On a Counterexample to Cantor's Diagonal Argument

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Abstract

A counterexample to Cantor's Diagonal Argument is revisited.

1 Introduction

In a recent article published in *ArXiv* E. Coiras [1] provides some counterexamples to Cantor's Diagonal Method (DM). He argues that for the DM to work the list of numbers have to be written down and that this set of numbers represented using positional fractional notation, $\mathbb{W}$, is not isomorphic to the set of the real numbers, $\mathbb{R}$. Coiras concludes that results obtained from the application of the Cantor's Diagonal Method in order to derive properties of $\mathbb{R}$ are not valid.

The section 2 briefly revisits the representation of the real numbers by binary numbers to show that the members of the written list of real numbers in the interval $[0, 1]$ used in the DM correspond one to one to the real numbers of this interval.

The section 3 reintroduces a list used in [2] to prove that Cantor's argument of 1891 fails.

The section 4 modify the list given in the previous section using an idea found in Coiras [1].

2 On the representation of the real numbers by binary numbers

There exist different ways to introduce the real numbers. One of these ways, called nonconstructive, assumes there exists a set $\mathbb{R}$ of undefined objects, called real numbers, which satisfying a number of properties that we use as axioms. All the properties of real numbers can be deduced from the these axioms.

Using an appropriate set of axioms for Euclidean geometry, each real number corresponds one to one to exactly one point on the straight line. Associated to the geometric representation of real numbers by means of points on a straight line there is the representation of real numbers by binary numbers. In this case for any point x on the interval [0,1] of the real line, there exists a real number represented by

$$f_1 \times 2^{-1} + f_2 \times 2^{-2} + f_3 \times 2^{-3} + f_4 \times 2^{-4} + \ldots \tag{1}$$

where

$$f_i \in \{0, 1\} \qquad \text{and} \qquad i \in \mathbb{N} \qquad \text{and} \qquad i \neq 0 \tag{2}$$

The representation (1) can be simplified to

$$.f_1 f_2 f_3 f_4 f_5 \ldots \tag{3}$$

where the first character of the sequence is the point. Each f_i of (3) is substituted by 0 or 1 to represent a given number. The i-th 0 or 1 on the right of "." corresponds to f_i.

3 A list constructed to show the inconsistency of Cantor's argument of 1891

Let L be the list

$$\begin{array}{ccll} n \in \mathbb{N} & \longleftrightarrow & x \in \mathbb{F} & \\ & & & \\ 0 & \longleftrightarrow & 0.000000 & \ldots \\ 1 & \longleftrightarrow & 0.100000 & \ldots \\ 2 & \longleftrightarrow & 0.010000 & \ldots \\ 3 & \longleftrightarrow & 0.110000 & \ldots \\ 4 & \longleftrightarrow & 0.001000 & \ldots \\ 5 & \longleftrightarrow & 0.101000 & \ldots \\ 6 & \longleftrightarrow & 0.011000 & \ldots \\ 7 & \longleftrightarrow & 0.111000 & \ldots \\ 8 & \longleftrightarrow & 0.000100 & \ldots \\ 9 & \longleftrightarrow & 0.100100 & \ldots \\ 10 & \longleftrightarrow & 0.010100 & \ldots \\ 11 & \longleftrightarrow & 0.110100 & \ldots \\ 12 & \longleftrightarrow & 0.001100 & \ldots \\ 13 & \longleftrightarrow & 0.101100 & \ldots \\ 14 & \longleftrightarrow & 0.011100 & \ldots \\ 15 & \longleftrightarrow & 0.111100 & \ldots \\ 16 & \longleftrightarrow & 0.000010 & \ldots \\ \ldots & & \ldots & \end{array} \tag{4}$$

where $\mathbb{F}$ is the set of real numbers in the interval $[0, 1]$. Let us notice that (i) $d_{1,1}$, $d_{3,1}$, $d_{5,1}$, $d_{7,1}$, $d_{9,1}$, $d_{11,1}$, $d_{13,1}$, $d_{15,1}$, $d_{17,1}$ ... are equal to 0 and $d_{2,1}$, $d_{4,1}$, $d_{6,1}$, $d_{8,1}$, $d_{10,1}$, $d_{12,1}$, $d_{14,1}$, $d_{16,1}$, $d_{18,1}$ ... are equal to 1, (ii) $d_{1,2}$, $d_{2,2}$, $d_{5,2}$, $d_{6,2}$, $d_{9,2}$, $d_{10,2}$, $d_{13,2}$, $d_{14,2}$, $d_{17,2}$ ... are equal to 0 and $d_{3,2}$, $d_{4,2}$, $d_{7,2}$, $d_{8,2}$, $d_{11,2}$, $d_{12,2}$, $d_{15,2}$, $d_{16,2}$, $d_{19,2}$ ... are equal to 1, (iii) $d_{1,3}$, $d_{2,3}$, $d_{3,3}$, $d_{4,3}$, $d_{9,3}$, $d_{10,3}$, $d_{11,3}$, $d_{12,3}$, $d_{17,3}$ ... are equal to 0 and $d_{5,3}$, $d_{6,3}$, $d_{7,3}$, $d_{8,3}$, $d_{13,3}$, $d_{14,3}$, $d_{15,3}$, $d_{16,3}$, ... are equal to 1, (iv) ... and so on ...

$$\begin{array}{cccc} d_{1,1} & d_{1,2} & d_{1,3} & \ldots \\ d_{2,1} & d_{2,2} & d_{2,3} & \ldots \\ d_{3,1} & d_{3,2} & d_{3,3} & \ldots \\ \ldots & & & \end{array} \tag{5}$$

More generally, considering the matrix (5) the procedure to obtain (4) is: the first column of the matrix is filled from top to bottom with a succession of the pattern 01. The second column is filled with a succession of the pattern 0011. The n-th column is filled from top to bottom with a succession of the pattern constituted of 2^{n-1} 0's followed by 2^{n-1} 1's.

Considering how the list was built we have that all real numbers of the interval $[0, 1]$ belongs to L. Let us notice the case of the number 1, this number can be represented by $0.\bar{1}$ using bar notation. Since that

$$lim_{n\to\infty}(2^{-1} + 2^{-2} + 2^{-3} + ... + 2^{-n}) = 0.\bar{1} \tag{6}$$

the number 1 belongs to L.

The list L was used in the last version of the article "The cardinality of the set of real numbers" [2] to show the inconsistency of the Cantor's argument of 1891 [3]. It is ease to see that applying the DM to L we obtain the number 1.

4 Using a modification of L

Let us modify the list L using an idea due to Coiras [1]. In the right column of L we change the positions of the first and second rows as shown in the list L' below:

$$
\begin{array}{ccll}
n \in \mathbb{N} & \longleftrightarrow & \quad x \in \mathbb{F} & \\
 & & & \\
0 & \longleftrightarrow & 0.100000 & \ldots \\
1 & \longleftrightarrow & 0.000000 & \ldots \\
2 & \longleftrightarrow & 0.010000 & \ldots \\
3 & \longleftrightarrow & 0.110000 & \ldots \\
4 & \longleftrightarrow & 0.001000 & \ldots \\
5 & \longleftrightarrow & 0.101000 & \ldots \\
6 & \longleftrightarrow & 0.011000 & \ldots \\
7 & \longleftrightarrow & 0.111000 & \ldots \\
8 & \longleftrightarrow & 0.000100 & \ldots \\
\ldots & & \quad \ldots &
\end{array}
\tag{7}
$$

Applying the DM to L' we obtain the number 0.011111... This number can be rewritten as 0.100000... Inspecting the list L' we found this number in the first line. As in previous section we found the number obtained using DM in the list. The Cantor's argument fails.

References

[1] E. Coiras, *Counterexamples to Cantorian Set Theory*, http://arxiv.org/pdf/1404.6447.pdf, April, (2014).

[2] Jailton C. Ferreira, *The cardinality of the set of real numbers*, http://arxiv.org/pdf/math/0108119.pdf , version 7, July, (2013).

[3] G. Cantor, *Über eine elementare Frage der Mannigfaltigkeitslehre*, Jahresbericht der Deutschen Mathematiker-Vereinigung, 1 (1891), 75-78.