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Coefficient inequalities for certain subclasses Of p-valent functions

R.B. Sharma and K. Saroja*
Department of Mathematics,
Kakatiya University, Warangal,
Andhra Pradesh - 506009, India.
rbsharma_005@yahoo.co.in
*sarojakasula@yahoo.com

ABSTRACT

The aim of the present paper is to introduce two new subclasses of p-valent functions with complex order. The coefficient inequalities and Fekete-Szego inequality for the functions in these classes are also obtained.

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Key Words: p-valent functions, complex order, coefficient inequalities, Fekete-Szego inequality.

* Corresponding Author.

1. Introduction

Let $\mathcal{A}_p$ denote the class of all p-valent functions *f* of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \qquad \ldots \qquad (1.1)$$

Which are regular in the open unit disc $U = \{z \in C : |z| < 1\}$

Here $\mathcal{A}_1 = \mathcal{A}$ and $p \in N$.

Let $M_p(\alpha)$ and $N_p(\alpha)$ be the classes consisting of the functions $f \in \mathcal{A}_p$ and satisfying the conditions

$$Re\left(\frac{zf'(z)}{f(z)}\right) < \alpha \qquad z \in U,\ \alpha > 1 \qquad \text{and}$$

$$Re\left(1+\frac{zf''(z)}{f'(z)}\right) < \alpha \ z \in U,\ \alpha > 1 \quad \text{respectively}$$

These classes were introduced by S.Owa and H.M.Srivastava [10] Y.Polatoglu, M.Bolcol, A.Sen and E.Yavuz [4] have studied the subordination results, coefficient inequalities, distortion properties, radius of starlikeness for the functions in $M_p(\alpha)$.

Several authors [3,4,5,7,9] have obtained the Fekete-Szego inequality for functions in various subclasses of analytic, p-valent, meromorphic functions.

In this paper, we define some subclasses of p-valent functions of complex order. We obtain the coefficient inequality and Fekete-Szego inequality, for the functions in these classes.

Definition 1.1: Let 'b' be a non-zero complex number and $\alpha > 1$. A function $f(z)$ of the form (1.1) is said to be in the class $M_p(b,\alpha)$ if

$$Re\left[1+\frac{1}{b}\left[\frac{zf'(z)}{f(z)}-1\right]\right] < \alpha, \quad z \in U \tag{1.2}$$

It is noted that

$M_p(1,\alpha) = M_p(\alpha)$ defined by S.Owa and H.M.Srivastava [10]

$M_1(1,\alpha)=M(\alpha)$ defined by S.Owa and J.Nishiwaki [2]

Definition 1.2: Let 'b' be a non-zero complex number and $\alpha>1$. A function $f(z)$ of the form (1.1) is said to be in the class $N_p(b,\alpha)$ if

$$Re\left[1+\frac{1}{b}\left[\frac{zf''(z)}{f'(z)}\right]\right]<\alpha, \qquad z\in U \tag{1.3}$$

It is noted that

$N_p(1,\alpha)=N_p(\alpha)$ defined by S.Owa and H.M.Srivastava [10]

$N_1(1,\alpha)=N(\alpha)$ defined by S.Owa and J.Nishiwaki [2]

To prove our results we require the following lemma.

Lemma (1.1) [9]: If $p(z)=1+c_1 z+c_2 z^2+......$ is a function with positive real part and $p(0)=1$ then for any complex number *v*, we have

$$\left|c_2-v\,c_1^2\right|\leq 2\ \text{Max}\ \{1,|\,2v-1|\}$$

This result is sharp for the functions

$$p(z)=\frac{1+z^2}{1-z^2}\ \text{ And }\ p(z)=\frac{1+z}{1-z}$$

In the next sections we obtain the coefficient inequality and Fekete-Szego inequality for the function *f* in the classes $M_p(b,\alpha)$ and $N_p(b,\alpha)$.

2. Coefficient inequalities

Theorem 2.1: If $f(z) \in M_p(b,\alpha)$ then

$$\left|a_{n+p}\right| \leq \frac{1}{n!}\prod_{j=0}^{n-1}\left[2\left[|b|(\alpha-1)+(p-1)\right]+j\right] \quad \ldots\ldots\ldots\ldots\ldots \tag{2.1}$$

Proof: Since $f(z) \in M_p(b,\alpha)$ then from the definition (1.1), we have

$$\operatorname{Re}\left[1+\frac{1}{b}\left[\frac{zf'(z)}{f(z)}-1\right]\right] < \alpha .$$

Define a function $p(z)$ such that

$$p(z)=\frac{\alpha-\left[1+\frac{1}{b}\left[\frac{zf'(z)}{f(z)}-1\right]\right]}{\alpha-\left[1+\frac{1}{b}(p-1)\right]}=1+\sum_{n=1}^{\infty}c_n z^n,\ z\in U \quad \ldots\ldots\ldots\ldots\ldots \tag{2.2}$$

Here $p(z)$ is a function with positive real part with $p(0)=1$.

Replacing $f(z)$, $zf'(z)$ with their equivalent expressions on both sides, we get

$$\left[1+\sum_{n=1}^{\infty}c_n z^n\right]\left[b\alpha-[b+p-1]\right]\left[z^p+\sum_{n=1}^{\infty}a_{n+p}z^{n+p}\right]$$

$$=[b\alpha-b+1]\left[z^p+\sum_{n=1}^{\infty}a_{n+p}z^{n+p}\right]-\left[pz^p+\sum_{n=1}^{\infty}(n+p)a_{n+p}z^{n+p}\right] \quad \ldots\ldots\ldots\ldots \tag{2.3}$$

Comparing the coefficient of z^{n+p} on both sides of equation (2.3),

We get,

$$-na_{n+p} = \left[b(\alpha-1)+(1-p)\right]\left[c_n + a_{p+1}c_{n-1} + a_{p+2}c_{n-2} + \ldots + a_{n-1+p}c_1\right] \quad \ldots\ldots\ldots\ldots(2.4)$$

Taking modulus on both sides of (2.4) and applying $|c_n| \le 2\ \forall n \ge 1$ we get

$$|a_{p+n}| \le 2\left[\frac{|b|(\alpha-1)+(p-1)}{n}\right]\left[1+|a_{p+1}|+|a_{p+2}|+\ldots+a_{n-2+p}+|a_{n-1+p}|\right] \quad \ldots\ldots\ldots\ldots(2.5)$$

For n = 1

$$|a_{p+1}| \le 2\left[|b|(\alpha-1)+(p-1)\right]$$

Thus the result holds true for n = 1

For n = 2

$$|a_{p+2}| \le 2\left[\frac{|b|(\alpha-1)+(p-1)}{2}\right]\left[1+2\left[|b|(\alpha-1)+(p-1)\right]\right]$$

Thus the result (2.1) is true for n = 2.

Suppose the result (2.1) is true for n = k

Now for n = k + 1, we have

$$|a_{p+k+1}| \le 2\left[\frac{|b|(\alpha-1)+(p-1)}{(k+1)}\right]\left[1+2\left[|b|(\alpha-1)+(p-1)\right]\right]+$$
$$2\left[\frac{|b|(\alpha-1)+(p-1)}{2}\right]\left[1+2\left[|b|(\alpha-1)+(p-1)\right]\right]+\ldots$$
$$+\ldots+\frac{1}{k!}\prod_{j=0}^{k-1}\left[2\left[|b|(\alpha-1)+(p-1)\right]+j\right]$$

$$|a_{p+k+1}| \le \frac{1}{(k+1)!}\prod_{j=0}^{k}\left[2\left[|b|(\alpha-1)+(p-1)\right]+j\right]$$

Thus the result (2.1) is true for $n = k+1$.

By mathematical induction the result (2.1) is true for all values of n.

This completes the proof of the theorem.

Theorem 2.2: If $f(z) \in N_p(b,\alpha)$ then

$$\left|a_{n+p}\right| \le \frac{p}{n!(n+p)} \prod_{j=0}^{n-1} \left[2\left[|b|(\alpha-1)+(p-1)+j \right] \right] \qquad \ldots\ldots\ldots\ldots\ldots \qquad (2.6)$$

Proof: Since $f(z) \in N_p(b,\alpha)$ then from the definition (1.2), we have

$$\operatorname{Re}\left[1+\frac{1}{b}\left[\frac{zf''(z)}{f'(z)}\right]\right] < \alpha .$$

Define a function $p(z)$ such that

$$p(z) = \frac{\alpha - \left[1+\frac{1}{b}\left[\frac{zf''(z)}{f'(z)}\right]\right]}{\alpha - \left[1+\frac{1}{b}(p-1)\right]} = 1+\sum_{n=1}^{\infty} c_n z^n \quad \ldots\ldots\ldots\ldots\ldots \qquad (2.7)$$

Here $p(z)$ is a function with positive real part and $p(0)=1$.

Replacing $f(z)$, $f'(z)$ & $f''(z)$ with their equivalent expressions in series on both sides, we get

$$\left[\alpha b - b + 1 - p\right]\left[pz^{p-1} + p\sum_{n=1}^{\infty} c_n z^{n+p-1} + \sum_{n=1}^{\infty} a_{n+p}(n+p) z^{p+n-1} + \left[\sum_{n=1}^{\infty} c_n z^n\right]\left[\sum_{n=1}^{\infty} a_{n+p}(n+p) z^{n+p-1}\right]\right]$$
$$= (\alpha b - b)\left[pz^{p-1} + \sum_{n=1}^{\infty} a_{n+p}(n+p) z^{n+p-1}\right] - \left[p(p-1)z^{p-1} + \sum_{n=1}^{\infty} a_{n+p}(n+p)(n+p-1) z^{n+p-1}\right]$$
(2.8)

Comparing the coefficient of z^{n+p-1} on both sides of equation (2.8),

We get,

$$-n(n+p)a_{n+p}=[\alpha b-b+1-p]\begin{bmatrix}pc_n+a_{p+1}c_{n-1}(p+1)+a_{p+2}(p+2)c_{n-2}+\ldots+\\ a_{n-2+p}(n-2+p)c_2+a_{n-1+p}c_1(n-1+p)\end{bmatrix}\ldots.(2.9)$$

Taking modulus on both sides of (2.9) and applying $|c_n|\le 2\ \forall n\ge 1$ we get

$$|a_{p+n}|\le 2\left[\frac{|b|(\alpha-1)+(p-1)}{n(n+p)}\right]\begin{bmatrix}p+(p+1)|a_{p+1}|+(p+2)|a_{p+2}|+\ldots+\\ (n-2+p)|a_{n-2+p}|+(n-1+p)|a_{n-1+p}|\end{bmatrix}\quad\ldots\ldots\ldots\ldots(2.10)$$

For n = 1

$$|a_{p+1}|\le 2\,\frac{[|b|(\alpha-1)+(p-1)].p}{(p+1)}$$

Thus the result (2.6) is true for n = 1.

For n = 2

$$|a_{p+2}|\le 2\left[\frac{|b|(\alpha-1)+(p-1)}{2(p+2)}\right]\left[p+2\left[|b|(\alpha-1)+(p-1)\right]p\right]$$

Thus the result (2.6) holds true for n = 2.

Suppose the result (2.6) is true for n = k

Now for n = k + 1

Consider

$$\begin{aligned}|a_{p+k+1}|\le{}& 2\left[\frac{|b|(\alpha-1)+(p-1)}{(k+1)(k+1+p)}\right]\left[p+2\left[|b|(\alpha-1)+(p-1)\right]\right]+\\ & 2\left[\frac{|b|(\alpha-1)+(p-1)}{2}\right]\left[p+2\left[|b|(\alpha-1)+(p-1)\right]\right]+\ldots\\ & +\ldots+\frac{p}{k!}\prod_{j=0}^{k}\left[2\left[|b|(\alpha-1)+(p-1)\right]+j\right]\end{aligned}$$

$$\Rightarrow |a_{p+k+1}|\le\frac{1}{(k+1)!(k+1+p)}\prod_{j=0}^{k}\left[2\left[|b|(\alpha-1)+(p-1)\right]+j\right]$$

Thus the result (2.6) is true for $n=k+1$.

By mathematical induction the result (2.6) is true for all values of n.

This completes the proof of the theorem.

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3. Fekete – Szego Inequalities

Theorem 3.1: If $f(z) \in Mp(b,\alpha)$ then for any complex number μ we have

$$\left|a_{p+2} - \mu a^2{}_{p+1}\right| \le \left[|b|(\alpha-1)+(p-1)\right]\max\left\{1,\left|2\left\{\left\{(1-\alpha)b+(p-1)\right\}\left\{2\mu-1\right\}\right\}-1\right|\right\}$$

And the result is sharp.

Proof: Since $f(z) \in M_p(b,\alpha)$ then from equation (2.4), we have

$$a_{p+1} = (-1)\left[b(\alpha-1)+(1-p)\right]c_1$$

$$= \left[(1-\alpha)b+(p-1)\right]c_1$$

And

$$a_{p+2} = \left(\frac{-1}{2}\right)\left[b(\alpha-1)+(1-p)\right]\left[c_2 + a_{p+1}c_1\right]$$

$$a_{p+2} = \frac{\left[b(1-\alpha)+(p-1)\right]}{2}\left[c_2 + \left[(1-\alpha)b+(p-1)\right]c_1^2\right]$$

For any complex number μ we have

$$a_{p+2} - \mu a^2{}_{p+1} = \frac{\left[b(1-\alpha)+(p-1)\right]}{2}\left[c_2 + \left[(1-\alpha)b+(p-1)\right]c_1^2\right] - \mu\left[(1-\alpha)b+(p-1)\right]^2 c_1^2$$

$$a_{p+2} - \mu a^2{}_{p+1} = \frac{\left[b(1-\alpha)+(p-1)\right]}{2}\left[c_2 + \left[(1-\alpha)b+(p-1)\right]c_1^2 - 2\mu\left[(1-\alpha)b+(p-1)\right]^2 c_1^2\right]$$

$$= \frac{\left[b(1-\alpha)+(p-1)\right]}{2}\left[c_2 - \left[(1-\alpha)b+(p-1)\right]\left[2\mu-1\right]c_1^2\right]$$

$$a_{p+2}-\mu a^2{}_{p+1}=\frac{\left[b(1-\alpha)+(p-1)\right]}{2}\left[c_2-vc_1^2\right]$$

Where $v=\left[(1-\alpha)b+(p-1)\right][2\mu-1]$

Taking modulus on both sides and by applying Lemma (1.1), we get

$$\left|a_{p+2}-\mu a^2{}_{p+1}\right|=\left|\frac{b(1-\alpha)+(p-1)}{2}\right|\left|c_2-vc_1^2\right|$$

$$\leq\left[|b|(\alpha-1)+(p-1)\right]\max\{1,|2v-1|\}$$

$$\leq\left[|b|(\alpha-1)+(p-1)\right]\max\left\{1,\left|2\left[(1-\alpha)b+(p-1)\right][2\mu-1]-1\right|\right\}$$

This proves the result (3.1). The result is sharp.

$$\left|a_{p+2}-\mu a^2{}_{p+1}\right|=|b|(\alpha-1)+(p-1) \qquad \text{if } p(z)=\frac{1+z^2}{1-z^2}$$

$$=\left[|b|(\alpha-1)+(p-1)\right]\left[\left|2\left[(1-\alpha)b+(p-1)\right][2\mu-]-1\right|\right]$$

$$\text{if } p(z)=\frac{1+z}{1-z}$$

This completes the proof of the theorem.

Theorem 3.2: If $f(z)\in N_p(b,\alpha)$ then for any complex number μ we have

$$\left|a_{p+2}-\mu a^2{}_{p+1}\right|\leq\frac{p}{(2+p)}\left[|b|(\alpha-1)+(p-1)\right]\max$$
$$\left\{1,\left|2\left[b(1-\alpha)+(p-1)\right]\left[2\mu p\frac{(2+p)}{(1+p)^2}-1\right]-1\right|\right\}$$

and the result is sharp.

Proof: If $f(z)\in N_p(b,\alpha)$ then from equation (2.9), we have

$$a_{p+1}=\frac{-\left[\alpha b-b+(1-p)\right]pc_1}{(1+p)}=\frac{\left[b(1-\alpha)+(p-1)\right]pc_1}{(1+p)} \tag{3.5}$$

And

$$a_{p+2} = \frac{-[\alpha b - b + 1 - p]}{2(2+p)}\left[pc_2 + a_{p+1}(p+1)c_1\right]$$

$$a_{p+2} = \frac{[b(1-\alpha)+(p-1)]}{2(2+p)}\left[pc_2 + [b(1-\alpha)+(p-1)]pc_1^2\right] \tag{3.6}$$

For any complex number μ we have

$$a_{p+2} - \mu a^2{}_{p+1} = \frac{p[b(1-\alpha)+(p-1)]}{2(2+p)}\left[c_2 + [b(1-\alpha)+(p-1)]c_1^2\right] - \mu\left[\frac{b(1-\alpha)+(p-1)}{(p+1)}\right]^2 p^2 c_1^2$$

$$a_{p+2} - \mu a^2{}_{p+1} = \frac{p[b(1-\alpha)+(p-1)]}{2(2+p)} + \left[\begin{matrix} c_2 + [b(1-\alpha)+(p-1)]c_1^2 - \\ 2\mu\left[\dfrac{2+p}{(p+1)^2}\right][b(1-\alpha)+(p-1)]pc_1^2 \end{matrix}\right]$$

$$a_{p+2} - \mu a^2{}_{p+1} = \frac{p[b(1-\alpha)+(p-1)]}{2(2+p)}\left[c_2 - vc_1^2\right]$$

Where $v = \left[2\mu p\left[\dfrac{2+p}{(1+p)^2}\right] - 1\right][b(1-\alpha)+(p-1)]$

Taking modulus on both sides and by applying Lemma (1.1), we get

$$\left|a_{p+2} - \mu a^2{}_{p+1}\right| = \left|\frac{p[b(1-\alpha)+(p-1)]}{2(2+p)}\right|\left|c_2 - vc_1^2\right|$$

$$\leq \frac{p\left[|b|(\alpha-1)+(p-1)\right]}{(2+p)}\max\{1,|2v-1|\}$$

$$\left|a_{p+2}-\mu a^{2}{}_{p+1}\right| \leq \frac{p}{2+p}\left[|b|(\alpha-1)+(p-1)\right]$$

$$\max\left\{1,\left|2\left[b(1-\alpha)+(p-1)\right]\left[2\mu p\frac{(2+p)}{(1+p)^{2}}-1\right]-1\right|\right\}$$

This proves the result (3.4) and is sharp, i.e.

$$\left|a_{p+2}-\mu a^{2}{}_{p+1}\right| = \frac{p}{2+p}\left[|b|(\alpha-1)+(p-1)\right] \quad \text{if } p(z)=\frac{1+z^{2}}{1-z^{2}}$$

$$=\frac{p}{(2+p)}\left[|b|(\alpha-1)+(p-1)\right]\left|2\left[b(1-\alpha)+(p-1)\right]\left[2\mu p\frac{(2+p)}{(1+p)^{2}}-1\right]-1\right|$$

$$\text{if } p(z)=\frac{1+z}{1-z}$$

This completes the proof of the theorem.

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