Asymmetric Ascoli-type theorems and filter exhaustiveness

A. Boccuto and X. Dimitriou *

Abstract

We prove an Ascoli-type theorem, giving a necessary and sufficient condition for forward compactness of sets of functions, defined and with values in asymmetric metric spaces.

1 Introduction

In the literature there have been many recent extensions of the classical Ascoli theorems. In the metric space context along several directions, for example in metric spaces endowed with asymmetric distance or in the context of ideal/filter exhaustiveness. Asymmetric distance has different applications in several branches of Mathematics, and is connected also with the study of various semicontinuity properties and related topics. Filter convergence and filter exhaustiveness have many developments in the very recent literature, for instance in limit and decomposition theorems for measures. We give an Ascoli-type theorem for the asymmetric case in the metric space setting, extending earlier results, in which equicontinuity is replaced by filter exhaustiveness. This tool allows us to give a necessary and sufficient condition for relative (forward) compactness of suitable function sets.

A filter $\mathcal{F}$ of $\mathbb{N}$ is called a *P-filter* iff for every sequence $(A_n)_n$ in $\mathcal{F}$ there exists another sequence $(B_n)_n$ in $\mathcal{F}$, such that the symmetric difference $A_n \triangle B_n$ is finite for all $n \in \mathbb{N}$ and $\bigcap_{n=1}^{\infty} B_n \in \mathcal{F}$.

*Authors' Address: A. Boccuto: Dipartimento di Matematica e Informatica, via Vanvitelli, 1 I-06123 Perugia, Italy, E-mail: boccuto@yahoo.it, antonio.boccuto@unipg.it

X. Dimitriou: Department of Mathematics, University of Athens, Panepistimiopolis, Athens 15784, Greece, Email: xenofon11@gmail.com, dxenof@windtools.gr, dxenof@math.uoa.gr

2010 A. M. S. Subject Classifications: Primary: 26E50, 28A12, 28A33, 28B10, 28B15, 40A35, 46G10, 54A20, 54A40.

Secondary: 06F15, 06F20, 06F30, 22A10, 28A05, 40G15, 46G12, 54H11, 54H12.

Key words and phrases: asymmetric metric space, (free) filter, filter forward/backward compactness, filter forward/backward convergence, filter forward/backward exhaustiveness, Ascoli theorem.

We now recall the main notions and properties about convergence, closure, compactness and exhaustiveness in the filter setting and in the asymmetric case.

An *asymmetric metric space* $X = (X, d)$ is any nonempty set endowed with an *asymmetric metric* or *asymmetric distance* $d : X \times X \to \mathbb{R}$, satisfying the following properties:

- $d(x, y) \geq 0$ for every x, $y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
- for each x, y, $z \in X$ we get $d(x, z) \leq d(x, y) + d(y, z)$.

Given an element $x_0 \in X$ and a positive real number r, we call *forward open ball* (resp. *backward open ball*) of center x_0 and radius r the set

$$B^+(x_0, r) := \{y \in X : d(x_0, y) < r\} \quad \text{(resp. } B^-(x_0, r) := \{y \in X : d(y, x_0) < r\}).$$

Let $\mathcal{F}$ be a free filter of $\mathbb{N}$ and choose $\overline{x} \in X$. A sequence $(s_n)_n$ in X is said to be $\mathcal{F}$*-forward bounded* (resp. $\mathcal{F}$*-backward bounded*) iff there is $r > 0$ such that $\{n \in \mathbb{N}\colon\, d(\overline{x}, s_n) \leq r\} \in \mathcal{F}$ (resp. $\{n \in \mathbb{N}\colon\, d(s_n, \overline{x}) \leq r\} \in \mathcal{F}$). We say that $(s_n)_n$ is *forward bounded* (resp. *backward bounded*) iff it is $\mathcal{F}_{\text{cofin}}$-forward bounded (resp. $\mathcal{F}_{\text{cofin}}$-backward bounded).

Let $x \in X$. A sequence x_n, $n \in \mathbb{N}$, in X is $\mathcal{F}$*-forward convergent* (resp. $\mathcal{F}$*-backward convergent*) *to* $x \in X$ iff

$$\{n \in \mathbb{N} : d(x, x_n) \leq \varepsilon\} \in \mathcal{F} \quad \text{(resp. } \{n \in \mathbb{N} : d(x_n, x) \leq \varepsilon\} \in \mathcal{F})$$

for every $\varepsilon > 0$. We say that $(x_n)_n$ *forward* (resp. *backward*) *converges to* x iff it $\mathcal{F}_{\text{cofin}}$-forward (resp. $\mathcal{F}_{\text{cofin}}$-backward) converges to x.

Let (X, d_X) and (Y, d_Y) be two asymmetric metric spaces, and choose $\overline{y} \in Y$. We say that a sequence $f_n : X \to Y$, $n \in \mathbb{N}$, is $\mathcal{F}$*-pointwise forward* (resp. *backward*) *bounded*, iff for every $x \in X$ there are a set $F_x \in \mathcal{F}$ and a positive real number k_x with

$$f_n(x) \in B^+(\overline{y}, k_x) \quad \text{(resp. } f_n(x) \in B^-(\overline{y}, k_x)) \text{ for each } n \in F_x. \tag{1}$$

Throughout this paper, the concept of (filter forward/backward) compactness is always intended as sequential compactness.

A subset $S \subset X$ is $\mathcal{F}$*-forward compact* (resp. $\mathcal{F}$*-backward compact*) iff every sequence in S has a subsequence, $\mathcal{F}$-forward (resp. $\mathcal{F}$-backward) convergent to an element of S. We say that $S \subset X$ is *forward compact* (resp. *backward compact*) iff it is $\mathcal{F}_{\text{cofin}}$-forward compact (resp. $\mathcal{F}_{\text{cofin}}$-backward compact).

Given $S \subset X$, the $\mathcal{F}$*-forward* (resp. $\mathcal{F}$*-backward*) *closure of* S in the set of all elements $x \in X$, such that there is a sequence $(s_n)_n$ in S, $\mathcal{F}$-forward (resp. $\mathcal{F}$-backward) convergent to x.

Remark 1.1. Observe that every $\mathcal{F}$-forward (resp. $\mathcal{F}$-backward) convergent sequence in an asymmetric metric space (X, d) has a forward (resp. backward) convergent subsequence. From this it follows that, for every free filter $\mathcal{F}$ of $\mathbb{N}$, $\mathcal{F}$-forward ($\mathcal{F}$-backward) compactness and closure are equivalent to forward (backward) compactness and closure respectively.

Let (X, d_X) and (Y, d_Y) be two asymmetric metric spaces, and $x \in X$. A sequence $f_n : X \to Y$, $n \in \mathbb{N}$, is said to be *$\mathcal{F}$-forward* (resp. *$\mathcal{F}$-backward*) *exhaustive at* x iff for every $\varepsilon > 0$ there exist $\delta > 0$ and a set $F \in \mathcal{F}$ (depending on ε and x) with $d_Y(f_n(x), f_n(y)) < \varepsilon$ (resp. $d_Y(f_n(y), f_n(x)) < \varepsilon$) whenever $n \in F$ and $y \in X$ with $d_X(x, y) < \varepsilon$. We say that $(f_n)_n$ is *$\mathcal{F}$-forward* (resp. *$\mathcal{F}$-backward*) *exhaustive on* X iff it is $\mathcal{F}$-forward (resp. $\mathcal{F}$-backward) exhaustive at every $x \in X$.

We say that a set $\mathcal{C} \subset Y^X$ is *forward* (resp. *backward*) *equicontinuous on* X iff for every $\varepsilon > 0$ and $x \in X$ there is $\delta > 0$ (depending on ε and x) such that for every $f \in \mathcal{C}$ and $y \in X$ with $d_X(x, y)$ we get $d_Y(f(x), f(y)) < \varepsilon$ (resp. $d_Y(f(y), f(x)) < \varepsilon$). Observe that equicontinuity is in general strictly stronger than filter exhaustiveness

Let $C(X, Y)$ denote the sets of all functions $f \in Y^X$ with the property that for every $\varepsilon > 0$ and $x \in X$ there exists $\delta > 0$ such that $f(y) \in B^+(f(x), \varepsilon)$ whenever $y \in B^+(x, \delta)$.

The *uniform metric* $\overline{\rho}$ on Y^X is defined by

$$\overline{\rho}(f, h) = \sup\{\overline{d_Y}(f(x), h(x)) : x \in X\}, \tag{2}$$

where $\overline{d_Y}(a, b) = \min\{d_Y(a, b), 1\}$, $a, b \in Y$. We say that a subset $\mathcal{C} \subset Y^X$ is *forward* (resp. *backward*) *totally bounded* iff, for every $\varepsilon > 0$, $\mathcal{C}$ is contained in the union of a finite number of forward (resp. backward) open balls of radius ε in the metric $\overline{\rho}$.

2 The main results

We begin with recalling the following

Proposition 2.1. *Let (X, d_X) and (Y, d_Y) be two asymmetric metric and forward compact spaces, and let $\mathcal{C} \subset C(X, Y)$ be forward equicontinuous. Then $\mathcal{C}$ is forward totally bounded with respect to $\overline{\rho}$.*

Proposition 2.2. *Let (X, d_X) and (Y, d_Y) be two asymmetric metric spaces such that Y is forward compact and forward convergence implies backward convergence in Y. Then $C(X, Y)$ is complete with respect to $\overline{\rho}$.*

We now state the next result.

Proposition 2.3. *Let (X, d_X) and (Y, d_Y) be asymmetric metric spaces such that X is forward compact, $\mathcal{F}$ be a free filter of $\mathbb{N}$, $\overline{y}$ be as in (1), and assume that forward closed and forward bounded sets in Y are forward compact. Let $f_n : X \to Y$, $n \in \mathbb{N}$, be a function sequence, $\mathcal{F}$-forward exhaustive and $\mathcal{F}$-pointwise forward bounded. Then there exist $F \in \mathcal{F}$ and a forward compact set $Z \subset Y$ with $f_n(X) \subset Z$ for every $n \in F$.*

We now turn to our main theorem.

Theorem 2.4. *Let (X, d_X) and (Y, d_Y) be asymmetric metric spaces, such that X is forward compact, $\overline{y}$ be as in (1), and $\mathcal{F}$ be a P-filter of $\mathbb{N}$. Assume that every forward closed and $\mathcal{F}$-forward bounded subset of Y is $\mathcal{F}$-forward compact, and forward convergence implies backward convergence in Y. Let $\overline{\rho}$ be as in (2), and $\mathcal{C} \subset C(X, Y)$ be such that*

2.4.1) *every sequence $(f_n)_n$ in $\mathcal{C}$ has a subsequence $(f_{n_r})_r$, $\mathcal{F}$-pointwise forward bounded in Y.*

Suppose moreover that

2.4.2) *every sequence in $\mathcal{C}$, pointwise $\mathcal{F}$-forward convergent in Y^X, has a $\mathcal{F}$-forward exhaustive subsequence.*

Then the set $\overline{\mathcal{C}}$, that is the forward closure of $\mathcal{C}$ with respect to $\overline{\rho}$, is forward compact.

Conversely, if $\overline{\mathcal{C}}$ is forward compact with respect to $\overline{\rho}$, then 2.4.1) and 2.4.2) hold.

Remarks 2.5. (a) Note that, in general, condition 2.4.2) is strictly weaker than forward/backward equicontinuity .

(b) Observe that in the asymmetric case, under the hypotheses of Theorem 2.4, in general forward compactness of the set $\overline{\mathcal{C}}$ does not imply forward equicontinuity of $\mathcal{C}$. Moreover, in general the condition that forward convergence implies backward convergence cannot be dropped.