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CHARACTERIZATION OF LATTICE SIGMA ALGEBRAS ON PRODUCT LATTICES

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ABSTRACT:
This manuscript describes that the class of super lattice measurable sets is closed under finite unions, countable unions, and countable intersections. It has been established that the product two lattice σ- algebras defined on a product lattice is lattice measurable and the elementary integration of these lattice measurable sets are equal. Further some characteristics of lattice σ- finite measures were identified.

Key words: Lattice σ- algebra, measure, lattice measure, σ-finite measure

ASM Classification numbers: 03G10, 28A05, 28A12

§1. INTRODUCTION:

In section 2, by Tanaka[9] we define the definition of lattice sigma algebra, lattice measure on a lattice sigma algebra by Anil kumar etrl[1,2] the definition of lattice measurable of the space, lattice measurable set, lattice measure space, lattice σ – finite measure are defined. Here we prove some elementary properties of lattice measurable sets.

Section 2 is devoted to the basic concepts which were making use of in the later text. The rationalization of lattice σ- algebra and lattice measure on lattice σ- algebra were organized. Further a classification of lattice measure space, lattice measurable set, lattice σ – finite measure space, lattice σ- finite measure were prearranged.

Section 3 establishes the results that the class of super lattice measurable sets is closed under finite unions, countable union, countable intersections. Further instituted a theorem that the product two lattice σ- algebras defined on a product lattice is lattice measurable. It has been obtained that the elementary integration of these lattice measurable sets are equal. Finally some characteristics of lattice σ- finite measures were observed.

§2. PRELIMINARIES

This section briefly reviews the well-known facts of Birkhoff's [3] lattice theory. The system $(L, \wedge, \vee)$, where L is a non empty set, $\wedge$ and $\vee$ are two binary operations on L, is called a lattice if $\wedge$ and $\vee$ satisfies, for any elements x, y, z, in L:(L1) commutative law: $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$. (L2) associative law: $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ and $x \vee (y \vee z) = (x \vee y) \vee z$. (L3) absorption law: $x \vee (y \wedge x) = x$ and $x \wedge (y \vee x) = x$. Hereafter, the lattice $(L, \wedge, \vee)$ will often be written as L for simplicity. A lattice $(L, \wedge, \vee)$ is called distributive if, for any x, y, z, in L. (L4) distributive law holds: $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ and $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$. A lattice L is called complete if, for any subset A of L, L contains the supremum $\vee A$ and the infimum $\wedge A$. If L is complete, then L itself includes the maximum and minimum elements which are often denoted by 1 and 0 or I and O respectively. A distributive lattice is called a Boolean lattice if for any element x in L, there exists a unique complement x^c such that $x \vee x^c = 1$ (L5) the law of excluded middle $x \wedge x^c = 0$ (L6) the law of non-contradiction.

Let L be a lattice and c: L → L be an operator. Then c is called a lattice complement in L if the following conditions are satisfied. (L5) and (L6); $\forall$ x ∈ L, $x \vee x^c = 1$ and $x \wedge x^c = 0$,(L7) the law of contrapositive; $\forall$ x, y ∈ L, $x \leq y$ implies $x^c \geq y^c$,(L8) the law of double negation;$\forall$ x ∈ L, $(x^c)^c = x$.Throughout this paper, we consider lattices as complete lattices which obey (L1) - (L8) except for (L6) the law of non-contradiction. Unless otherwise stated, X is the entire set and L is a lattice of any subsets of X.

Definition2.1: If a lattice L satisfies the following conditions, then it is called a lattice σ-Algebra;

(1) $\forall h \in L, h^c \in L$

(2) if $h_n \in L$ for n = 1, 2, 3 ....., then $\bigvee_{n=1}^{\infty} h_n \in L$.

We denote σ (L) = ß, as the lattice σ-Algebra generated by L.

Example 2.1: [[4] Halmos (1974)]. 1. {ϕ, X} is a lattice σ-Algebra.
2. P(X) power set of X is a lattice σ-Algebra.

Example2.2: Let X = $\Re$ and L = {measurable subsets of $\Re$ } with usual ordering (≤). Here L is a lattice and σ (L) = ß is a lattice σ - algebra generated by L.

Example2.3: Let X be any non-empty set, L = {All topologies on X}. Here L is a complete lattice but not σ - algebra.

Example2.4: [[4] Halmos (1974)]. Let X = $\Re$ and L = {E < $\Re$ / E is finite or E^c is finite}. Here L is lattice algebra but not lattice σ - algebra.

Definitition2.2: The ordered pair (X, ß) is said to be lattice measurable space.

Example2.5: Let X = $\Re$ and L = {All Lebesgue measurable sub sets of $\Re$ }. Then it can be verified that ($\Re$, ß) is a lattice measurable space.

Definitition2.3: If the mapping μ: ß → R ∪ {∞} satisfies the following properties, then μ is called a lattice measure on the lattice σ-Algebra σ (L).

(1) μ (ϕ) = μ (0) = 0.
(2) For all h, g ∈ ß, such that μ(h), μ(g) ≥ 0 and h ≤ g ⇒ μ(h) ≤ μ(g).
(3) For all h, g ∈ ß, μ (h ∨ g) + μ (h ∧ g) = μ (h) + μ (g).
(4) If $h_n \subset$ ß, n ∈ N such that $h_1 \leq h_2 \leq ... \leq h_n \leq ....$, then $\mu (\bigvee_{n=1}^{\infty} h_n) = \lim \mu (h_n)$.

Note2.1: Let μ_1 and μ_2 be lattice measures defined on the same lattice σ-Algebra ß. If one of them is finite, then the set function μ (E) = μ_1 (E) - μ_2 (E), E ∈ ß is well defined and is countably additive on ß.

Example2.6: [[6]Royden (1981)]: Let X be any set and ß = P(X) be the class of all sub sets of X. Define for any A ∈ ß, μ(A) = +∞ if A is infinite = |A| if A is finite, where |A| is the number of elements in A. Then μ is a countable additive set function defined on ß and hence μ is a lattice measure on ß.

Definition2.4: A set A is said to be lattice measurable set or lattice measurable if A belongs to ß.

Example2.7: [Anilkumar etrl[1,2] 2011] The interval (a, ∞) is a lattice measurable under usual ordering.

Example2.8: [Anilkumar etrl[1,2] 2011] [0, 1] < $\Re$ is lattice measurable under usual ordering. Let X= $\Re$, L= {lebesgue measurable subsets of $\Re$ }with usual ordering (≤) clearly σ (L) is a lattice σ-algebra generated by L. Here [0,1] is a member of σ(L). Hence it is a Lattice measurable set.

Example2.9: [Anilkumar etrl**[1,2]** 2011] Every Borel lattice is a lattice measurable.

Definition2.5: The lattice measurable space (X, ß) together with a lattice measure μ is called a lattice measure space and it is denoted by (X, ß , μ).

Example2.10: $\Re$ is a set of real numbers μ is the lattice Lebesgue measure on $\Re$ and ß is the family of all Lebesgue measurable subsets of real numbers. Then ($\Re$, ß, μ) is a lattice measure space.

Example2.11: $\Re$ be the set of real numbers and ß is the class of all Borel lattices, μ be a lattice Lebesgue measure on $\Re$ then ($\Re$, ß, μ) is a lattice measure space.

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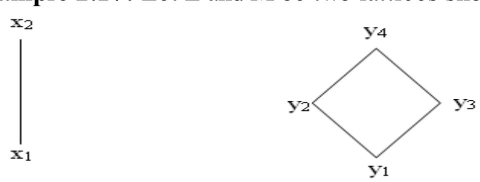
Definition2.6: Let (X, ß, μ) be a lattice measure space. If μ(X) is finite then μ is called lattice finite measure.
Example2.12: The lattice Lebesgue measure on the closed interval [0, 1] is a lattice finite measure.
Example2.13: When a coin is tossed, either head or tail comes when the coin falls. Let us assume that these are the only possibilities. Let X = {H, T}, H for head and T for tail. Let ß = {ϕ, {H}, {T}, X}. Define the mapping P: ß $\rightarrow$[0, 1] by P (ϕ) = 0, P ({H}) = P ({T}) = ½, P (X) = 1. Then P is a lattice finite measure on the lattice measurable space (X, ß).
Definition2.7: If μ is a lattice finite measure, then (X, ß, μ) is called a lattice finite measure space.
Example2.14: Let ß be the class of all Lebesgue measurable sets of [0, 1] and μ be a lattice Lebesgue measure on [0, 1]. Then ([0, 1], ß, μ) is a lattice finite measure space.
Definition2.8: Let (X, ß, μ) be a lattice measure space. If there exists a sequence of lattices measurable sets { x_n } such that

(i) X = $\bigvee_{n=1}^{\infty} x_n$ and (ii) $\mu(x_n)$ is finite then μ is called a lattice σ – finite measure.

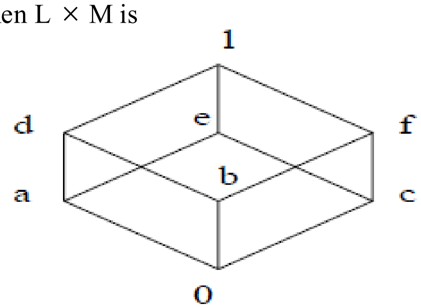
Example2.15: The lattice Lebesgue measure on ($\Re$, μ) is a lattice σ – finite measure since $\Re = \bigvee_{n=1}^{\infty}$ (-n, n) and μ ((-n,n)) = 2n is finite for every n.
Definition2.9: If μ be a lattice σ – finite measure, then (X, ß, μ) is called lattice σ – finite measure space.

Example2.16: Let ß be the class of all Lebesgue measurable sets on $\Re = \bigvee_{n=1}^{\infty}$ (-n, n) and μ be a lattice Lebesgue measure on $\Re$, then ($\Re$, ß, μ) is a lattice σ – finite measure space.
Definition 2.10: The lattice measure m defined on S $\times$ T above is called the product of the lattice measures μ and λ and is denoted by $\mu \times \lambda$.
Definition 2.11: Let X and Y be two lattices. Then their Cartesian product denoted by X $\times$ Y is defined as X $\times$ Y = {(x, y)/x $\in$ X, y $\in$ Y}. It is called product lattice.
Example 2.17: Let L and M be two lattices shown in the figures below



(L) (M)

Then L $\times$ M is



Where l = (x_2, y_4), d = (x_2, y_2), e = (x_1, y_4), f = (x_2, y_3), a = (x_1, y_2), b = (x_2, y_1), c = (x_1, y_3) and O = (x_1, y_1).
Definition 2.12: If A < X, B < Y then A $\times$ B < X $\times$ Y. Any lattice of the form A $\times$ B is called super lattice in X $\times$ Y.
Example 2.18: If A $\subset$ B and C $\subset$ D then (A $\times$C) $\subset$ (B $\times$ D)
Let(x, y) be any element of A $\times$C. Then by definition of product lattice we have
x $\in$ A, y $\in$ C.
But it is given that A $\subset$ B and C $\subset$ D.
Therefore x $\in$ B and y $\in$ D.
That is (x, y) is an element of B $\times$ D. Hence (A $\times$C) $\subset$ (B $\times$ D).

Remark 2.1: Let (X, S), (Y, T) be lattice measurable spaces.
Then S is a lattice σ - algebra in X and T is a lattice σ - algebra in Y.
Definition 2.13: If A $\in$ S and B $\in$ T, then the lattice of the form A $\times$ B is called super lattice measurable set.
Example 2.19: Every member of S $\times$ T is a super lattice measurable set.
Definition 2.14: If $Q = R_1 \vee R_2 \vee \ldots\ldots \vee R_n$ where each R_i is a super lattice measurable set and $R_i \wedge R_j = \phi$ for $i \neq j$, then Q is called elementary lattice. The class of all elementary lattices is denoted by L_E.
Remark 2.2: S $\times$ T is defined to be smallest lattice σ - algebra in X $\times$ Y which contains every super lattice measurable set.

Definition 2.15: If $A_i, B_i \in \sigma(L)$ such that $A_i < A_{i+1}, B_i > B_{i+1}$ for i = 1, 2, 3, …. and $A = \bigvee_{i=1}^{\infty} A_i$, $B = \bigwedge_{i=1}^{\infty} B_i$, then $A \in \sigma(L)$ and $B \in \sigma(L)$. This lattice σ - algebra $\sigma(L)$ is a monotone class.
Example 2.20: X $\times$ Y is a monotone class.
Definition 2.16: Let $E < X \times Y$ where $x \in X$, $y \in Y$. We define x – section lattice of E by $E_x = \{y/ (x, y) \in E\}$ and y – section lattice of $E_y = \{x/(x, y) \in E\}$.
Note 2.2: $E_x < Y$ and $E_y < X$.
Definition 2.17: [5] Let σ(L) be a lattice σ-algebra of sub sets of a set X. A function $\mu: \sigma(L) \to [0, \infty]$ is called a positive lattice measure defined on σ(L) if
(1) $\mu(\phi) = 0$
(2) $\mu(\bigvee_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$ where $\{A_n\}$ is a disjoint countable collection of members of σ(L) and $\mu(A) < \infty$ for at least one $A \in \sigma(L)$.
Example 2.21: (i) Counting measure: Let X be a non – empty set. Let σ(L) = P(X). Define $\mu: \sigma(L) \to [0, \infty]$ by |E| = number of lattice measurable sets in E, if E is finite, ∞ if E is infinite. Then μ is a positive lattice measure on P(X) called the positive lattic e counting measure on X.
(ii) Unit mass at x_0: Let X be a non – empty set. Let σ(L) = P(X). Fix $x_0 \in X$.
Define $\mu: \sigma(L) \to [0, \infty]$ by $\mu(E) = 1$ if $x_0 \in E = 0$ if $x_0 \notin E$
then μ is a positive lattice measure on P(X) is called unit measure concentrated at x_0.
Theorem 2.1: [5] If E $\in$ S $\times$ T, then $E_x \in T$ and $E_y \in S$ for every $x \in X$ and $y \in Y$.
Theorem 2.2: [5] S $\times$ T is the smallest monotone class which contains all elementary lattices.
Theorem 2.3: [7] Suppose $\{f_n\}$ is a sequence of complex lattice measurable functions on X such that f(x) = lim $f_n(x)$ exists for every $x \in X$. If there is a function $g \in L^1$ such that $|f_n(x)| \leq g(x)$ where n = 1, 2, 3, ……. $x \in X$,
then (1) $f \in L^1$ (2) $\lim \int_X |f_n - f| d\mu = 0$. (3) $\lim \int_X f_n \, d\mu = \int_X f \, d\mu$.
Theorem 2.4: [7] Let $\{f_n\}$ be a sequence of lattice measurable functions on X such that $0 \leq f_1(x) \leq f_2(x) \ldots\ldots \leq \infty$ for every $x \in X$ and $f_n(x) \to f(x)$ as $n \to \infty$ for every $x \in X$. Then f is lattice measurable and $\int_X f_n \, d\mu \to \int_X f \, d\mu$ as $n \to \infty$.

Result 2.1: [1] First Valuation Theorem: Suppose that $\{E_k\}$ is monotonic increasing sequence of lattice measurable sets and $E = \bigvee_{k=1}^{\infty} E_k$ then $m(E) = \operatorname{Lt}_{n\to\infty} m(E_n)$.
Result 2.2: [1] Second Valuation Theorem: Suppose that $\{E_k\}$ is a monotonic decreasing sequence of lattice measurable sets and $E = \bigwedge_{k=1}^{\infty} E_k$, then $m(E) = \operatorname{Lt}_{n\to\infty} m(E_n)$.
Theorem 2.5: [5] Let μ be a positive lattice measure defined on a lattice σ-algebra σ(L). Then μ satisfies first valuation theorem (Result 2.1) and second valuation theorem(Result 2.2) that is
(1) Let $A = \bigvee_{n=1}^{\infty} A_n$, $A_n \in \sigma(L)$. Let $A_1 < A_2$ …….. Then $\mu(A_n) \to \mu(A)$ as $n \to \infty$.

(2) If $A = \bigwedge_{n=1}^{\infty} A_n$, $A_n \in \sigma(L)$ and $A_1 > A_2 \ldots$.. with $\mu(A_1)$ finite. Then $\mu(A_n) \rightarrow \mu(A)$ as $n \rightarrow \infty$.

§3. CHARACTERIZATION OF LATTICE SIGMA ALGEBRAS ON PRODUCT LATTICES

Definition 3.1: Let f: $X \times Y \rightarrow Z$ is topological space. For each $x \in X$, define $f_x: Y \rightarrow Z$ by $f_x(y) = f(x, y)$. Then f_x is called Y-lattice measurable function. For each $y \in Y$, define $f_y : X \rightarrow Z$ by $f_y(x) = f(x, y)$. Then f_y is called X – lattice measurable function.

Theorem 3.1: Let f be an $(S \times T)$ lattice measurable function on $X \times Y$, Then

1) For each $x \in X$, f_x is a T – lattice measurable function
2) For each $y \in Y$, f_y is a S – lattice measurable function.

Proof. Let V be an open set in Z. Let $Q = \{(x, y) \in X \times Y : f(x, y) \in V\}$
Since f is $S \times T$ lattice measurable, $Q \in S \times T$. $Q_x = \{y: (x, y) \in Q\} = \{y: f(x,y) \in V\} = \{y: f_x (y) \in V \}$ By theorem 2.1 $Q_x \in T$. Therefore f_x is a T – lattice measurable function. A similar argument shows that f_y is an S –lattice measurable function.

Result 3.1: If $\Phi(x) = \lambda(Q_x)$, $\psi(y) = \mu(Q_y)$ where $Q_x \in S$ and $Q_y \in T$ for all $x \in X$, $y \in Y$ and $K = \{Q \in S \times T$: Φ is S -lattice measurable, ψ is T –lattice measurable and $\int_X f\, d\mu = \int_Y \psi\, d\lambda$ } ------(1). Then every super lattice measurable set belong to K where K is the class of super lattice measureable set satisfying (1).

Proof. Let $Q = A \times B$, $A \in S$, $B \in T$.
Then $Q \in S \times T$. Also, $Q_x = B$ if $x \in A$
$= \phi$ if $x \notin A$
Therefore $\lambda(Q_x) = \lambda(B)\, \chi_A(x)$.
In a similar way, $Q_y = A$ if $y \in B$
$= \phi$ if $y \notin B$
$\mu(Q_y) = \mu(A)\, \chi_B(y)$
Therefore $\Phi(x) = \lambda(B)\, \chi_A(x)$, $\psi(y) = \mu(A)\, \chi_B(y)$. Since $A \in S$, Φ is S –lattice measurable and since $B \in T$, ψ is T - lattice measurable.

Also $\int_X \Phi\, d\mu = \int_X \lambda(B)\chi_A(x)\, d\mu = \lambda(B)\, \mu(A)$

$\int_Y \psi\, d\lambda = \int_Y \mu(A)\chi_B(y) d\lambda = \mu(A)\, \lambda(B)$

Therefore $\int_X \Phi\, d\mu = \int_Y \psi\, d\lambda$.

Thus, every super lattice measurable set belongs to K.

Result 3.2: If $Q_1 < Q_2 < \ldots\ldots\ldots Q_n \in K$ and if $Q = \bigvee_{i=1}^{n} Q_i$ then $Q \in K$ (or) finite union of members of K is again a member of K.

Proof: Since $Q_i \in S \times T$, and since $S \times T$ is a lattice σ - algebra, we get $Q \in S \times T$.
Let $\Phi_i(x) = \lambda(Q_{ix})$, $\psi_i(y) = \mu(Q_{iy})$, then as $Q_i \in K$, we get Φ_i is S – lattice measurable, ψ_i is T –lattice measurable for every i and $\int_X \Phi_i\, d\mu = \int_Y \psi_i\, d\lambda$. Since μ and λ are positive lattice measures, $\lambda(Q_{ix}) \rightarrow \lambda(\bigvee_i Q_{ix})$ and $\mu(Q_{iy}) = \mu(\bigvee_i Q_{iy})$ as $i \rightarrow \infty$.(by theorem 2.5(1)) Since $Q_x = \vee Q_{ix}$, $Q_y = \vee Q_{iy}$, we get $\lambda(Q_{ix}) \rightarrow \lambda(Q_x)$ and $\mu(Q_{iy}) \rightarrow \mu(Q_y)$, that is $\Phi_i \rightarrow \Phi$ and $\psi_i \rightarrow \psi$ as $i \rightarrow \infty$.

Since $\{\Phi_i\}$ are S -lattice measurable, $\{\psi_i\}$ are T –lattice measurable (by theorem 2.4), we get that Φ is S –lattice measurable, ψ is

T –lattice measurable and $\int_X \Phi_i\, d\mu \to \int_X \Phi\, d\mu$, $\int_Y \psi_i\, d\lambda \to \int_Y \psi\, d\lambda$. Since $\int_X \Phi_i\, d\mu = \int_Y \psi_i\, d\lambda$, for every i, we get that $\int_X \Phi\, d\mu = \int_Y \psi\, d\lambda$. Therefore Q $\in$ K.

Result 3.3: If $\{Q_i\}$ is a disjoint countable collection of members of K and if $Q = \vee_i Q_i$ then $Q \in K$. (or) countable union of member of K is again a member of K.

Proof: Let $Q_1, Q_2, \ldots\ldots Q_n$ be n disjoint numbers of K. Let $Q = Q_1 \vee Q_2 \vee \ldots\ldots Q_n$. As $Q_i \in S \times T$ is a lattice σ - algebra, we get $Q \in S \times T$. Let $\Phi_i(x) = \lambda(Q_{ix})$, $\psi_i(y) = \mu(Q_{iy})$. Then Φ_i's are S –lattice measurable and ψ_i's are T –lattice measurable for all i, $1 \le i \le n$ and $\int_X \Phi_i\, d\mu = \int_Y \psi_i\, d\lambda$. $Q_x = \vee_{i=1}^{n} Q_{ix}$, $Q_y = \vee_{i=1}^{n} Q_{iy}$. Let $\Phi(x) = \lambda(Q_x)$, $\psi(y) = \mu(Q_y)$. Then $\Phi(x) = \lambda(\vee_{i=1}^{n} Q_{ix}) = \sum_{i=1}^{n} \lambda(Q_{ix})$ (Therefore Q_{ix}'s are disjoint) $\psi(y) = \mu(\vee_{i=1}^{n} Q_{iy}) = \sum_{i=1}^{n} \mu(Q_{iy})$ (Therefore Q_{iy}'s are disjoint) That is $\Phi(x) = \sum_{i=1}^{n} \Phi_i(x)$, $\psi(y) = \sum_{i=1}^{n} \psi_i(y)$. Therefore $\Phi(x)$ is S –lattice measurable and $\psi(y)$ is T –lattice measurable. Now $\chi_Q = \sum_{i=1}^{n} \chi_{Q_i}$ (Therefore Q is the disjoint union of Q_i's). Now $\lambda(Q_x) = \int_Y \chi_Q(x, y)\, d\lambda(y)$

$$= \int_Y \left(\sum_{i=1}^{n} \chi_{Q_i}(x, y) \right) d\lambda(y)$$

Therefore $\int_X \Phi\, d\mu = \int_X d\mu\ (x) \int_Y \chi_Q(x, y)\, d\lambda\ (y)$

$$= \int_X d\mu\ (x) \int_Y \sum_{i=1}^{n} \chi_{Q_i}(x, y)\, d\lambda\ (y) = \int_X d\mu\ (x) \left(\sum_{i=1}^{n} \int_Y \chi_{Q_i}(x, y)\, d\lambda\ (y) \right) = \sum_{i=1}^{n} \int_X d\mu\,(x) \int_Y \chi_{Q_i}(x, y)\, d\lambda\,(y) = \sum_{i=1}^{n} \int_X \Phi_i\, d\mu =$$

$$\sum_{i=1}^{n} \int_Y \psi_i\, d\lambda \ (\text{Since } Q_i \in K) = \sum_{i=1}^{n} \int_Y \psi_i\, d\lambda\,(y) \int_X \chi_{Q_i}(x, y)\, d\mu\,(x) = \int_Y d\lambda\ (y) \int_X \sum_{i=1}^{n} \chi_{Q_i}(x, y)\, d\mu\ (x) = \int_Y d\lambda\ (y) \int_X \chi_Q(x, y)\, d\mu\,(x)$$

$$= \int_Y \psi\, d\lambda$$

Therefore $Q \in K$. Let $Q = \vee_{i=1}^{\infty} Q_i$, $Q_i \in K$, Q_i is disjoint. Then $Q_1 < Q_1 \vee Q_2 < \ldots\ldots < Q_i \vee \ldots\ldots\ldots Q_n < Q_n \ldots\ldots\ldots$ Let $Q_1 = w_1$, $Q_1 \vee Q_2 = w_2$, ……. $Q_1 \vee \ldots\ldots\ldots Q_n = w_n$ etc. Then $w_1, w_2, \ldots\ldots w_n \ldots.$ are in K. (Since they are finite union of disjoint members of K). Also $w_1 < w_2 < \ldots\ldots < w_n < \ldots.$ and $Q = \vee_{i=1}^{\infty} w_i$. Hence by result 2, $Q \in K$.

Result 3.4: If $\mu(A) < \infty$ and $\lambda(B) < \infty$, and if $A \times B > Q_1 > Q_2 > Q_3 \ldots\ldots\ldots$ $Q = \wedge_{i=1}^{\infty} Q_i$, $Q_i \in K$ for every i, then $Q \in K$ (or) countable intersection of members of K is again a member of K.

Proof: Since $Q_i \in K$, and since $S \times T$ is a lattice σ - algebra, we get $Q = \wedge_{i=1}^{\infty} Q_i \in S \times T$. Let $\Phi_i(x) = \lambda(Q_{ix})$, $\psi_i(y) = \mu(Q_{iy})$. Then as $Q_i \in K$, we get, Φ_i is a S –lattice measurable and ψ_i is a T –lattice measurable for every i and $\int_X \Phi_i\, d\mu = \int_Y \psi_i\, d\lambda$. Now μ and λ are positive lattice measures. Also $A \times B > Q_1 > Q_2 > \ldots\ldots\ldots$

$\lambda(Q_{1x}) \le \lambda((A \times B)_x) = \lambda(B)\ \chi_A(x) \le \lambda(B) < \infty$. $\mu(Q_{1y}) \le \mu((A \times B)_y) = \mu(A)\ \chi_B(y) \le \mu(A) < \infty$. Therefore by the

theorem, (by theorem 2.5(2)). We get,
$\lambda(Q_{1x}) \to \lambda(Q_x)$, $\mu(Q_{1y}) \to \mu(Q_y)$ that is $\Phi_i \to \Phi$, $\psi_i \to \psi$, as $i \to \infty$ where $\Phi(x) = \lambda(Q_x)$, $\psi(y) = \mu(Q_y)$. Now $\{\Phi_i\}$ are S –lattice measurable, $\{\psi_i\}$ are T –lattice measurable. Also, if $g(x) = \lambda((A \times B)_x)$, $h(y) = \mu((A \times B)_y)$ then $\Phi_i \le g$, $\psi_i \le h$ for all i, clearly g is S – lattice measurable and h is T –lattice measurable (by result 3.1).
Therefore by theorem 2.3. Φ is S –lattice measurable and ψ is T –lattice measurable and $\lim_{n\to\infty} \int_X \Phi_n \, d\mu = \int_X \Phi \, d\mu$, $\lim_{n\to\infty} \int_Y \psi_n \, d\lambda = \int_Y \psi \, d\lambda$ But $\int_X \Phi_n \, d\mu = \int_Y \psi_n \, d\lambda$, for every n. Therefore $\int_X \Phi \, d\mu = \int_Y \psi \, d\lambda$.
Therefore $Q \in K$.

Theorem 3.2: Let (X, S, μ), (Y, T, λ) be lattice σ - finite measure spaces. Suppose $Q \in S \times T$.If $\Phi(x) = \lambda(Q_x)$, $\psi(y) = \mu(Q_y)$ for all $x \in X$, $y \in Y$ then Φ is S –lattice measurable, ψ is T lattice measurable and $\int_X \Phi \, d\mu = \int_Y \psi \, d\lambda$.

Proof. From the hypothesis, we have that μ and λ are positive lattice measures on S and T respectively and $X = \bigvee_{n=1}^{\infty} X_n$, $\mu(X_n) < \infty$, $Y = \bigvee_{m=1}^{\infty} Y_m$, $\lambda(Y_m) < \infty$.Since $Q_x \in T$, $Q_y \in S$ we can find $\lambda(Q_x)$ and $\mu(Q_y)$. Let $K = \{Q \in S \times T :$ Φ is S –lattice measurable, ψ is T –lattice measurable and $\int_X f \, d\mu = \int_Y \psi \, d\lambda\}$. Define $Q_{mn} = Q \wedge (X_n \times Y_m)$ (m, n = 1, 2, 3, …….). Let $ß = \{Q \in S \times T$: $Q_{mn} \in K$ for all choices of m and n$\}$.
[Since $X = \bigvee_{n=1}^{\infty} X_n$, $Y = \bigvee_{m=1}^{\infty} Y_m$, X_n's are disjoint, Y_m's are disjoint, $\mu(X_n) < \infty$, $\mu(Y_m) < \infty$ for all m, n.] Then from result 3.2 and result 3.4 we get that ß is a monotone class.
(Note that if $Q \in ß$, then $Q \in S \times T$ and $Q_{mn} \in S \times T$ and $Q_{mn} \in K$ for all m, n).
But Q_{mn}'s are disjoint. Also $Q = \vee Q_i$. Therefore $Q \in K$ (by result 3.3) if $Q \in ß$ such that $Q_i < Q_{i+1}$ for i = 1,2,3,…… then $Q_i \in ß$ and hence $\vee Q_i \in K$ (by result 3.2) let $Q = \vee Q_i$. Then $Q_{mn} = \bigvee_{i=1}^{\infty} (Q_i)_{mn}$. As $(Q_i)_{mn} \in K$ for all m, n and since these are disjoint $Q_{mn} \in K$ (by result 3.4). Hence $Q \in ß$. A similar argument shows that if $Q_i \in ß$ and $Q_i > Q_{i+1}$ i = 1, 2, 3,….. then $\wedge Q_i \in ß$. For this we use result 3.4. We also observe that $Q_i < X \times Y$ implies $Q_i < X_n \times Y_m$. Also $\mu(X_n) < \infty$, $\mu(Y_m) < \infty$. Result 3.1 and result 3.3 shows that ß contains all elementary lattices. But $ß < S \times T$ (by definition of ß). By theorem 2.2. $ß = S \times T$. Thus $Q_{mn} \in K$ for all $Q \in S \times T$ and for all choices of m, n.
As $Q = \vee Q_{mn}$, Q_{mn} being disjoint we get by result 3.3, $Q \in K$.
Therefore for every $Q \in S \times T$ we get Φ is S –lattice measurable and ψ is T –lattice measurable and $\int_X \Phi \, d\mu = \int_Y \psi \, d\lambda$. Hence the theorem.

Remark 3.1: Since $\lambda(Q_x) = \int_Y \chi_Q(x, y) d\lambda(y)$ $(x \in X)$ and $\mu(Q_y) = \int_X \chi_Q(x, y) d\mu(x)$ $(y \in Y)$
$\int_X \Phi \, d\mu = \int_Y \psi \, d\lambda$ gives $\int_X d\mu(x) \int_Y \chi_Q(x, y) d\lambda(y) = \int_Y d\lambda(y) \int_X \chi_Q(x, y) d\mu(x)$.

Result 3.5: Let (X, S, μ) and (Y, T, λ) be lattice σ - finite measure spaces. For any $Q \in S \times T$ define $m(Q) = \int_X \lambda(Q_X) \, d\mu(x) = \int_X \mu(Q_Y) \, d\lambda(y)$. Then m is a lattice measure on a lattice σ - algebra $S \times T$.

Proof: Clearly m(Q) is in $[0, \infty]$. Let $\{A_i\}_{i=1}^{\infty}$ be a disjoint countable collection of lattice measurable sets of $S \times T$. Let $A = \bigvee_{i=1}^{\infty} A_i$.

Let $\Phi(x) = \lambda(A_x) = \lambda(\bigvee_{i=1}^{\infty} A_{i_x}) = \sum_{i=1}^{\infty} \lambda(A_{i_x}) = \sum_{i=1}^{\infty} \Phi_i(x)$ where $\Phi_i(x) = \lambda(A_{i_x})$. Therefore $\Phi = \sum_{i=1}^{\infty} \Phi_i$. $m(A) = \int_X \lambda(A_X)\, d\mu(x) = \int_X \Phi\, d\mu$

$$= \int_X \sum_{i=1}^{\infty} \Phi_i\, d\mu = \sum_{i=1}^{\infty} \int_X \Phi_i\, d\mu = \sum_{i=1}^{\infty} \int_X \lambda(A_{i_x})\, d\mu(x)$$

$$= \sum_{i=1}^{\infty} m(A_i).$$

Therefore m is a lattice measure on the lattice σ – algebra $S \times T$.

Result 3.6: $\mu \times \lambda$ is lattice σ - finite measure.

Proof: $X = \bigvee_{i=1}^{\infty} X_n$, $Y = \bigvee_{i=1}^{\infty} Y_m$, X_n's are disjoint, Y_m's are disjoint and $\mu(X_n) < \infty$, $\mu(Y_m) < \infty$ for all m, n. Obviously $X_n \in S$ and $Y_m \in T$. Therefore $X_n \times Y_m$ is a super lattice measurable set and hence $X_n \times Y_m \in S \times T$ for all m, n. Also $\mu \times \lambda\ (X_n \times Y_m) = \mu \times \lambda(Q)$ where $Q = X_n \times Y_m = \int_X \lambda(Q_X)\, d\mu(x)$

$$= \int_X \lambda(Y_m)\, \chi_{X_n}(x)\, d\mu(x) = \lambda(Y_m)\ \mu(X_n) < \infty.$$

Since $\lambda(Y_m) < \infty$ and $\mu(X_n) < \infty$, therefore $X \times Y = \bigvee_{m,n} X_n \times Y_m$ and $\mu \times \lambda(X_n \times Y_m) < \infty$ for all m, n. Hence $\mu \times \lambda$ is lattice σ - finite measure.

Conclusion:

This manuscript express that the class of super lattice measurable sets is closed under finite unions, countable unions, and countable intersections. It has been ascertained that the product two lattice σ- algebras defined on a product lattice is lattice measurable and the elementary integration of these lattice measurable sets are made equal. Further some characteristics of lattice σ-finite measures were acknowledged.

REFERENCES.

[1] D.V.S.R. Anil kumar, J. Venkateswara Rao, E.S.R. Ravi kumar, Characterization of Class of Measurable Borel Lattices, International Journal of Contemporary Mathematical Sciences, Volume 6(2011), no. 9, 439 – 446.
[2] D.V.S.R. Anil kumar, J. Venkateswara Rao, J. Pramada (2011), Radon – Nikodym Theorem and its uniqueness of Signed Lattice Measure, International Journal Of Mathematics And Computation, Volume 12, No. S11; pp.19-27.
[3] Birkhoff. G (1967), Lattice Theory 3rd ed., AMS Colloquim Publications, Colloguim Publications, Rhode Island, New Delhi.
[4] Halmos. P.R (1974), Measure Theory,Springer, New York.
[5] J. pramada, J. Venkateshwara rao and D.V.S.R. Anil Kumar, 2011. Characterization of Class of Super Lattice Measurable Sets. Journal of Applied Sciences, 11: 3525-3529. DOI: 10.3923/jas.2011.3525.3529 URL: http://scialert.net/abstract/?doi=jas.2011.3525.3529
[6] Royden. H.L.(1981), Real Analysis, 3rd ed., Macmillan Publishing, New York.
[7] Rudin, W., 1987. Real and Complex Analysis. 3rd Edn. McGraw – Hill, UK.
[8] Szasz Gabor(1963), Introduction to lattice theory, academic press, New York and London
[9] Tanaka. J(2009), Hahn Decomposition Theorem of Signed Lattice Measure, arXiv:0906.0147Vol1 [Math.CA] 31 May 2009.