.18}$$

Cauchy introduced the following formula.
(Cauchy's integral formula)

$$f(s) = \oint_{S^1} \frac{f(\mathrm{t})}{(t - \mathrm{s})}\frac{dt}{2\pi i} \tag{2.19}$$

S^1 is the closed path.

2.4 Quaternionic analysis

Karl Rudolf Fueter [10] introduced the following equation as the analogue of Cauchy - Riemann equation for quaternionic analysis in 1934.
(Cauchy - Riemann - Fueter differential equation)

$$\frac{\partial f}{\partial \tau} + i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z} = 0 \tag{2.20}$$

We write the above equation as follows shortly.

$$\frac{\partial f}{\partial \bar{s}} = 0 \tag{2.21}$$

Fueter introduced the following formula as the analogue of Cauchy's integral formula.
(Cauchy - Fueter's integral formula)

$$f(s) = \oint_{S^3} \frac{(t - \mathrm{s})^{-1}}{|t - \mathrm{s}|^2} f(t)\frac{Dt}{2\pi^2} \tag{2.22}$$

Here S^3 is the three-dimensional closed surface.

The detail of the quaternionic analysis described in Sudbery's paper[11] in 1979.

In this paper, we introduce the following new formula.
(Integral formula of quaternion)

$$f(s) = \oint_{S^3} \frac{-dt^3}{2\pi^2} \frac{f(t)}{(t-s)^3} \tag{2.23}$$

Here S^3 is the three-dimensional closed surface.

2.5 Mellin transform

Hjalmar Mellin[12] published Mellin transform in 1904.
(Mellin transform)

$$f(s) = M[F(x)] \tag{2.24}$$

$$f(s) = \int_0^\infty x^{s-1} F(x) dx \tag{2.25}$$

We express the inverse Mellin transform by the following contour integration.
(Inverse Mellin transform)

$$F(z) = M^{-1}[f(s)] \tag{2.26}$$

$$F(z) = \oint_{S^3} \frac{-ds^3}{2\pi^2} \frac{f(s)}{z^{s+2}} \tag{2.27}$$

S^3 is the three-dimensional closed surface. The surface circles around all poles of the integrand. For example, we suppose the surface S^3 as follows. The white circles mean poles.

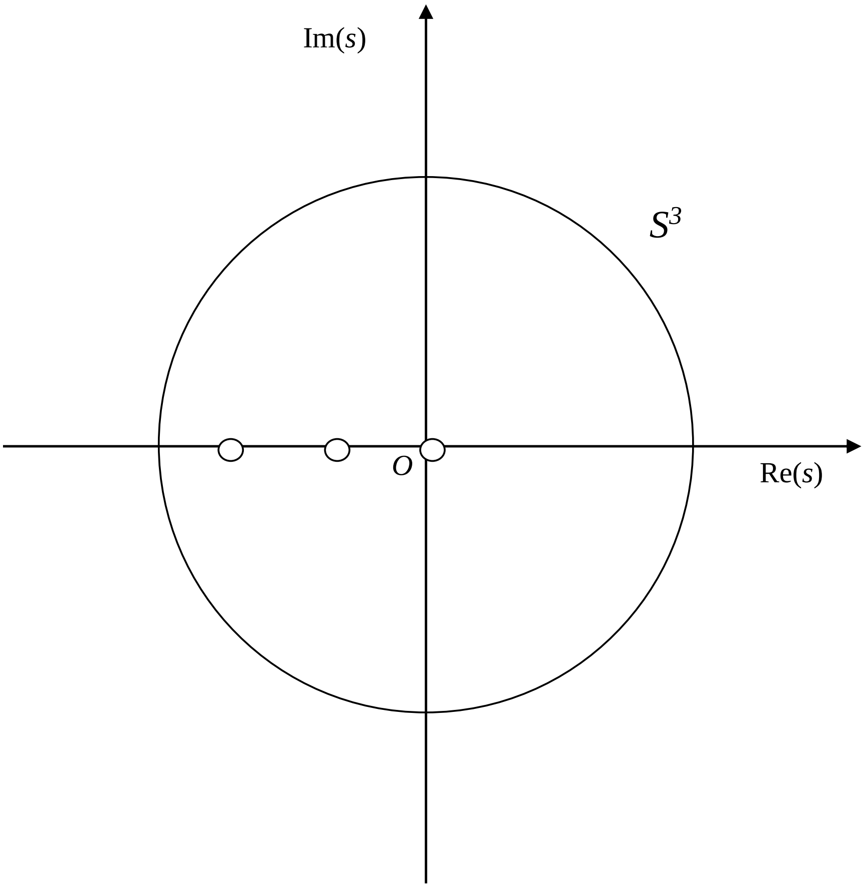

Figure 2.1: The closed surface S^3

2.6 Hurewicz's *Z*-transform

Witold Hurewicz [13] published *Z*-transform in 1947. When the function *F* (*z*) is holomorphic over the domain *D* = {0 <|z|< *R*}, the function can be transformed to the series which converges uniformly in wider sense over the domain.
(*Z*-transform)

$$F(z) = Z[f(n)] \tag{2.28}$$

$$F(z) = \sum_{n=\infty}^{\infty} \frac{f(n)}{z^n} \tag{2.29}$$

$$D = \{0 < |z| < R\} \tag{2.30}$$

The inverse *Z*-transform of quaternion is shown below.
(Inverse *Z*-transform)

$$f(n) = Z^{-1}[F(z)] \tag{2.31}$$

$$f(n) = \oint_{S^3} \frac{-dz^3}{2\pi^2 z^3} z^n F(z) \tag{2.32}$$

The three-dimensional closed surface S^3 is shown below.

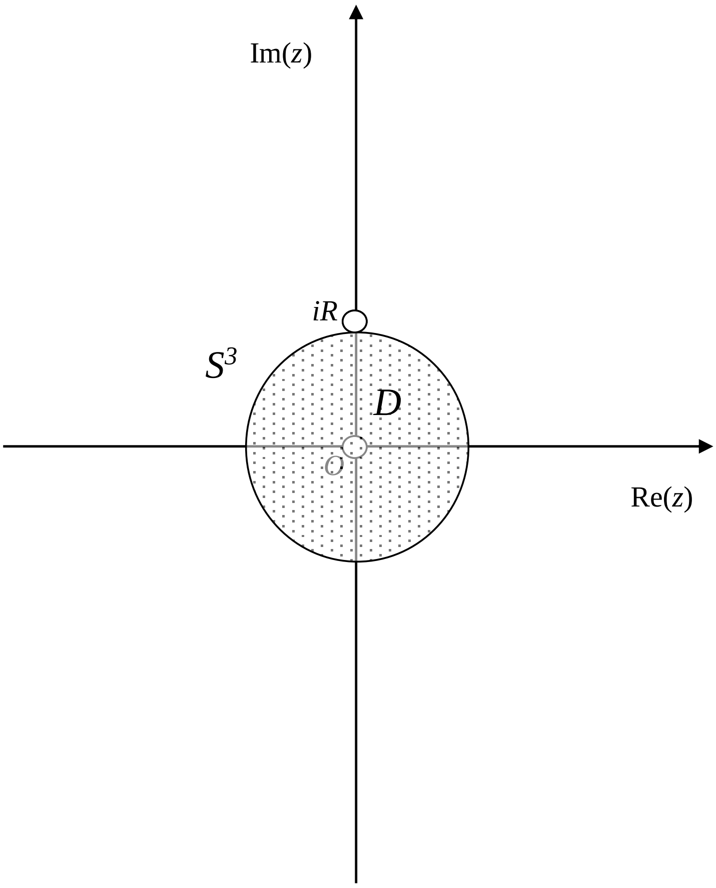

Figure 2.2: The closed surface S^3

2.7 Residue theorem of quaternion

We have the following residue theorem of the complex number.
(Residue theorem of quaternion)

$$\oint_C F(z) \frac{dz}{2\pi i} = \operatorname*{Res}_{z \to c+} F(z)\, dz \tag{2.33}$$

We introduced the residue theorem of the quaternion as follows.

We suppose that the function $F(z)$ has the isolated singularities c in the three-dimensional closed surface S^3 and is holomorphic except for the isolated singularities. Then, we have the following formula.
(Residue theorem of quaternion)

$$\oint_{S^3} \frac{-dz^3}{2\pi^2} F(z) = \operatorname*{Res}_{z \to c+} F(z)\, dz^3 \qquad (2.34)$$

We suppose that the function $F(z)$ has isolated singularities c_k in the three-dimensional closed surface S^3 and is holomorphic except for the isolated singularities. Then, we have the following formula.
(Residue theorem of quaternion)

$$\oint_{S^3} \frac{-dz^3}{2\pi^2} F(z) = \sum_{k=1}^{n} \operatorname*{Res}_{z \to c_k+} F(z)\, dz^3 \qquad (2.35)$$

2.8 Euler's gamma function

Leonhard Euler[14] introduced the gamma function as a generalization of the factorial in 1729. The gamma function is defined by the following equation.
(Definitional integral formula of the gamma function)

$$\Gamma(s) = \int_0^{\infty} x^{s-1} e^{-x} dx \qquad (2.36)$$

We introduced the integral representation of quaternion of gamma function.
(Contour integration of gamma function)

$$\frac{1}{\Gamma(1-s)} = \oint_{\gamma^3} \frac{-dz^3}{2\pi^2 z^3} z^s e^z \qquad (2.37)$$

The three-dimensional closed surface γ^3 is shown in the following figure. The white circles mean poles.

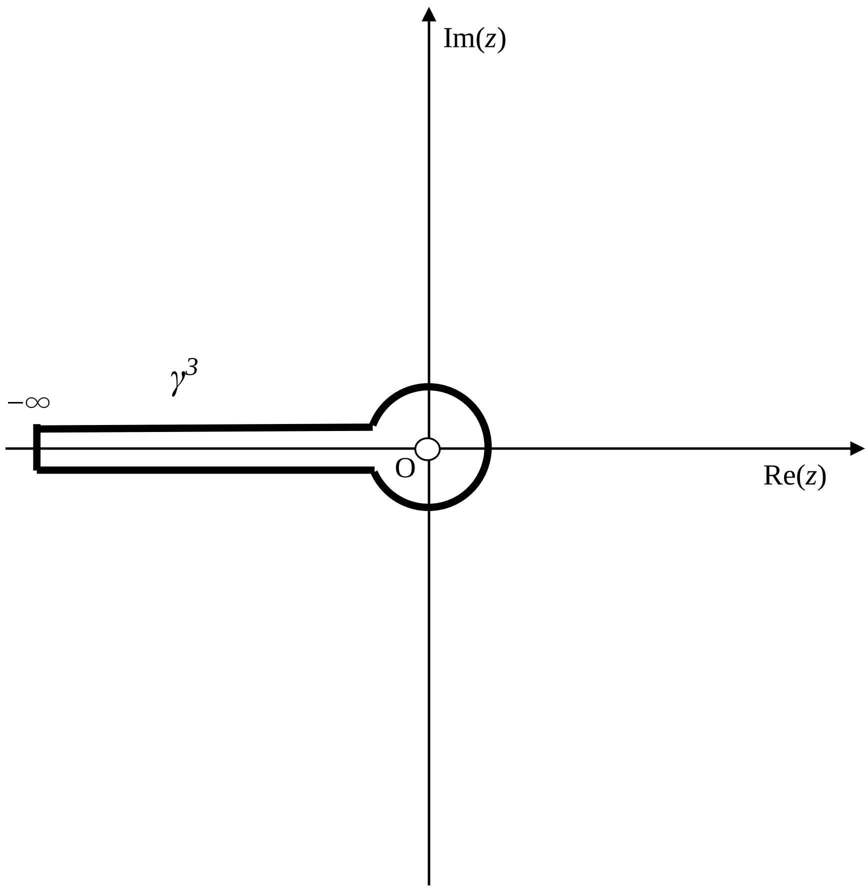

Figure 2.3: The closed surface γ^3

2.9 Euler's beta function

Leonhard Euler introduced the beta function in 1768 in his book[15]. We express the Beta function by using the gamma functions.
(Definitional formula of the beta function)

$$B(s,t) = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)} \qquad (2.38)$$

2.10 Riemann zeta function

Bernhard Riemann[16] expanded the argument of the zeta function to the complex number in 1859. The definitional series of the function is shown below.
(The definitional series)

$$\zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots = \sum_{k=1}^{\infty} \frac{1}{k^s} \qquad (2.39)$$

The function is also defined by the following formula.
(Definitional integral formula)

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} x^{s-1} \frac{e^{-x}}{1 - e^{-x}} dx \tag{2.40}$$

We interpret the above formula as the following Mellin transform.
(Mellin transform)

$$g(s) = M[G(x)] \tag{2.41}$$

$$g(s) = \int_0^{\infty} x^{s-1} G(x) dx \tag{2.42}$$

$$G(x) = \frac{e^{-x}}{1 - e^{-x}} \tag{2.43}$$

$$g(s) = \zeta(s)\Gamma(s) \tag{2.44}$$

The inverse Mellin transform of the function is shown below. The variable c_k is the isolated singularities.
(Inverse Mellin transform)

$$G(x) = M^{-1}[g(s)] \tag{2.45}$$

$$G(x) = \oint_{S^3} \frac{-ds^3}{2\pi^2} \frac{g(s)}{x^{s+2}} \tag{2.46}$$

$$G(x) = \frac{e^{-x}}{1 - e^{-x}} \tag{2.47}$$

$$g(s) = \zeta(s)\Gamma(s) \tag{2.48}$$

The contour integration of the zeta function is shown below.
(The contour integration)

$$\frac{\zeta(s)}{\Gamma(1-s)} = \oint_{\gamma^3} \frac{-dz^3}{2\pi^2 z^3} \frac{z^s e^z}{1 - e^z} \tag{2.49}$$

We interpret the above formula as the following the inverse *Z*-transform.
(Inverse *Z*-transform)

$$h(s) = Z^{-1}[H(z)] \qquad (2.50)$$

$$h(s) = \oint_{\gamma^3} \frac{-dz^3}{2\pi^2 z^3} z^s H(z) \qquad (2.51)$$

$$H(z) = \frac{e^z}{1 - e^z} \qquad (2.52)$$

$$h(s) = \frac{\zeta(s)}{\Gamma(1 - s)} \qquad (2.53)$$

The closed surface γ^3 is shown in the following figure. The white circles mean poles.

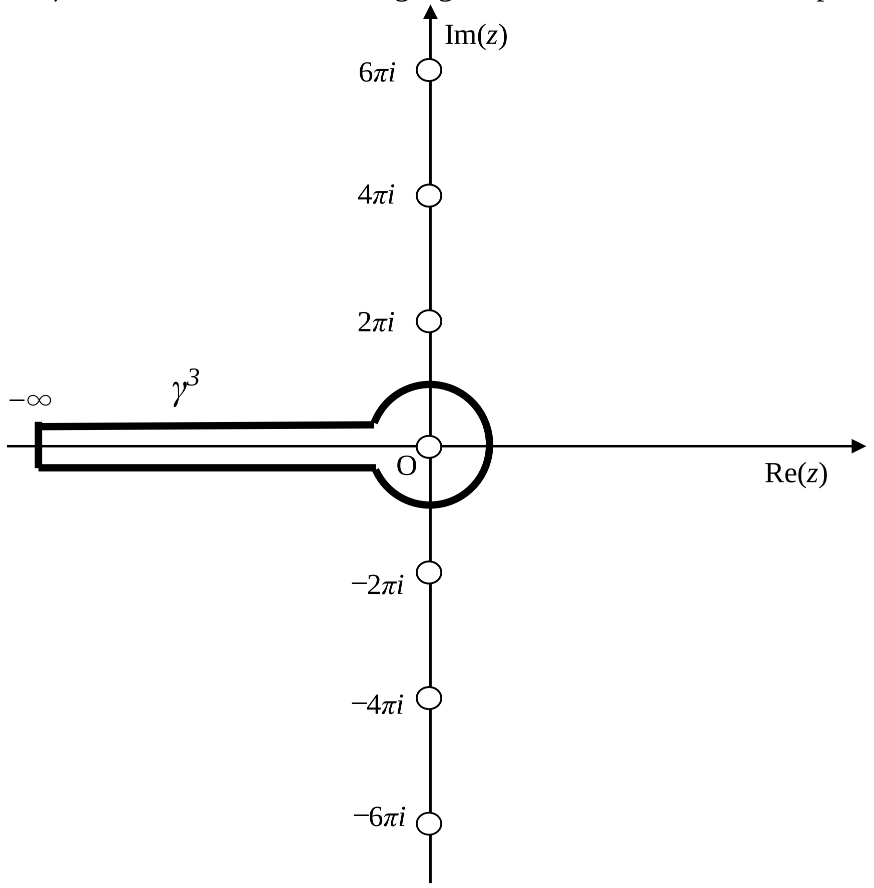

Figure 2.4: The closed surface γ^3

The Z-transform of the function as follows.
(Z-transform)

$$H(z) = Z[h(s)] \quad (2.54)$$

$$H(z) = \sum_{s=-\infty}^{\infty} \frac{h(s)}{z^s} \quad (2.55)$$

$$H(z) = \frac{e^z}{1 - e^z} \quad (2.56)$$

$$h(s) = \frac{\zeta(s)}{\Gamma(1 - s)} \quad (2.57)$$

The generating functions of Mellin transform and Z-transform have the following relations.

$$H(z) = -e^z G(z) \quad (2.58)$$

$$H(z) = G(-z) \quad (2.59)$$

Riemann showed the following reflection formula.
(Riemann's reflection formula)

$$\zeta(1 - s) = \frac{2}{(2\pi)^s} \Gamma(s) \cos\left(\frac{\pi}{2} s\right) \zeta(s) \quad (2.60)$$

Riemann proposed the following conjecture.
(Riemann hypothesis)
Nontrivial zeros all have real part 1/2.

We express the examples of nontrivial zeros ρ_1 and ρ_2 in the following figure and equation. The black circles are zeros and the white circle means a pole.

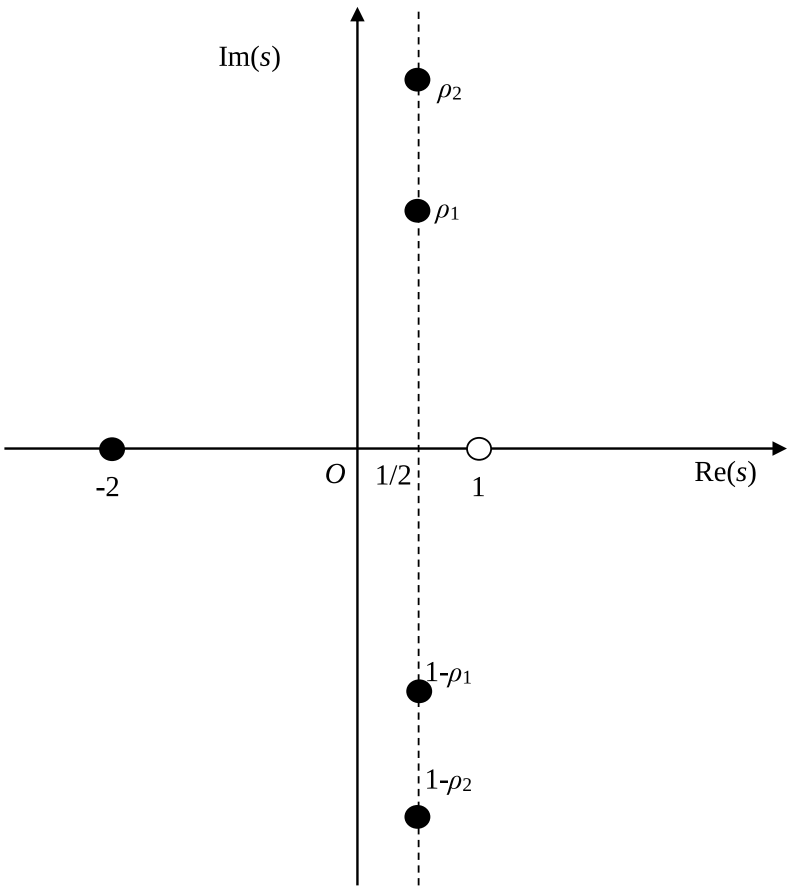

Figure 2.5: Nontrivial zeros of zeta function

$$\rho_1 = \frac{1}{2} + i(14.13\cdots) \tag{2.61}$$

$$\rho_2 = \frac{1}{2} + i(21.02\cdots) \tag{2.62}$$

Since the proof of the Riemann hypothesis has not been successful, it has been an important mathematical issue.

2.11 Bernoulli polynomials

Jakob Bernoulli introduced Bernoulli numbers in 1713 in his book[17]. Seki Takakazu also introduced Bernoulli numbers in 1712 in his book[18] independently. Bernoulli numbers are defined by Bernoulli polynomials. The definition of Bernoulli polynomials is shown below.
(Bernoulli polynomials)

$$\frac{xe^{qx}}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n(q)}{n!} x^n \tag{2.63}$$

The above series are called "formal power series" because it does not converge over the whole domain. The convergent radius is 2π because the minimum distance between origin and poles is 2π for the generating function.

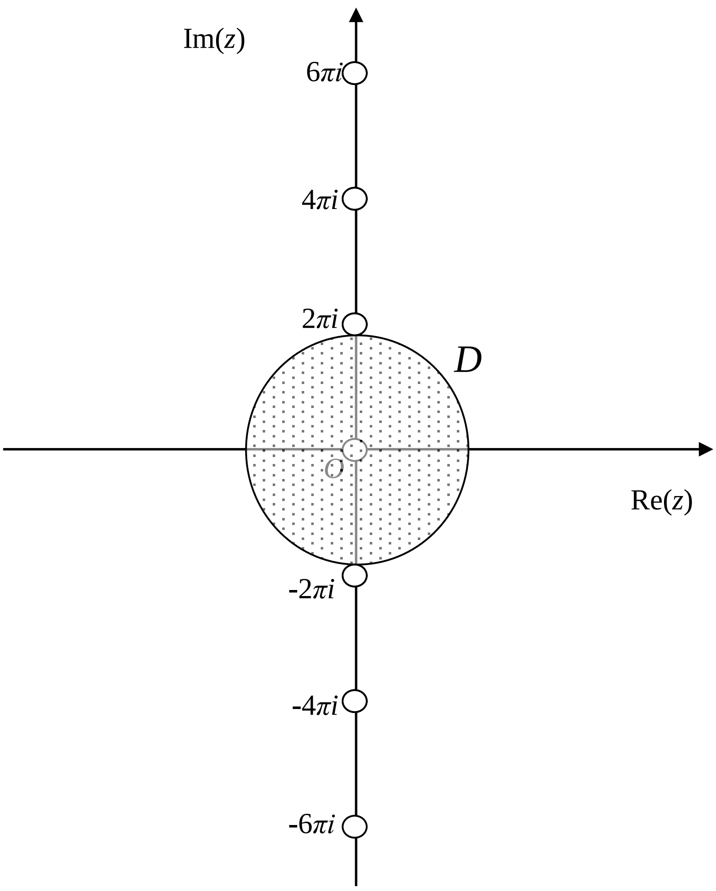

Figure 2.6: The convergent radius of Bernoulli polynomials

2.12 Bernoulli numbers

We suppose that $B_n(q)$ is Bernoulli polynomials. There are the following two kinds of definitions of Bernoulli numbers B_n.

$$B_n = B_n(0) \tag{2.64}$$

$$B_n = B_n(1) \tag{2.65}$$

In this paper, in order to unite with the definition of Bernoulli function explained later, the latter definition is adopted. At the former and the latter, there is the following difference by $n = 1$.

$$B_n(0) = -\frac{1}{2} \tag{2.66}$$

$$B_n(1) = \frac{1}{2} \tag{2.67}$$

Bernoulli polynomials $B_n(1)$ equals to $B_n(0)$ except $n = 1$. The definition of Bernoulli numbers is shown below.
(Definitional series of Bernoulli numbers)

$$\frac{xe^x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n \tag{2.68}$$

Bernoulli numbers has the following formula for even positive integer n.
(Reflection formula of Bernoulli numbers)

$$\zeta(n) = \frac{(2\pi)^n}{2} \frac{1}{n!} \ (-1)^{\frac{n}{2}+1} B_n \tag{2.69}$$

Bernoulli numbers has the following formula for natural number n.
(Formula of Bernoulli numbers)

$$\zeta(-n) = -\frac{B_{n+1}}{n+1} \tag{2.70}$$

In this paper, we express the Z-transform of Bernoulli numbers as shown below.

$$H(z) = Z[h(s)] \tag{2.71}$$

$$H(z) = \sum_{s=-\infty}^{\infty} \frac{h(s)}{\mathrm{z}^s} \tag{2.72}$$

$$H(z) = \frac{e^z}{1 - e^z} \tag{2.73}$$

$$h(s) = \frac{-B_{1-s}}{\Gamma(-s)} \tag{2.74}$$

3 Derivation of the reflection integral equation

3.1 The framework of the method to derivation

The inverse Mellin transform of zeta function is shown below.

$$G(z) = M^{-1}[g(s)] \tag{3.1}$$

$$G(z) = \oint_{S^3} \frac{-ds^3}{2\pi^2} \frac{g(s)}{z^{s+2}} \tag{3.2}$$

$$G(z) = \frac{e^{-z}}{1 - e^{-z}} \tag{3.3}$$

$$g(s) = \zeta(s)\Gamma(s) \tag{3.4}$$

The inverse Z-transform of the function is shown below.

$$h(s) = Z^{-1}[H(z)] \tag{3.5}$$

$$h(s) = \oint_{\gamma^3} \frac{-dz^3}{2\pi^2 z^3} z^s H(z) \tag{3.6}$$

$$H(z) = \frac{e^{z}}{1 - e^{z}} \tag{3.7}$$

$$h(s) = \frac{\zeta(s)}{\Gamma(1 - s)} \tag{3.8}$$

The generating functions of Mellin transform and Z-transform have the following relations.

$$H(z) = -e^{z} G(z) \tag{3.9}$$

$$H(z) = G(-z) \tag{3.10}$$

The framework of the method to derivation is shown below.

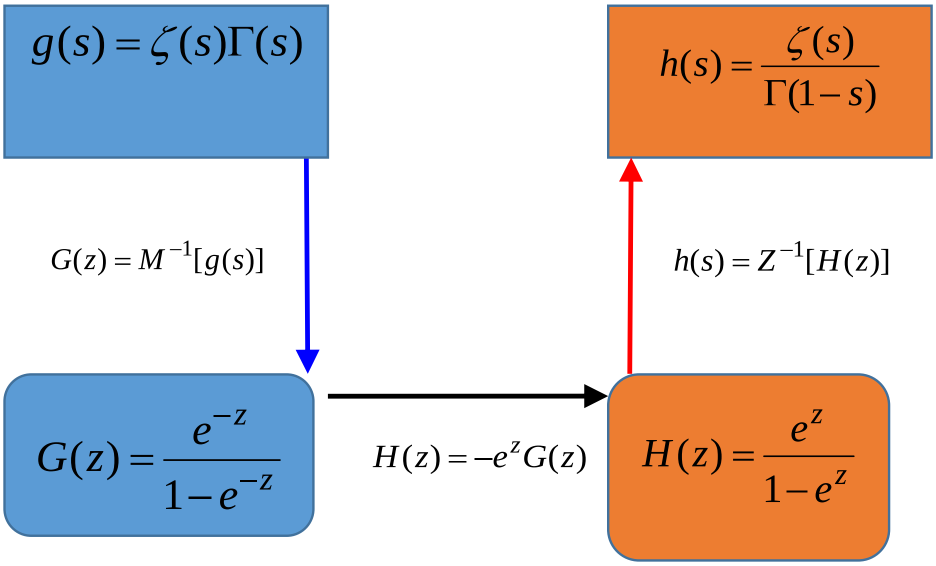

Figure 3.1: The framework of the method to derivation

We obtain the reflection integral equation by the above framework.
(Reflection integral equation)

$$\zeta(1-s) = \oint_{S^3} \frac{dt^3}{2\pi^2} \frac{B(s,t+2)}{(t+1)t} \zeta(t) \qquad (3.11)$$

This paper explains this derivation method.

3.2 Derivation of the reflection integral equation from the inverse Mellin transform

Inverse Mellin transform of the zeta function is shown below.

$$G(z) = M^{-1}[g(t)] \qquad (3.12)$$

$$G(z) = \oint_{S^3} \frac{-dt^3}{2\pi^2} \frac{g(t)}{z^{t+2}} \qquad (3.13)$$

$$G(z) = \frac{e^{-z}}{1-e^{-z}} \qquad (3.14)$$

$$g(t) = \zeta(t)\Gamma(t) \qquad (3.15)$$

The three-dimensional closed surface S^3 of the inverse Mellin transform needs to circle around all poles of the integrand. Then we adopt the closed surface S^3 as follows. The white circles mean poles.

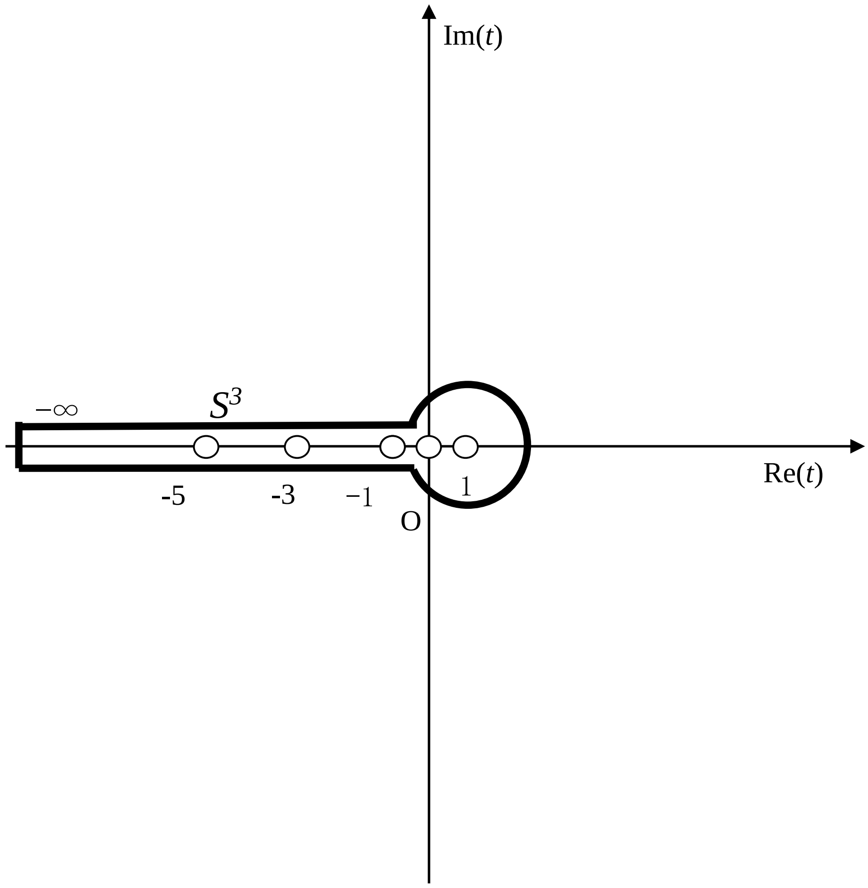

Figure 3.2: The closed surface S^3

On the other hand, Inverse *Z*-transform of the function is shown below.
(Inverse *Z*-transform)

$$h(s) = Z^{-1}[H(z)] \tag{3.16}$$

$$h(s) = \oint_{\gamma^3} \frac{-dz^3}{2\pi^2 z^3} z^s H(z) \tag{3.17}$$

$$H(z) = \frac{e^z}{1 - e^z} \tag{3.18}$$

$$h(s) = \frac{\zeta(s)}{\Gamma(1 - s)} \tag{3.19}$$

The three-dimensional closed surface γ^3 is shown in the following figure. The white circles mean poles.

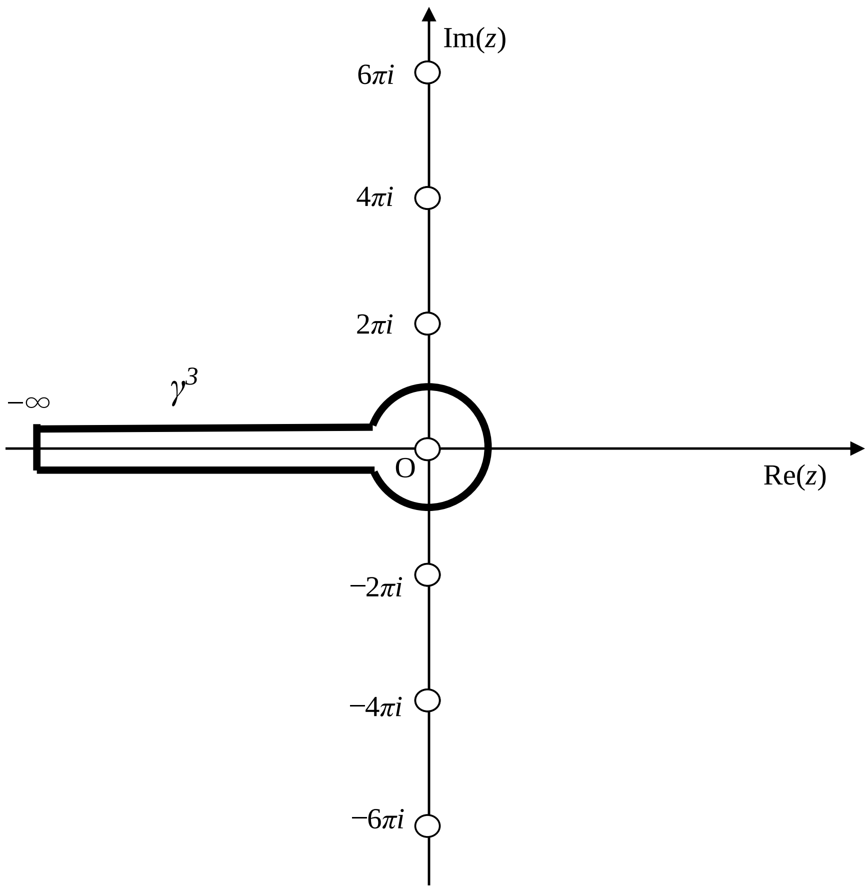

Figure 3.3: The closed surface γ

We deform the equation of the inverse *Z*-transform as follows.

$$h(s) = \oint_{\gamma^3} \frac{-dz^3}{2\pi^2 z^3} z^s \{-e^z G(z)\} \qquad (3.20)$$

We replace *s* to 1-*s* as follows.

$$h(1-s) = \oint_{\gamma^3} \frac{-dz^3}{2\pi^2 z^3} \frac{1}{z^{1-s}} \{-e^z G(z)\} \qquad (3.21)$$

We obtain the following equation by substituting the equation of the inverse Mellin transform.

$$h(1-s) = \oint_{\gamma^3} \frac{-dz^3}{2\pi^2 z^3} \frac{1}{z^{1-s}} \left\{-e^z \oint_{S^3} \frac{-dt^3}{2\pi^2} \frac{g(t)}{z^{t+2}}\right\} \qquad (3.22)$$

In order to integrate the above equation for the variable *z*, we deform the above equation as follows.

$$h(1-s) = \oint_{S^3} \frac{dt^3}{2\pi^2} \left(\oint_{\gamma^3} \frac{-e^z}{z^{s+t+4}} \frac{dz^3}{2\pi^2} \right) g(t) \qquad (3.23)$$

We apply the following formula to the above equation.
(Contour integral formula of the gamma function)

$$\frac{1}{\Gamma(u+2)} = \oint_{\gamma^3} \frac{-dz^3}{2\pi^2 z^3} \frac{e^z}{z^{u+1}} \tag{3.24}$$

The three-dimensional closed surface γ^3 is shown in the following figure.
Then we obtain get the following equation.

$$h(1-s) = \oint_{S^3} \frac{dt^3}{2\pi^2} \frac{g(t)}{\Gamma(s+t+2)} \tag{3.25}$$

$$\frac{\zeta(1-s)}{\Gamma(s)} = \oint_{S^3} \frac{dt^3}{2\pi^2} \frac{\Gamma(t)\zeta(t)}{\Gamma(s+t+2)} \tag{3.26}$$

$$\zeta(1-s) = \oint_{S^3} \frac{dt^3}{2\pi^2} \frac{\Gamma(s)\Gamma(t)\zeta(t)}{\Gamma(s+t+2)} \tag{3.27}$$

$$\zeta(1-s) = \oint_{S^3} \frac{dt^3}{2\pi^2} \frac{\Gamma(s)\Gamma(t+2)\zeta(t)}{\Gamma(s+t+2)(t+1)t} \tag{3.28}$$

Here, we simplify the above equation by using the following the beta function.

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{3.29}$$

As the result, we obtain the following equation.
(Reflection integral equation)

$$\zeta(1-s) = \oint_{S^3} \frac{dt^3}{2\pi^2} \frac{B(s,t+2)}{(t+1)t} \zeta(t) \tag{3.30}$$

The three-dimensional closed surface S^3 is shown in the following figure. The white circles mean poles.

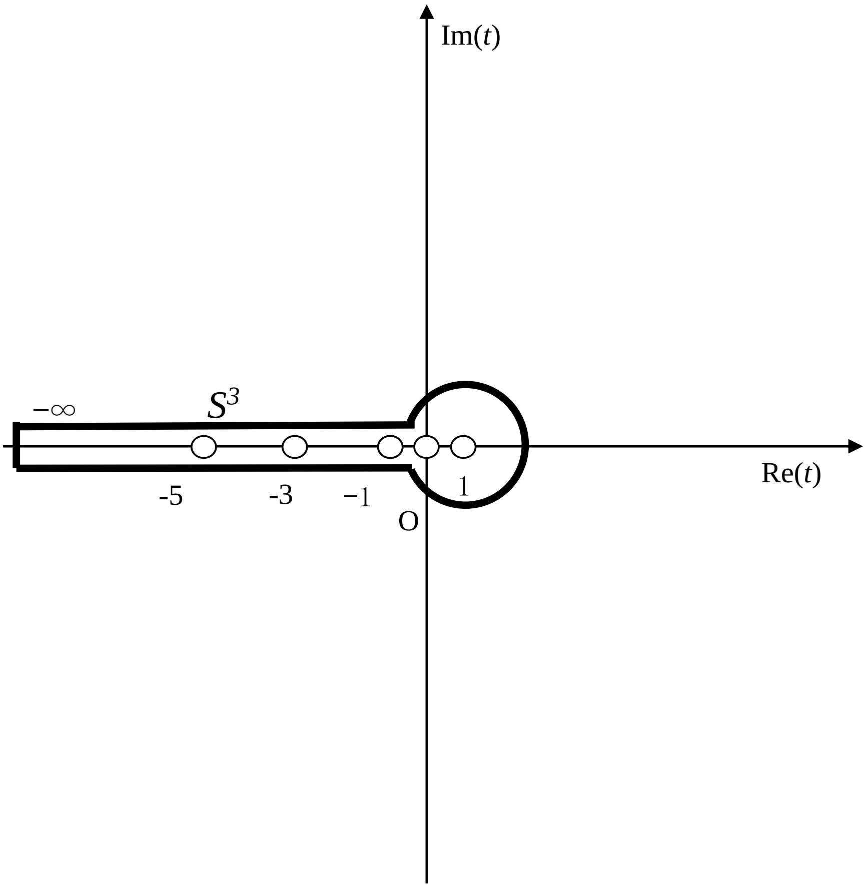

Figure 3.4: The closed surface S^3

4 Conclusion

We obtained the following results in this paper.
- We derived reflection integral equation.

5 Future issues

The future issues are shown below.

- To study the eigenvalues of integral equation.

6 Appendix

6.1 Integral formula of quaternion

The deriving method of the integral formula of quaternion is shown below.

We consider the 3-dimensional sphere surrounding the point *s*.
We divide the sphere to the small 3-dimensional cube.
The number of the cube is *n*.
We number the each cube, *m*. The positon of the each cube is t_m.

The value of the function of positon *s* is a mean value of the function of all point t_m on the sphere.

$$f(s) = \sum_{m=1}^{n} \frac{1}{n} f(t_m) \qquad (6.1)$$

We describe the position by the polar coordinate as follows.

$$t = r \exp(i\phi + j\chi + k\psi) \qquad (6.2)$$

Each side of the cube of the positon t_m is shown below.

$$\delta_\phi t = it\ \delta\phi \qquad (6.3)$$

$$\delta_\chi\ t = jt\ \sin\phi\ \ \delta\chi \qquad (6.4)$$

$$\delta_\psi t = kt\ \sin\phi \sin\chi\ \ \delta\psi \qquad (6.5)$$

$$\delta t^3 = \ \delta_\phi t \delta_\chi t \delta_\psi t = ijk\ t^3 \sin^2\phi \sin\chi\ \delta\phi\delta\chi\delta\psi \qquad (6.6)$$

The volume $|\delta V|$ of the cube is shown below.

$$|\delta V| = |\delta_\phi t\ \delta_\chi t\ \delta_\psi t| \qquad (6.7)$$

On the other hand, the volume $|V|$ of the sphere of the radius R is shown below.

$$|V| = 2\pi^2 R^3 \qquad (6.8)$$

We express the radius *R* as follows.

$$R = |t - s| \qquad (6.9)$$

We express the number *n* of the cube as follows.

$$\frac{1}{n} = \frac{|\delta V|}{|V|} \qquad (6.10)$$

In order to remove the symbol of absolute value, we conform the direction of the volume *V* and the volume δV.
Then, we divide the side of cube by *i*, *j*, and *k*.

$$\delta'_{\phi} t = \frac{\delta_{\phi} t}{i} \tag{6.11}$$

$$\delta'_{\chi} t = \frac{\delta_{\chi} t}{j} \tag{6.12}$$

$$\delta'_{\psi} t = \frac{\delta_{\psi} t}{k} \tag{6.13}$$

We express the operation by the following figure.

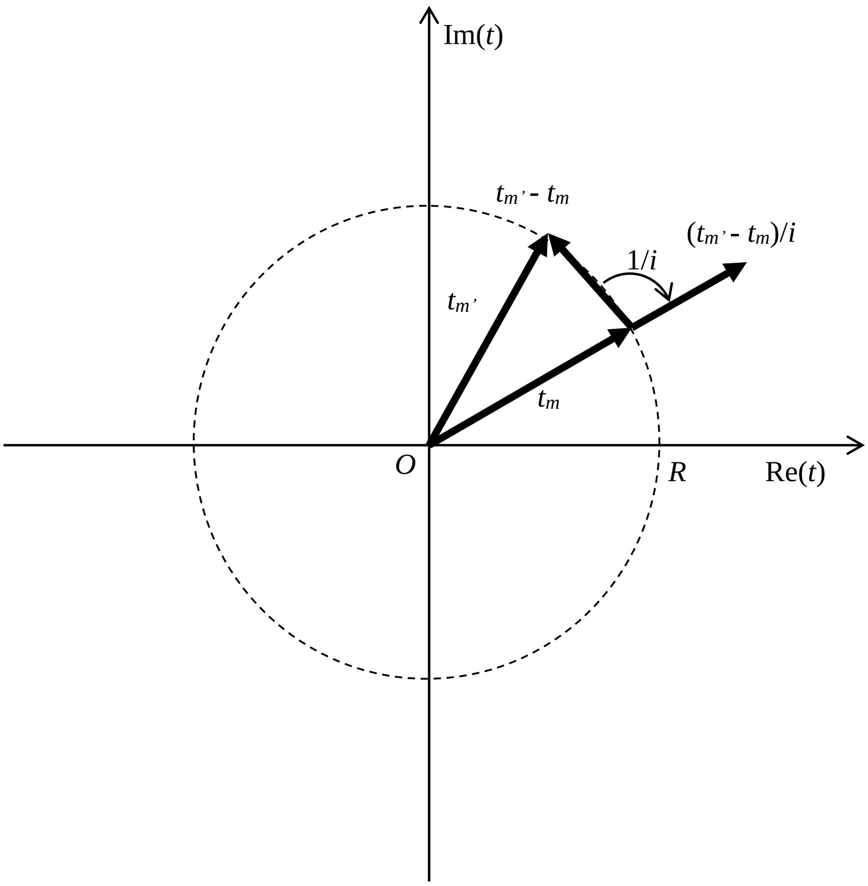

Figure 6.1: Integral formula of quaternion

We express the number n of the cube as follows.

$$\frac{1}{n} = \frac{|\delta V|}{|V|} = \frac{|\delta'_{\phi} t\, \delta'_{\chi} t\, \delta'_{\psi} t|}{2\pi^2 |t-s|^3} \tag{6.14}$$

$$\frac{1}{n} = \frac{\delta'_{\phi} t\, \delta'_{\chi} t\, \delta'_{\psi} t}{2\pi^2 (t-s)^3} \tag{6.15}$$

$$\frac{1}{n} = \frac{\delta_{\phi} t\, \delta_{\chi} t\, \delta_{\psi} t}{2\pi^2 (t-s)^3 ijk} \tag{6.16}$$

We express the function $f(s)$ as follows.

$$f(s) = \sum_{m=1}^{n} \frac{\delta_{\phi} t_m\, \delta_{\chi} t_m\, \delta_{\psi} t_m}{2\pi^2 (t-s)^3 ijk} f(t_m) \tag{6.17}$$

$$f(s) = \sum_{m=1}^{n} \frac{-\delta t_m^3}{2\pi^2 (t-s)^3} f(t_m) \tag{6.18}$$

We obtain the follownig integral formula of quaternion from the above formula.
(Integral formula of quaternion)

$$f(s) = \oint_{S^3} \frac{-dt^3}{2\pi^2} \frac{f(t)}{(t-s)^3} \tag{6.19}$$

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