The logic of Complexity

When one and one do not add up to two

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Contents

Part I_ some preliminary ideas

Complex

- Etymologically it comes from 'complexus': that which is woven together

- We use the term 'complex' to designate phenomena in which "the whole is different than the sum of its parts"

Non-complex = independent

- The opposite concept is not simple as composed by few parts or easy to understand, but as non-compound or independent; *i.e.: that which is not woven*

- We consider a phenomena as not being complex when 'the whole is equal to the sum of its parts'

Complexity = non linearity

- In mathematical terms, the complexity or not of an aspect of a phenomena can be determined by reviewing the relation it has with its constituent aspects:

- Linear [independent]

$$f(ax + by) = af(x) + bf(y)$$

- Non linear [complex]

$$f(ax + by) \neq af(x) + bf(y)$$

Complexity=non linearity

- Or in other words:
  - Linear [independent] i.e.: $1+1=2$
  - Non linear [complex] i.e.: $1+1\neq2$

Linearity and non-linearity are not mutually exclusive

- A phenomena can combine complex and non-complex aspects. For instance, a stock purchase:
  - The economic cost of the purchase varies linearly with the amount of purchased stocks; it is directly proportional to such number.
  - The later variation of stocks prices is non-linear; it follows 'chaotic' rules.
  - The utility we obtain from the capital gains is non-linear; it has diminishing marginality.

Degree of truth

- Arises in the context of Fuzzy Logic /Fuzzy Set Theory [Zadeh, 1965] to characterize concepts that can be partially true when referred to an object.
  - Classic Logic only admits two truth values: true or false; white or black
  - Fuzzy Logic accepts 'degrees of truth'; it equates accepting that besides white and black there can be many 'shades of grey'

Nearly decomposable concepts

- Underlies the proposal of L-Fuzzy Sets [Goguen, 1967]: the degree of truth in relation to a statement, can usually be considered as a combination of degrees of truth referred to several partial statements implied in the first statement.

  - For instance, to assess [decide] to what extent I am happy, I may need to assess three partial statements: health, love and money

Nearly decomposable concepts

- The degree of truth of a global concept must be assessed based on the degree of truth of some partial concepts, that interact in a non linear way; i.e.: in a ‘complex’ manner.
- Therefore, we designate them as ‘nearly decomposable concepts’

Nearly decomposable concepts

- There are many nearly decomposable concepts:
- Democracy, Sustainability, depression, happiness, talent, quality, etc..
- For more clarity, we continue with the example of happiness, which admits an easy decomposition as:

Happiness

Health - Love - Money

Part II- Two types of concepts

Logic and Duality

- We can only state that something is true if we can also state that it is false; only what can be false can be true.
- Any quality that we can refer to an object requires the opposite quality to exist.
- Truth opposes Falseness; Low opposes High; Probable opposes Improbable; Happiness opposes Unhappiness,…

Two types of concepts

- When we review each concept with its complementary [opposite] concept, we see a big difference between the two of them:
  - To be Happy we need to have 'health, love and money'.
  - To be Unhappy, it suffices that we lack 'health, love or money'.

Two types of concepts:

- From Set Theory we can model the former statement as:

$$\text{Happiness} = health \cap love \cap money$$

$$\text{Unhappiness} = \neg health \cup \neg love \cup \neg money$$

- The first is an intersection, while the second is a union.

Two types of concepts:

- And in terms of calculation, they differ considerably:

$$Happiness = min\{health, love, money\}$$

$$Unhappiness = max\{\neg health, \neg love, \neg money\}$$

- It implies equating Happiness to the minimum of the three values, and Unhappiness to the maximum of their complements.
- An ASYMMETRY emerges between both concepts

Two types of concepts

- However, the above modelization does not provide a satisfactory result in many situations:
  - For instance, a situation in which Health=0,2; Love=0,8; Money=0,8, is clearly preferred to another in which Health=0,2; Love=0,2 and Money=0,2
  - The minimum value of the three has not modified, but the 'Happiness Degree' surely has reduced from the first situation to the second. Union and intersection operations from Set Theory cannot deal with non-linearity.

Two types of concepts

- Additionally, Set Theory cannot explain why there is such difference between the two type of concepts.

- To understand it and propose adequate aggregation formulas, we need to review it from Communication Theory point of view [Shannon, 1949]

Communication Theory

- Proposes measuring the amount of information conveyed by a message based on the amount of uncertainty that we can reduce by receiving it.
- It relates to the improbability of receiving such message which in turn depends on the context.

Communication Theory

- A highly expected message provides very little information.
- A hardly expected message, provides a lot of information.

Communication Theory

- For instance; a colleague offers us [for a price] telling us which question is going to be asked in an exam.
- ¿Would anyone be willing to pay the same amount of money if there are only two possible questions than if there are 200 possible questions?
- In both situations we will be receiving the same message; the question that is going to be asked in the exam. However, it is likely that in the second case we may be willing to pay more money than in the first one. Why?

Communication Theory

- The first approach to understand it is from the idea of duality; reviewing not only what we know is going to happen, but also what we get to know that is not going to happen. In both cases, we get to know that a question 'X' is going to be asked but ….
  - In the first case, we also get to know that 1 other possible question is not going to be asked.
  - In the second case we also get to know that 199 other possible questions are not going to be asked.
- In the second case, the message received allows us to deny 198 more possible statements; we have obviously received more information.

Communication Theory

- Another way to understand it is from the idea of probability:
  - In the first case, the probability of hitting the subject is 50%.
  - In the second case, the probability of hitting the subject is 0,5%
- It is more unlikely that we hit the subject in the second case if our colleague does not tell us which one it will be. It may be compared to a bet; the lower the chance to win, the higher the prize in case of winning.

Communication Theory

- Based on the above, Communication Theory proposes Entropy [Shannon, 1949] to measure the amount of information provided by a message:

$$H = \sum_{i=1}^{n} p_i * \log_2 p_i$$

Communication Theory

- Additionally, Communication Theory proposes two other interesting formulations:
  - Conditional Entropy

$$H_X[I] = -\sum_{i,j} p(i,j) * log_2\, p_i(j)$$

  - Mutual Information

$$I[I;x] = H[x] - H_x[I]$$

Certainty Degree and Uncertainty Degree

- we propose our formulations building on three ideas:

  - Entropy measures ‘uncertainty’
  - Mutual information allows us to measure the matching degree between two objects.
  - If one of the objects is a concept, mutual information allow us to measure the matching degree of an object and a meaning.

Certainty Degree and Uncertainty Degree

- We can use the formula of Mutual Information to measure the ‘matching degree’ of a global concept and those partial concepts in which we have decomposed it:
- We do it for two ‘special’ concepts:
  - x=‘certainty’
  - Non-x=‘uncertainty’
- We designate the obtained values as: Certainty Degree and Uncertainty Degree

Certainty Degree and Uncertainty Degree

- Certainty Degree

$$I\,[I,x]_{\%} = C_c\,[I]_{\%} = \bar{\mathrm{p}}_{\mathrm{i}} * \sum_{1=1}^{n} \frac{p_i * \log_2 p_i}{\bar{\mathrm{p}}_{\mathrm{i}} * \log_2 \bar{p}_i}$$

- Uncertainty Degree

$$I\,[I,\neg x]_{\%} = C_{\neg c}\,[I]_{\%} = 1 - \bar{\mathrm{p}}_{\mathrm{i}} * \sum_{1=1}^{n} \frac{p_i * \log_2 p_i}{\bar{\mathrm{p}}_{\mathrm{i}} * \log_2 \bar{p}_i}$$

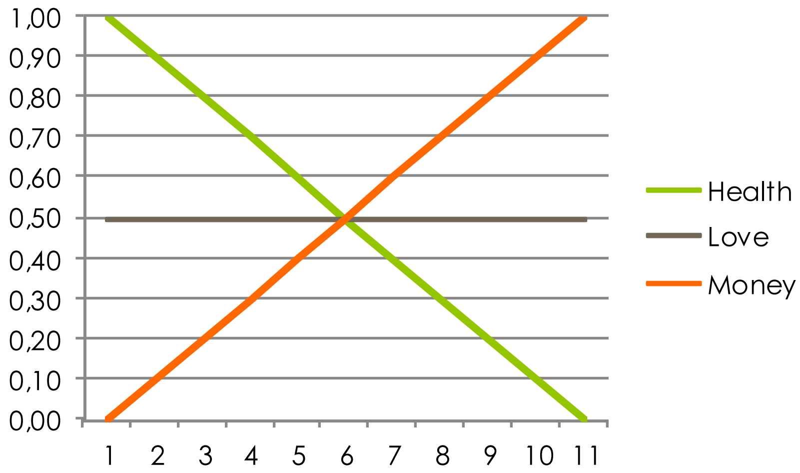
Certainty Degree vs Uncertainty Degree

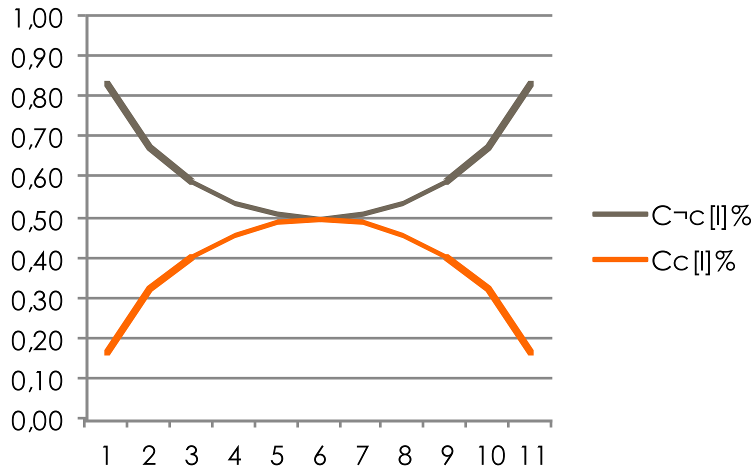
- Certainty Degree is the complementary value of Uncertainty Degree.

- Their graphic representations are also complementary.

Certainty Degree vs Uncertainty Degree

- If we continue with the example of Happiness, we obtain …..





Part III_ Conclusions and applications

Certainty vs Uncertainty

- Conclusions have great importance because ….
  - In reality one and one hardly ever add up to two.
  - We see that one and one can add up to 'more' or to 'less' than two.
  - The result is not random, but follows a rule which relates to the meaning of the measured concepts.

Certainty vs Uncertainty

- When a concept implies certainty, one and one add to less than two.
- When a concept implies uncertainty, one and one add to more than two.

Certainty vs Uncertainty

- The importance of these two meanings becomes huge:
  - Almost every concept shares at least certain 'meaning' with Certainty or with Uncertainty, and consequently the reviewed issues must be applied when measuring them.
  - As formulations have been built on Entropy formula, the conclusions can be interpreted in terms of meaning [subject] but also in physical terms [object]

Concepts that imply certainty

- In terms of meaning, they comprise the following qualities:
  - Control
  - Predictability
  - Knowledge
  - Desirability
- In physical terms [thermodynamic] they imply departure from thermal equilibrium; i.e.: organization.

Concepts that imply uncertainty

- In terms of meaning they comprise the opposed qualities:
  - Absence of control,
  - Unpredictability,
  - Ignorance
  - Undesirability
- In physical terms [thermodynamic]they imply approaching thermal equilibrium; i.e.: disorganization.

Possible applications

- Decision Theory:
  - Decisions are made based on the 'utility' that is obtained from different 'action courses'. Utility is usually measured in a similar way to the logical decomposition revised above.
  - Utility is a concept that implies departure from thermal equilibrium [action] or control; i.e.: certainty.
  - Information aggregation when measuring utility shall be done using the formulas for concepts that imply certainty.

Possible applications

- Systems Theory:
  - There is a large number of phenomena that can be modelled as systems: ecosystems; cities; companies; International Alliances, ….
  - Emergent Properties in systems refer to concepts that imply departure from thermal equilibrium [self organization or dissipative structures] hence certainty.
  - Information aggregation when measuring degree of emergence of different properties in systems shall be done using the formulas for concepts that imply certainty

Other possible applications

- Assessing the Degree of Truth of any diffuse statement relating a 'nearly decomposable concept'.
- It can be a large number of concepts that currently we find difficult to measure:
  - Depression
  - Difficulty of undertaking a task
  - Talent,
  - Extent to what a political system is democratic, …

References

- ALVIRA, RICARDO
  - *A mathematical Theory of Sustainability and Sustainable Development*
  - *A unified Complexity Theory [currently available in Spanish as 'Una Teoria Unificada de la Complejidad']*
- BOOLE, GEORGE [1854] *An Investigation of The Laws of Thought, on which are Founded the Mathematical Theories of Logic and Probabilities*
- GOGUEN, J. A. [1967] "L-Fuzzy Sets". *Journal of Mathematical Analysis and Applications* 18, 145-174
- SHANNON, CLAUDE [1948] *A Mathematical Theory of Communication*
- ZADEH, LOFTI A. [1965] "Fuzzy Sets". *Information and Control*, 8, 338-353 (1965)

That's all. Thanks for your attention !

- Comments are welcome

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- Other documents of the author are available at:

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