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A SIMILARITY TRANSFORMATION FOR THE ROTATION MATRIX

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ABSTRACT. It is shown explicitly a similarity transformation to diagonalize the rotation matrix.

Keywords: Rotation matrix, Similarity transformation, Cayley-Klein parameters.

1. Introduction

The 3- space rotations can be performed with the matrix [1]:

$$\underset{\sim}{R} = \begin{pmatrix} \frac{1}{2}\left(\alpha^2+\alpha^{x^2}-\beta^2-\beta^{x^2}\right) & -\frac{i}{2}\left(\alpha^2-\alpha^{x^2}+\beta^2-\beta^{x^2}\right) & -\left(\alpha\beta+\alpha^x\beta^x\right) \\ \frac{i}{2}\left(\alpha^2-\alpha^{x^2}-\beta^2+\beta^{x^2}\right) & \frac{1}{2}\left(\alpha^2+\alpha^{x^2}+\beta^2+\beta^{x^2}\right) & -i\left(\alpha\beta-\alpha^x\beta^x\right) \\ \alpha\beta^x+\alpha^x\beta & i\left(\alpha^x\beta-\alpha\beta^x\right) & \alpha\alpha^x-\beta\beta^x \end{pmatrix} , \qquad \textbf{(1)}$$

where α and β are the Cayley-Klein parameters under the condition:

$$\alpha\alpha^x+\beta\beta^x=1 , \qquad \textbf{(2)}$$

which implies the orthogonal character of (1):

$$\underset{\sim}{R}\,\underset{\sim}{R}^T=\underset{\sim}{I} \qquad , \qquad \det \underset{\sim}{R}=1. \qquad \textbf{(3)}$$

The Euler-Olinde Rodrigues coefficients [2],[3] $a_1,\ldots,a_4$ defined by:

$$a_1=\frac{i}{2}\left(\beta-\beta^x\right),\quad a_2=-\frac{1}{2}\left(\beta+\beta^x\right),\quad a_3=\frac{i}{2}\left(\alpha-\alpha^x\right),\quad a_4=\frac{1}{2}\left(\alpha+\alpha^x\right), \qquad \textbf{(4)}$$

verify:

$$a_1^2 + a_2^2 + a_3^2 + a_4^2 = 1 \quad , \tag{5}$$

by virtue of (2), so that (1) adopts the form:

$$\underset{\sim}{R} = \begin{pmatrix} 1-2\left(a_2^2+a_3^2\right) & 2\left(a_1a_2-a_3a_4\right) & 2\left(a_1a_3+a_2a_4\right) \\ 2\left(a_1a_2+a_3a_4\right) & 1-2\left(a_1^2+a_3^2\right) & 2\left(a_2a_3-a_1a_4\right) \\ 2\left(a_1a_3-a_2a_4\right) & 2\left(a_1a_4+a_2a_3\right) & 1-2\left(a_1^2+a_2^2\right) \end{pmatrix} \quad , \tag{6}$$

The rotation matrix represents [4] a rotation of the Cartesian axes through a Φ angle around the axis defined by the unitary vector $\left(l_1, l_2, l_3\right)$, such that [3]:

$$a_j = l_j Sin\left(\frac{\Phi}{2}\right), \; j = 1,2,3 \quad , \quad a_4 = Cos\left(\frac{\Phi}{2}\right), \tag{7}$$

consistent with (5), since:

$$l_1^2 + l_2^2 + l_3^2 = 1 \; . \tag{8}$$

Then (6) simplifies to:

$$\underset{\sim}{R} = \begin{pmatrix} l_1^2\gamma + Cos\Phi & l_1l_2\gamma - l_3Sin\Phi & l_1l_3\gamma + l_2Sin\Phi \\ l_1l_2\gamma + l_3Sin\Phi & l_2^2\gamma + Cos\Phi & l_2l_3\gamma - l_1Sin\Phi \\ l_1l_3\gamma - l_2Sin\Phi & l_2l_3\gamma + l_1Sin\Phi & l_3^2\gamma + Cos\Phi \end{pmatrix}, \gamma = 1 - Cos\Phi \, , \tag{9}$$

which is be studied in the present work.

In next Section, the eigenvalue problem of (9) is used to write an $\underset{\sim}{R}$ as a product of three matrices, which in turn explicitly shows the existence of a similarity transformation to diagonalize any rotation matrices.

2. Eigenvalues and eigenvectors of $\underset{\sim}{R}$.

The relation:

$$\underset{\sim}{R}\,\vec{u} = \lambda\,\vec{u} \; , \tag{10}$$

expresses the eigenvalue problem for the rotation matrix, such that is quite known [5] that:

$$\lambda_1 = 1 \quad , \quad \vec{u}_1 = \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix} \quad ,$$

$$\lambda_2 = e^{-i\Phi} \quad , \quad \lambda_3 = \lambda_2^x = e^{i\Phi} \quad , \tag{11}$$

however, it is very difficult to find explicitly $\vec{u}_2$. In fact, the eigenvector $\vec{u}_2$ can be any of the following three vectors, proportional among them:

$$\begin{pmatrix} l_2^2 + l_3^2 \\ -l_1 l_2 + i l_3 \\ -l_1 l_3 - i l_2 \end{pmatrix} \quad , \quad \begin{pmatrix} -l_1 l_2 - i l_3 \\ l_1^2 + l_3^2 \\ -l_2 l_3 + i l_1 \end{pmatrix} \quad , \quad \begin{pmatrix} -l_1 l_3 + i l_2 \\ -l_2 l_3 - i l_1 \\ l_1^2 + l_2^2 \end{pmatrix} \quad , \tag{12}$$

and it is chosen as the most convenient according to the values of $l_1, l_2 \; and \; l_3$, for instance, if $l_1 = l_3 = 0$, $l_2 = 1$, the second vector is the trivial solution $\vec{u}_2 = \upsilon$, and in that case it can be used any other vector of (12). Without losing generality, here it will be used:

$$\vec{u}_2 = \begin{pmatrix} l_2^2 + l_3^2 \\ -l_1 l_2 + i l_3 \\ -l_1 l_3 - i l_2 \end{pmatrix} \quad , \quad \vec{u}_3 = \bar{\vec{u}}_2 = \begin{pmatrix} l_2^2 + l_3^2 \\ -l_1 l_2 - i l_3 \\ -l_1 l_3 + i l_2 \end{pmatrix} \quad . \tag{13}$$

The problem of eigenvalues of a matrix gives algebraic information if it is also solved for the corresponding transpose matrix of (9) [6]:

$$\underset{\sim}{R}^T \vec{v} = \lambda \vec{v} \; . \tag{14}$$

then getting the same eigenvalues (11), with:

$$\vec{v}_1 = \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} \frac{1}{2} \\ -\frac{}{2\left(l_2^2+l_3^2\right)} \\ \frac{-l_1 l_3 + i l_2}{2\left(l_2^2+l_3^2\right)} \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} \frac{1}{2} \\ \frac{}{2\left(l_2^2+l_3^2\right)} \\ -\frac{l_1 l_3 + i l_2}{2\left(l_2^2+l_3^2\right)} \end{pmatrix}, \qquad \textbf{(15)}$$

which together with (8) and (13) imply:

$$\vec{v}_j \ \vec{u}_k = \delta_{jk} \ . \qquad \textbf{(16)}$$

Then it is natural to construct the matrices:

$$\underset{\sim}{U} = \left(\vec{u}_1\ \vec{u}_2\ \vec{u}_3\right) = \begin{pmatrix} l_1 & l_2^2+l_3^2 & l_2^2+l_3^2 \\ l_2 & -l_1 l_2 + i l_3 & -l_1 l_2 - i l_3 \\ l_3 & -l_1 l_3 - i l_2 & -l_1 l_3 + i l_2 \end{pmatrix},$$

$$\underset{\sim}{\Lambda} = Diag\left(\lambda_1, \lambda_2, \lambda_3\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-i\Phi} & 0 \\ 0 & 0 & e^{i\Phi} \end{pmatrix}, \qquad (17)$$

$$\underset{\sim}{V} = \left(\vec{v}_1, \vec{v}_2, \vec{v}_3\right) = \begin{pmatrix} l_1 & \frac{1}{2} & \frac{1}{2} \\ l_2 & -\frac{l_1 l_2 + i l_3}{2\left(l_2^2+l_3^2\right)} & \frac{-l_1 l_2 + i l_3}{2\left(l_2^2+l_3^2\right)} \\ l_3 & \frac{-l_1 l_3 + i l_2}{2\left(l_2^2+l_3^2\right)} & -\frac{l_1 l_3 + i l_2}{2\left(l_2^2+l_3^2\right)} \end{pmatrix},$$

and because the property (16) it results:

$$\underset{\sim}{V}^T = \underset{\sim}{U}^{-1} \ . \qquad \textbf{(18)}$$

Notice the existence of $\underset{\sim}{U}^{-1}$ because when choosing $\vec{u}_2$ given by (13) it has been supposed that $l_2\ and\ l_3$ do not simultaneously cancel, so that $\det \underset{\sim}{U} = -2i\left(l_2^2 + l_3^2\right) \neq 0$.

General theorems of Linear Algebra [6] lead to the following factorization of the rotation matrix:

$$\underset{\sim}{R} = \underset{\sim}{U}\,\Lambda\,\underset{\sim}{V}^T \overset{(18)}{=} \underset{\sim}{U}\,\Lambda\,\underset{\sim}{U}^{-1} \quad . \qquad \textbf{(19)}$$

which is easily seen through (8), (9) and (17). From (19) it is immediate that:

$$\underset{\sim}{U}^{-1}\, R\, \underset{\sim}{U} = \Lambda = Diag\left(1, e^{-i\Phi}, e^{i\Phi}\right) \quad , \qquad \textbf{(20)}$$

that is, the $\underset{\sim}{U}$ matrix given explicitly by (17) generates a similarity transformation which diagonalizes $\underset{\sim}{R}$, q.e.d.

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