International Journal of Mathematical Engineering and Science
ISSN : 2277-6982 Volume 1 Issue 12 (December 2012)
http://www.ijmes.com/ https://sites.google.com/site/ijmesjournal/

A hybrid iterative scheme for a general system of variational inequalities based on mixed nonlinear mappings

S. Imnang [1]

[1] Department of Mathematics and Statistics, Faculty of Science,
Thaksin University, Phatthalung Campus, Phatthalung, 93110, Thailand
[1] Centre of Excellence in Mathematics, CHE, Si Ayutthaya Road, Bangkok 10400, Thailand
[1] E-mail address: suwicha.n@hotmail.com

Abstract. The purpose of this paper is to study the strong convergence of a hybrid iterative scheme for finding a common element of the set of a general system of variational inequalities for α-inverse- strongly monotone mapping and relaxed (c,d)-cocoercive mapping, the set of solutions of a mixed equilibrium problem and the set of common fixed points of a finite family of nonexpansive mappings in a real Hilbert space. Using the demi-closedness principle for nonexpansive mapping, we prove that the iterative sequence converges strongly to a common element of these three sets under some control conditions. Our results extend recent results announced by many others.

Keywords: Nonexpansive mapping; A general system of variational inequality problem; Mixed equilibrium problem; Demi-closedness principle.

2000 Mathematics Subject Classification: 47H10, 49J40, 47H05 47H09, 46B20.

1 Introduction

Let H be a real Hilbert space with inner product $\langle ., . \rangle$ and C be a nonempty closed convex subset of H. Recall that $T : C \to C$ is nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|,$$

for all $x, y \in C$. The fixed point set of T is denoted by $F(T) := \{x \in C : Tx = x\}$.

Let $A : C \to H$ be a nonlinear mapping. Then A is called

(i) *monotone*, if
$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in C$;

(ii) α*-strongly monotone*, if there exists a positive real number $\alpha > 0$ such that
$\langle Ax - Ay, x - y\rangle \geq \alpha\|x - y\|^2, \quad \forall x, y \in C$;

(iii) *L-Lipschitz continuous* (or *Lipschitzian*), if there exists a constant $L \geq 0$ such that
$\|Ax - Ay\| \leq L\|x - y\|, \quad \forall x, y \in C$;

(iv) α*-inverse-strongly monotone*, if there exists a positive real number $\alpha > 0$ such that
$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C$.

It is obvious that every α-inverse-strongly monotone mapping A is monotone and Lipschitz continuous. It is known that if T is a nonexpansive mapping of C into itself, then $A = I - T$ is 1/2-inverse strongly monotone, where I is the identity mapping of H.

A mapping A is called *relaxed c-cocoercive*, if there exists a constant $c > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq (-c)\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

A mapping A is called *relaxed (c,d)-cocoercive*, if there exist two constants $c, d > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq (-c)\|Ax - Ay\|^2 + d\|x - y\|^2, \quad \forall x, y \in C.$$

For $c = 0$, A is d-strongly monotone. This class of mappings is more general than the class of strongly monotone mappings. As a result, we have the following implication: d-strong monotonicity $\Rightarrow$ relaxed (c, d)-cocoercivity.

For a given nonlinear operator $A : C \to H$, we consider the following variational inequality problem of finding $x^* \in C$ such that

$$\langle Ax^*, x - x^*\rangle \geq 0, \quad \forall x \in C. \tag{1.1}$$

The set of solutions of the variational inequality (1.1) is denoted by $VI(C, A)$. Variational inequality theory has emerged as an important tool in studying a wide class of obstacle, unilateral, free, moving, equilibrium problems arising in several branches of pure and applied sciences in a unified and general framework. The variational inequality problem has been extensively studied and continued in the literature, see, Piri [12], Qin et al. [13], Shehu [14], Wangkeeree and Preechasilp [19], Yao et al. [21], Yao et al. [23] and relevant references cited therein.

Next, we focus on a general system of variational inequality problems [in short, GSVI] which is considered by Ceng et al. [2]: find $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda Ay^* + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in C, \\ \langle \mu Bx^* + y^* - x^*, x - y^* \rangle \geq 0, \quad \forall x \in C, \end{cases} \tag{1.2}$$

where $A, B : C \to H$ are two nonlinear mappings, $\lambda > 0$ and $\mu > 0$ are two constants. In particular, if $A = B$, then GSVI (1.2) reduces to find $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda Ay^* + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in C, \\ \langle \mu Ax^* + y^* - x^*, x - y^* \rangle \geq 0, \quad \forall x \in C, \end{cases} \tag{1.3}$$

which is defined by Verma [17], and is called the new system of variational inequalities. Further, if we add the requirement that $x^* = y^*$, then problem (1.3) reduces to the classical variational inequality $VI(C, A)$. Ceng et al. [2] introduced and studied a relaxed extragradient method for finding a common element of the set of solutions of GSVI (1.2) for the α and β-inverse-strongly monotone mappings and the set of fixed points of a nonexpansive mapping in a real Hilbert space. Some related works, we refer to see [3, 5, 8, 9, 18, 22].

Recently, in 2012, Ceng et al. [3] considered an iterative method for the system of GSVI (1.2) and obtained a strong convergence theorem for the two different systems of GSVI (1.2) and the set of fixed points of a strict pseudocontraction mapping in a real Hilbert space.

Let $\varphi : C \to \mathbb{R} \bigcup \{+\infty\}$ be a proper extended real-valued function and F be a bifunction from $C \times C$ to $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. Ceng and Yao [4] considered the following mixed equilibrium problem (in short, MEP):

$$\text{Find } \ x \in C \ \text{ such that } \ F(x, y) + \varphi(y) \geq \varphi(x), \quad \forall y \in C. \tag{1.4}$$

The set of solution of MEP (1.4) is denoted by $MEP(F, \varphi)$. It is easy to see that x is a solution of MEP (1.4) implies that $x \in \mathrm{dom}\varphi = \{x \in C \mid \varphi(x) < +\infty\}$.

If $\varphi = 0$, then the MEP (1.4) becomes the following equilibrium problem:

$$\text{Find } \ x \in C \ \text{ such that } \ F(x, y) \geq 0, \quad \forall y \in C. \tag{1.5}$$

International Journal of Mathematical Engineering and Science
ISSN : 2277-6982 Volume 1 Issue 12 (December 2012)
http://www.ijmes.com/ https://sites.google.com/site/ijmesjournal/

The set of solution of (1.5) is denoted by $EP(F)$.

If $F = 0$, then the MEP (1.4) reduces to the convex minimization problem:

$$\text{Find } x \in C \text{ such that } \varphi(y) \geq \varphi(x), \quad \forall y \in C.$$

If $\varphi = 0$ and $F(x, y) = \langle Ax, y - x\rangle$ for all $x, y \in C$, where A is a mapping from C into H, then MEP (1.4) reduces to the classical variational inequality and $EP(F) = VI(C, A)$. For solving problem MEP (1.4), Ceng and Yao [4] introduced a hybrid iterative scheme for finding a common element of the set $MEP(F, \varphi)$ and the set of common fixed points of finite many nonexpansive mappings in a Hilbert space. Some related works, we refer to see [8, 14, 18, 21].

Recently, in 2012, Kumam and Katchang [9] introduced an iterative algorithm for finding a common element of the set of solutions of a system of mixed equilibrium problems, the set of solutions of a general system of variational inequalities for Lipschitz continuous and relaxed cocoercive mappings, the set of common fixed points for nonexpansive semigroups and the set of common fixed points for an infinite family of strictly pseudocontractive mappings in Hilbert spaces.

Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself. In 1999, Atsushiba and Takahashi [1] defined the mapping W_n as follows:

$$\begin{aligned}
U_{n,1} &= \lambda_{n,1}T_1 + (1 - \lambda_{n,1})I,\\
U_{n,2} &= \lambda_{n,2}T_2U_{n,1} + (1 - \lambda_{n,2})I,\\
U_{n,3} &= \lambda_{n,3}T_3U_{n,2} + (1 - \lambda_{n,3})I,\\
&\vdots\\
U_{n,N-1} &= \lambda_{n,N-1}T_{N-1}U_{n,N-2} + (1 - \lambda_{n,N-1})I,\\
W_n = U_{n,N} &= \lambda_{n,N}T_NU_{n,N-1} + (1 - \lambda_{n,N})I,
\end{aligned}$$

where $\{\lambda_{n,i}\}_i^N \subseteq [0, 1]$. This mapping is called the W-mapping generated by $T_1, T_2, \ldots, T_N$ and $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N}$. In 2000, Takahashi and Shimoji [16] proved that if X is a strictly convex Banach space, then $F(W_n) = \bigcap_{i=1}^N F(T_i)$, where $0 < \lambda_{n,i} < 1, \quad i = 1, 2, \ldots, N$.

In 2009, Kangtunyakarn and Suantai [7] introduced a new mapping called the S-mapping. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself. For each $n \in \mathbb{N}$, and $j = 1, 2, \ldots, N$, let $\alpha_j^{(n)} = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j})$ be such that $\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j} \in [0, 1]$ with $\alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$. They defined the new mapping

$S_n : C \to C$ as follows:

$$
\begin{aligned}
U_{n,0} &= I,\\
U_{n,1} &= \alpha_1^{n,1} T_1 U_{n,0} + \alpha_2^{n,1} U_{n,0} + \alpha_3^{n,1} I,\\
U_{n,2} &= \alpha_1^{n,2} T_2 U_{n,1} + \alpha_2^{n,2} U_{n,1} + \alpha_3^{n,2} I,\\
U_{n,3} &= \alpha_1^{n,3} T_3 U_{n,2} + \alpha_2^{n,3} U_{n,2} + \alpha_3^{n,3} I,\\
&\vdots\\
U_{n,N-1} &= \alpha_1^{n,N-1} T_{N-1} U_{n,N-2} + \alpha_2^{n,N-1} U_{n,N-2} + \alpha_3^{n,N-1} I,\\
S_n = U_{n,N} &= \alpha_1^{n,N} T_N U_{n,N-1} + \alpha_2^{n,N} U_{n,N-1} + \alpha_3^{n,N} I.
\end{aligned}
$$

The mapping S_n is called the S-mapping generated by $T_1, T_2, \ldots, T_N$ and $\alpha_1^{(n)}$, $\alpha_2^{(n)}, \ldots, \alpha_N^{(n)}$. Nonexpansivity of each T_i ensures the nonexpansivity of S_n.

Motivated and inspired by Ceng et al. [2], Ceng et al. [3], Ceng and Yao [4] and Kangtunyakarn and Suantai [7], we introduce a hybrid iterative scheme for finding a common element of the set of solutions of GSVI (1.2) for $\alpha-$inverse -strongly monotone mapping and relaxed (c,d)-cocoercive mapping, the set of solutions of MEP (1.4) and the set of common fixed points of a finite family of nonexpansive mappings in a real Hilbert space. Starting with an arbitrary $v \in C$ and let $x_1 \in C$, we define the sequences $\{x_n\}, \{u_n\}$ and $\{y_n\}$ by

$$
\begin{cases}
F(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C,\\
y_n = P_C(u_n - \mu B u_n),\\
x_{n+1} = a_n v + b_n x_n + (1 - a_n - b_n) S_n P_C(y_n - \lambda A y_n), \quad n \geq 1,
\end{cases}
\tag{1.6}
$$

where $\lambda > 0$ and $\mu > 0$ are two constants, $\{r_n\} \subset (0, \infty)$ and $\{a_n\}, \{b_n\} \subset [0, 1]$. Using the demi-closedness principle for nonexpansive mappings, we show that the sequence $\{x_n\}$ converges strongly to a common element of those three sets under some control conditions. Our results extend recent results announced by many others.

2 Preliminaries

In this section, we recall the well known results and give some useful lemmas that will be used in the next section.

Let C be a nonempty closed convex subset of a real Hilbert space H. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C.$$

International Journal of Mathematical Engineering and Science
ISSN : 2277-6982 Volume 1 Issue 12 (December 2012)
http://www.ijmes.com/ https://sites.google.com/site/ijmesjournal/

P_C is called the *metric projection* of H onto C. It is well known that P_C is a nonexpansive mapping of H onto C and satisfies

$$\langle x-y, P_Cx - P_Cy\rangle \geq \|P_Cx - P_Cy\|^2, \quad \forall x, y \in H. \tag{2.1}$$

Obviously, this immediately implies that

$$\|(x-y)-(P_Cx - P_Cy)\|^2 \leq \|x-y\|^2 - \|P_Cx - P_Cy\|^2, \quad \forall x, y \in H. \tag{2.2}$$

Recall that, P_Cx is characterized by the following properties: $P_Cx \in C$ and

$$\begin{aligned} &\langle x - P_Cx, y - P_Cx\rangle \leq 0, \\ &\|x-y\|^2 \geq \|x - P_Cx\|^2 + \|P_Cx - y\|^2, \end{aligned} \tag{2.3}$$

for all $x \in H$ and $y \in C$; see Goebel and Kirk [6] for more details.

For solving the mixed equilibrium problem, let us give the following assumptions for the bifunction F, φ and the set C:
(A1) $F(x, x) = 0$ for all $x \in C$;
(A2) F is monotone, i.e. $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;
(A3) For each $y \in C$, $x \mapsto F(x, y)$ is weakly upper semicontinuous;
(A4) For each $x \in C$, $y \mapsto F(x, y)$ is convex;
(A5) For each $x \in C$, $y \mapsto F(x, y)$ is lower semicontinuous;
(B1) For each $x \in H$ and $r > 0$, there exist a bounded subset $D_x \subseteq C$ and $y_x \in C$ such that for any $z \in C \setminus D_x$,

$$F(z, y_x) + \varphi(y_x) + \frac{1}{r}\langle y_x - z, z - x\rangle < \varphi(z).$$

(B2) C is a bounded set.

In the sequel, we shall need to use the following lemmas.

Lemma 2.1. ([11]) *Let C be a nonempty closed convex subset of H. Let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A5) and let $\varphi : C \to \mathbb{R}\bigcup\{+\infty\}$ be a proper lower semicontinuous and convex function. Assume that either (B1) or (B2) holds. For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows.*

$$T_r(x) = \left\{z \in C : F(z, y) + \varphi(y) + \frac{1}{r}\langle y - z, z - x\rangle \geq \varphi(z), \quad \forall y \in C\right\}$$

for all $x \in H$. Then the following conclusions hold:
(1) *For each $x \in H$, $T_r(x) \neq \emptyset$;*
(2) *T_r is single-valued;*
(3) *T_r is firmly nonexpansive, i.e. for any $x, y \in H$,*

$$\|T_r(x) - T_r(y)\|^2 \leq \langle T_rx - T_ry, x - y\rangle;$$

International Journal of Mathematical Engineering and Science
ISSN : 2277-6982 Volume 1 Issue 12 (December 2012)
http://www.ijmes.com/ https://sites.google.com/site/ijmesjournal/

(4) $F(T_r) = MEP(F, \varphi)$;
(5) $MEP(F, \varphi)$ *is closed and convex.*

Lemma 2.2. ([20]) Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that

$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n$,

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(ii) $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.3. ([10]) *Let $(H, \langle ., . \rangle)$ be an inner product space. Then, for all $x, y, z \in H$ and $\alpha, \beta, \gamma \in [0, 1]$ with $\alpha + \beta + \gamma = 1$, we have*

$$\|\alpha x + \beta y + \gamma z\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x - y\|^2 - \alpha\gamma\|x - z\|^2 - \beta\gamma\|y - z\|^2.$$

Lemma 2.4. ([15]) *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{b_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty} b_n < 1$. Suppose $x_{n+1} = (1 - b_n)y_n + b_n x_n$ for all integers $n \geq 1$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

Lemma 2.5. ([6]) *Demi-closedness principle. Assume that T is a nonexpansive self-mapping of a nonempty closed convex subset C of a real Hilbert space H. If T has a fixed point, then $I - T$ is demi-closed: that is, whenever $\{x_n\}$ is a sequence in C converging weakly to some $x \in C$ (for short, $x_n \rightharpoonup x \in C$), and the sequence $\{(I - T)x_n\}$ converges strongly to some y (for short, $(I - T)x_n \to y$), it follows that $(I - T)x = y$. Here I is the identity operator of H.*

The following lemma is an immediate consequence of an inner product.

Lemma 2.6. *In a real Hilbert space H, there holds the inequality*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in H.$$

Lemma 2.7. ([7]) *Let C be a nonempty closed convex subset of a strictly convex Banach space X. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$ and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j)$, $j = 1, 2, \ldots, N$, where $\alpha_1^j, \alpha_2^j, \alpha_3^j \in [0, 1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j \in (0, 1)$ for all $j = 1, 2, \ldots, N-1$, $\alpha_1^N \in (0, 1]$ and α_2^j, $\alpha_3^j \in [0, 1)$ for all $j = 1, 2, \ldots, N$. Let S be the S-mapping generated by $T_1, T_2, \ldots, T_N$ and $\alpha_1, \alpha_2, \ldots, \alpha_N$. Then $F(S) = \bigcap_{i=1}^N F(T_i)$.*

Lemma 2.8. ([7]) *Let C be a nonempty closed convex subset of a Banach space X. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself and for all $n \in \mathbb{N}$ and all $j \in \{1, 2, \dots, N\}$, let $\alpha_j^{(n)} = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j})$, $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j)$ where $\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j} \in [0,1]$, $\alpha_1^j, \alpha_2^j, \alpha_3^j \in [0,1]$, $\alpha_1^{n,j}+\alpha_2^{n,j}+\alpha_3^{n,j} = 1$ and $\alpha_1^j+\alpha_2^j+\alpha_3^j = 1$. Suppose $\alpha_i^{n,j} \to \alpha_i^j$ as $n \to \infty$ for all $i \in \{1, 3\}$ and all $j = 1, 2, 3, \dots, N$. Let S and S_n be the S-mappings generated by $T_1, T_2, \dots, T_N$ and $\alpha_1, \alpha_2, \dots, \alpha_N$ and $T_1, T_2, \dots, T_N$ and $\alpha_1^{(n)}, \alpha_2^{(n)}, \dots, \alpha_N^{(n)}$, respectively. Then $\lim_{n\to\infty} \|S_n x - Sx\| = 0$ for every $x \in C$.*

Lemma 2.9. ([2]) *For given $x^*, y^* \in C$, (x^*, y^*) is a solution of GSVI (1.2) if and only if x^* is a fixed of the mapping $G : C \to C$ defined by*

$$G(x) = P_C[P_C(x - \mu Bx) - \lambda A P_C(x - \mu Bx)], \quad \forall x \in C,$$

where $y^* = P_C(x^* - \mu B x^*)$.

Throughout this paper, the set of fixed points of the mapping G is denoted by $GSVI(C, A, B)$.

3 Main Results

In this section, we prove a strong convergence theorem of the hybrid iterative scheme (1.6) to a common element of the set of solutions of GSVI (1.2) for $\alpha-$inverse -strongly monotone mapping and relaxed (c,d)-cocoercive mapping, the set of solutions of MEP (1.4) and the set of common fixed points of a finite family of nonexpansive mappings in a real Hilbert space.

Next, we prove some lemmas which are very useful for our consideration.

Lemma 3.1. *Let $A : C \to H$ be α-inverse-strongly monotone and let $B : C \to H$ be L-Lipschitzian and relaxed (c', d')-cocoercive. Let the mapping $G : C \to C$ be defined by*

$$G(x) = P_C[P_C(x - \mu Bx) - \lambda A P_C(x - \mu Bx)], \quad \forall x \in C.$$

If $\lambda \in (0, 2\alpha]$ and $0 < \mu \leq \frac{2(d' - c' L^2)}{L^2}$. Then G is nonexpansive.

Proof. For any $x, y \in C$, we have

$$\begin{aligned}\|G(x)-G(y)\| &= \|P_C[P_C(x-\mu Bx)-\lambda AP_C(x-\mu Bx)] \\ &\quad - P_C[P_C(y-\mu By)-\lambda AP_C(y-\mu By)]\|^2 \\ &\leq \|P_C(x-\mu Bx)-\lambda AP_C(x-\mu Bx) \\ &\quad - \big(P_C(y-\mu By)-\lambda AP_C(y-\mu By)\big)\| \\ &= \|(I-\lambda A)P_C(I-\mu B)x-(I-\lambda A)P_C(I-\mu B)y\|.\end{aligned}$$

It is well known that if $A : C \to H$ be α-inverse-strongly monotone and $B : C \to H$ be L-Lipschitzian and relaxed (c', d')-cocoercive, then $I - \lambda A$ and $I - \mu B$ are nonexpansive, where $\lambda \in (0, 2\alpha]$ and $0 < \mu \leq \frac{2(d'-c'L^2)}{L^2}$. It follows that $(I - \lambda A)P_C(I - \mu B)$ is nonexpansive, which implies that G is nonexpansive. □

Theorem 3.2. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let F be a function from $C\times C$ to $\mathbb{R}$ satisfying (A1)-(A5) and $\varphi : C \to \mathbb{R}\bigcup\{+\infty\}$ be a proper lower semicontinuous and convex function. Let $A : C \to H$ be α-inverse-strongly monotone mapping and let $B : C \to H$ be a L-Lipschitzian and relaxed (c', d')-cocoercive mapping. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive self-mappings of C such that $\Omega = \bigcap_{i=1}^N F(T_i)\bigcap GSVI(C, A, B)\bigcap MEP(F, \varphi) \neq \emptyset$. For all $j \in \{1, 2, \ldots, N\}$, let $\alpha_j^{(n)} = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j})$ be such that $\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j} \in [0, 1]$, $\alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$, $\{\alpha_1^{n,j}\}_{j=1}^{N-1} \subset [\eta_1, \theta_1]$ with $0 < \eta_1 \leq \theta_1 < 1$, $\{\alpha_1^{n,N}\} \subset [\eta_N, 1]$ with $0 < \eta_N \leq 1$ and $\{\alpha_2^{n,j}\}_{j=1}^N, \{\alpha_3^{n,j}\}_{j=1}^N \subset [0, \theta_2]$ with $0 \leq \theta_2 < 1$. Let S_n be the S-mappings generated by $T_1, T_2, \ldots, T_N$ and $\alpha_1^{(n)}, \alpha_2^{(n)}, \ldots, \alpha_N^{(n)}$. Assume that either (B1) or (B2) holds and that v is an arbitrary point in C. Let $x_1 \in C$ and $\{x_n\}, \{u_n\}, \{y_n\}$ be the sequences defined by*

$$\begin{cases} F(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = P_C(u_n - \mu Bu_n), \\ x_{n+1} = a_n v + b_n x_n + (1 - a_n - b_n)S_n P_C(y_n - \lambda Ay_n), \quad n \geq 1, \end{cases}$$

where $\lambda \in (0, 2\alpha)$ and $0 < \mu < \frac{2(d'-c'L^2)}{L^2}$. Suppose that the following conditions hold:

(C1) $\lim_{n\to\infty} a_n = 0$ *and* $\sum_{n=1}^\infty a_n = \infty$;

(C2) $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty} b_n < 1$;

(C3) $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$;

(C4) $\lim_{n\to\infty} |\alpha_1^{n+1,i} - \alpha_1^{n,i}| = 0$ *for all* $i \in \{1, 2, \ldots, N\}$ *and* $\lim_{n\to\infty} |\alpha_3^{n+1,j} - \alpha_3^{n,j}| = 0$ *for all* $j \in \{2, 3, \ldots, N\}$.

Then $\{x_n\}$ converges strongly to $\overline{x} = P_\Omega v$ and $(\overline{x}, \overline{y})$ is a solution of GSVI (1.2), where $\overline{y} = P_C(\overline{x} - \mu B\overline{x})$.

Proof. Let $x^* \in \Omega$ and $\{T_{r_n}\}$ be a sequence of mappings defined as in Lemma 2.1. It follows from Lemma 2.9 that

$$x^* = P_C[P_C(x^* - \mu Bx^*) - \lambda AP_C(x^* - \mu Bx^*)].$$

Put $y^* = P_C(x^* - \mu Bx^*)$ and $t_n = P_C(y_n - \lambda Ay_n)$, then $x^* = P_C(y^* - \lambda Ay^*)$ and

$$x_{n+1} = a_n v + b_n x_n + (1 - a_n - b_n)S_n t_n.$$

By nonexpansiveness of $I - \lambda A$, $I - \mu B$, P_C and T_{r_n}, we have

$$\begin{aligned}\|t_n - x^*\|^2 &= \|P_C(I - \lambda A)y_n - P_C(I - \lambda A)y^*\|^2 \\ &\le \|y_n - y^*\|^2 = \|P_C(I - \mu B)u_n - P_C(I - \mu B)x^*\|^2 \\ &\le \|u_n - x^*\|^2 = \|T_{r_n}x_n - T_{r_n}x^*\|^2 \le \|x_n - x^*\|^2, \end{aligned} \tag{3.1}$$

which, implies that

$$\begin{aligned}\|x_{n+1} - x^*\| &= \|a_n v + b_n x_n + (1 - a_n - b_n)S_n t_n - x^*\| \\ &\le a_n\|v - x^*\| + b_n\|x_n - x^*\| + (1 - a_n - b_n)\|t_n - x^*\| \\ &\le a_n\|v - x^*\| + b_n\|x_n - x^*\| + (1 - a_n - b_n)\|x_n - x^*\| \\ &\le \max\{\|v - x^*\|, \|x_1 - x^*\|\}.\end{aligned}$$

Thus, $\{x_n\}$ is bounded. Consequently, the sequences $\{u_n\}$, $\{y_n\}$, $\{t_n\}$, $\{Ay_n\}$, $\{Bu_n\}$ and $\{S_n t_n\}$ are also bounded. Also, observe that

$$\begin{aligned}\|t_{n+1} - t_n\| &= \|P_C(y_{n+1} - \lambda Ay_{n+1}) - P_C(y_n - \lambda Ay_n)\| \\ &\le \|y_{n+1} - y_n\| \\ &= \|P_C(u_{n+1} - \mu Bu_{n+1}) - P_C(u_n - \mu Bu_n)\| \\ &\le \|u_{n+1} - u_n\|.\end{aligned} \tag{3.2}$$

On the other hand, from $u_n = T_{r_n}x_n \in \mathrm{dom}\varphi$ and $u_{n+1} = T_{r_{n+1}}x_{n+1} \in \mathrm{dom}\varphi$, we have

$$F(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall y \in C, \tag{3.3}$$

and

$$F(u_{n+1}, y) + \varphi(y) - \varphi(u_{n+1}) + \frac{1}{r_{n+1}}\langle y - u_{n+1}, u_{n+1} - x_{n+1}\rangle \ge 0, \quad \forall y \in C. \tag{3.4}$$

Putting $y = u_{n+1}$ in (3.3) and $y = u_n$ in (3.4), we have

$$F(u_n, u_{n+1}) + \varphi(u_{n+1}) - \varphi(u_n) + \frac{1}{r_n}\langle u_{n+1} - u_n, u_n - x_n\rangle \ge 0,$$

and

$$F(u_{n+1}, u_n) + \varphi(u_n) - \varphi(u_{n+1}) + \frac{1}{r_{n+1}}\langle u_n - u_{n+1}, u_{n+1} - x_{n+1}\rangle \geq 0.$$

From the monotonicity of F, we obtain that

$$\left\langle u_{n+1} - u_n, \frac{u_n - x_n}{r_n} - \frac{u_{n+1} - x_{n+1}}{r_{n+1}}\right\rangle \geq 0,$$

and hence

$$\left\langle u_{n+1} - u_n, u_n - u_{n+1} + u_{n+1} - x_n - \frac{r_n}{r_{n+1}}(u_{n+1} - x_{n+1})\right\rangle \geq 0.$$

Then, we have

$$\begin{aligned}\|u_{n+1} - u_n\|^2 &\leq \left\langle u_{n+1} - u_n, x_{n+1} - x_n + (1 - \frac{r_n}{r_{n+1}})(u_{n+1} - x_{n+1})\right\rangle \\ &\leq \|u_{n+1} - u_n\|\left\{\|x_{n+1} - x_n\| + |1 - \frac{r_n}{r_{n+1}}|\|u_{n+1} - x_{n+1}\|\right\},\end{aligned}$$

and hence

$$\|u_{n+1} - u_n\| \leq \|x_{n+1} - x_n\| + \frac{1}{r_{n+1}}|r_{n+1} - r_n|\|u_{n+1} - x_{n+1}\|. \tag{3.5}$$

It follows from (3.2) and (3.5) that

$$\|t_{n+1} - t_n\| \leq \|x_{n+1} - x_n\| + \frac{1}{r_{n+1}}|r_{n+1} - r_n|\|u_{n+1} - x_{n+1}\|. \tag{3.6}$$

Let $x_{n+1} = b_n x_n + (1 - b_n)z_n$. Then, we obtain

$$\begin{aligned}z_{n+1} - z_n &= \frac{x_{n+2} - b_{n+1}x_{n+1}}{1 - b_{n+1}} - \frac{x_{n+1} - b_n x_n}{1 - b_n} \\ &= \frac{a_{n+1}v + (1 - a_{n+1} - b_{n+1})S_{n+1}t_{n+1}}{1 - b_{n+1}} - \frac{a_n v + (1 - a_n - b_n)S_n t_n}{1 - b_n} \\ &= \frac{a_{n+1}}{1 - b_{n+1}}(v - S_{n+1}t_{n+1}) + \frac{a_n}{1 - b_n}(S_n t_n - v) + S_{n+1}t_{n+1} - S_n t_n.\end{aligned} \tag{3.7}$$

Next, we estimate $\|S_{n+1}t_{n+1} - S_n t_n\|$.

International Journal of Mathematical Engineering and Science
ISSN : 2277-6982 Volume 1 Issue 12 (December 2012)
http://www.ijmes.com/ https://sites.google.com/site/ijmesjournal/

For each $k \in \{2, 3, \dots, N\}$, we have

$$\begin{aligned}
\|U_{n+1,k}t_n - U_{n,k}t_n\| &= \|\alpha_1^{n+1,k}T_kU_{n+1,k-1}t_n + \alpha_2^{n+1,k}U_{n+1,k-1}t_n + \alpha_3^{n+1,k}t_n \\
&\quad - \alpha_1^{n,k}T_kU_{n,k-1}t_n - \alpha_2^{n,k}U_{n,k-1}t_n - \alpha_3^{n,k}t_n\| \\
&= \|\alpha_1^{n+1,k}(T_kU_{n+1,k-1}t_n - T_kU_{n,k-1}t_n) \\
&\quad + (\alpha_1^{n+1,k} - \alpha_1^{n,k})T_kU_{n,k-1}t_n + (\alpha_3^{n+1,k} - \alpha_3^{n,k})t_n \\
&\quad + \alpha_2^{n+1,k}(U_{n+1,k-1}t_n - U_{n,k-1}t_n) + (\alpha_2^{n+1,k} - \alpha_2^{n,k})U_{n,k-1}t_n\| \\
&\leq \alpha_1^{n+1,k}\|U_{n+1,k-1}t_n - U_{n,k-1}t_n\| + |\alpha_1^{n+1,k} - \alpha_1^{n,k}|\|T_kU_{n,k-1}t_n\| \\
&\quad + |\alpha_3^{n+1,k} - \alpha_3^{n,k}|\|t_n\| + \alpha_2^{n+1,k}\|U_{n+1,k-1}t_n - U_{n,k-1}t_n\| \\
&\quad + |\alpha_2^{n+1,k} - \alpha_2^{n,k}|\|U_{n,k-1}t_n\| \\
&= (\alpha_1^{n+1,k} + \alpha_2^{n+1,k})\|U_{n+1,k-1}t_n - U_{n,k-1}t_n\| \\
&\quad + |\alpha_1^{n+1,k} - \alpha_1^{n,k}|\|T_kU_{n,k-1}t_n\| + |\alpha_3^{n+1,k} - \alpha_3^{n,k}|\|t_n\| \\
&\quad + |\alpha_2^{n+1,k} - \alpha_2^{n,k}|\|U_{n,k-1}t_n\| \\
&\leq \|U_{n+1,k-1}t_n - U_{n,k-1}t_n\| + |\alpha_1^{n+1,k} - \alpha_1^{n,k}|\|T_kU_{n,k-1}t_n\| \\
&\quad + |\alpha_3^{n+1,k} - \alpha_3^{n,k}|\|t_n\| + |(\alpha_1^{n,k} - \alpha_1^{n+1,k}) + (\alpha_3^{n,k} - \alpha_3^{n+1,k})|\|U_{n,k-1}t_n\| \\
&\leq \|U_{n+1,k-1}t_n - U_{n,k-1}t_n\| + |\alpha_1^{n+1,k} - \alpha_1^{n,k}|\|T_kU_{n,k-1}t_n\| \\
&\quad + |\alpha_3^{n+1,k} - \alpha_3^{n,k}|\|t_n\| + |\alpha_1^{n,k} - \alpha_1^{n+1,k}|\|U_{n,k-1}t_n\| \\
&\quad + |\alpha_3^{n,k} - \alpha_3^{n+1,k}|\|U_{n,k-1}t_n\| \\
&= \|U_{n+1,k-1}t_n - U_{n,k-1}t_n\| + |\alpha_1^{n+1,k} - \alpha_1^{n,k}|(\|T_kU_{n,k-1}t_n\| + \|U_{n,k-1}t_n\|) \\
&\quad + |\alpha_3^{n+1,k} - \alpha_3^{n,k}|(\|t_n\| + \|U_{n,k-1}t_n\|). \qquad (3.8)
\end{aligned}$$

It follow from (3.8) that

$$\begin{aligned}
\|S_{n+1}t_n - S_nt_n\| &= \|U_{n+1,N}t_n - U_{n,N}t_n\| \\
&\leq \|U_{n+1,1}t_n - U_{n,1}t_n\| + \sum_{j=2}^{N}|\alpha_1^{n+1,j} - \alpha_1^{n,j}|(\|T_jU_{n,j-1}t_n\| + \|U_{n,j-1}t_n\|) \\
&\quad + \sum_{j=2}^{N}|\alpha_3^{n+1,j} - \alpha_3^{n,j}|(\|t_n\| + \|U_{n,j-1}t_n\|) \\
&= |\alpha_1^{n+1,1} - \alpha_1^{n,1}|\|T_1t_n - t_n\| \\
&\quad + \sum_{j=2}^{N}|\alpha_1^{n+1,j} - \alpha_1^{n,j}|(\|T_jU_{n,j-1}t_n\| + \|U_{n,j-1}t_n\|) \\
&\quad + \sum_{j=2}^{N}|\alpha_3^{n+1,j} - \alpha_3^{n,j}|(\|t_n\| + \|U_{n,j-1}t_n\|).
\end{aligned}$$

International Journal of Mathematical Engineering and Science
ISSN : 2277-6982 Volume 1 Issue 12 (December 2012)
http://www.ijmes.com/ https://sites.google.com/site/ijmesjournal/

This together with the condition $(C4)$, we obtain

$$\lim_{n\to\infty} \|S_{n+1}t_n - S_n t_n\| = 0. \tag{3.9}$$

It follows from (3.6) that

$$\begin{aligned}\|S_{n+1}t_{n+1} - S_n t_n\| &\le \|t_{n+1} - t_n\| + \|S_{n+1}t_n - S_n t_n\| \\ &\le \|x_{n+1} - x_n\| + \frac{1}{r_{n+1}}|r_{n+1} - r_n|\|u_{n+1} - x_{n+1}\| \\ &\quad + \|S_{n+1}t_n - S_n t_n\|. \end{aligned} \tag{3.10}$$

By (3.7) and (3.10), we have

$$\begin{aligned}\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| &\le \frac{a_{n+1}}{1-b_{n+1}}\|v - S_{n+1}t_{n+1}\| + \frac{a_n}{1-b_n}\|S_n t_n - v\| \\ &\quad + \|S_{n+1}t_{n+1} - S_n t_n\| - \|x_{n+1} - x_n\| \\ &\le \frac{a_{n+1}}{1-b_{n+1}}\|v - S_{n+1}t_{n+1}\| + \frac{a_n}{1-b_n}\|S_n t_n - v\| \\ &\quad + \frac{1}{r_{n+1}}|r_{n+1} - r_n|\|u_{n+1} - x_{n+1}\| \\ &\quad + \|S_{n+1}t_n - S_n t_n\|.\end{aligned}$$

This together with (C1)-(C3) and (3.9), we obtain that

$$\limsup_{n\to\infty} \|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \le 0.$$

Hence, by Lemma 2.4, we get $\|x_n - z_n\| \to 0$ as $n \to \infty$. Consequently,

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} (1 - b_n)\|z_n - x_n\| = 0. \tag{3.11}$$

From (C3), (3.2) and (3.5), we also have $\|u_{n+1} - u_n\| \to 0$, $\|t_{n+1} - t_n\| \to 0$ and $\|y_{n+1} - y_n\| \to 0$, as $n \to \infty$.
Since

$$x_{n+1} - x_n = a_n(v - x_n) + (1 - a_n - b_n)(S_n t_n - x_n),$$

therefore

$$\|S_n t_n - x_n\| \to 0 \quad \text{as} \quad n \to \infty. \tag{3.12}$$

Next, we prove that $\lim_{n\to\infty} \|x_n - u_n\| = 0$. From Lemma 2.1(3), we have

$$\begin{aligned}\|u_n - x^*\|^2 = \|T_{r_n}x_n - T_{r_n}x^*\|^2 &\le \langle T_{r_n}x_n - T_{r_n}x^*, x_n - x^*\rangle \\ &= \langle u_n - x^*, x_n - x^*\rangle = \frac{1}{2}\big\{\|u_n - x^*\|^2 + \|x_n - x^*\|^2 - \|x_n - u_n\|^2\big\}.\end{aligned}$$

Hence

$$\|u_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \|x_n - u_n\|^2. \tag{3.13}$$

From Lemma 2.3, (3.1) and (3.13), we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\leq a_n\|v - x^*\|^2 + b_n\|x_n - x^*\|^2 + (1 - a_n - b_n)\|t_n - x^*\|^2 \\ &\leq a_n\|v - x^*\|^2 + b_n\|x_n - x^*\|^2 + (1 - a_n - b_n)\|u_n - x^*\|^2 \\ &\leq a_n\|v - x^*\|^2 + b_n\|x_n - x^*\|^2 \\ &\quad + (1 - a_n - b_n)\big[\|x_n - x^*\|^2 - \|x_n - u_n\|^2\big] \\ &\leq a_n\|v - x^*\|^2 + \|x_n - x^*\|^2 - (1 - a_n - b_n)\|x_n - u_n\|^2.\end{aligned}$$

It follows that

$$\begin{aligned}&(1 - a_n - b_n)\|x_n - u_n\|^2 \leq a_n\|v - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\ &\leq a_n\|v - x^*\|^2 + (\|x_n - x^*\| + \|x_{n+1} - x^*\|)\|x_{n+1} - x_n\|.\end{aligned}$$

From the conditions (C1), (C2) and (3.11), we obtain

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{3.14}$$

Since

$$\|S_n t_n - u_n\| \leq \|S_n t_n - x_n\| + \|x_n - u_n\|,$$

it follows from (3.12) and (3.14) that

$$\lim_{n\to\infty} \|S_n t_n - u_n\| = 0. \tag{3.15}$$

Next, we show that $\|Ay_n - Ay^*\| \to 0$ and $\|Bu_n - Bx^*\| \to 0$ as $n \to \infty$. From (3.1), we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\leq a_n\|v - x^*\|^2 + b_n\|x_n - x^*\|^2 + (1 - a_n - b_n)\|t_n - x^*\|^2 \\ &= a_n\|v - x^*\|^2 + b_n\|x_n - x^*\|^2 \\ &\quad + (1 - a_n - b_n)\|P_C(y_n - \lambda Ay_n) - P_C(y^* - \lambda Ay^*)\|^2 \\ &\leq a_n\|v - x^*\|^2 + b_n\|x_n - x^*\|^2 \\ &\quad + (1 - a_n - b_n)\|(y_n - \lambda Ay_n) - (y^* - \lambda Ay^*)\|^2 \\ &\leq a_n\|v - x^*\|^2 + b_n\|x_n - x^*\|^2 \\ &\quad + (1 - a_n - b_n)\big[\|y_n - y^*\|^2 + \lambda(\lambda - 2\alpha)\|Ay_n - Ay^*\|^2\big] \\ &\leq a_n\|v - x^*\|^2 + \|x_n - x^*\|^2 \\ &\quad + (1 - a_n - b_n)\lambda(\lambda - 2\alpha)\|Ay_n - Ay^*\|^2,\end{aligned}$$

and

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &\le a_n\|v-x^*\|^2+b_n\|x_n-x^*\|^2+(1-a_n-b_n)\|t_n-x^*\|^2\\
&\le a_n\|v-x^*\|^2+b_n\|x_n-x^*\|^2+(1-a_n-b_n)\|y_n-y^*\|^2\\
&\le a_n\|v-x^*\|^2+b_n\|x_n-x^*\|^2\\
&\quad +(1-a_n-b_n)\|(u_n-\mu Bu_n)-(x^*-\mu Bx^*)\|^2\\
&\le a_n\|v-x^*\|^2+b_n\|x_n-x^*\|^2\\
&\quad +(1-a_n-b_n)\big[\|u_n-x^*\|^2-2\mu\langle u_n-x^*,Bu_n-Bx^*\rangle\\
&\quad +\mu^2\|Bu_n-Bx^*\|^2\big]\\
&\le a_n\|v-x^*\|^2+b_n\|x_n-x^*\|^2\\
&\quad +(1-a_n-b_n)\big[\|u_n-x^*\|^2+2\mu c^{'}\|Bu_n-Bx^*\|^2\\
&\quad -2\mu d^{'}\|u_n-x^*\|^2+\mu^2\|Bu_n-Bx^*\|^2\big]\\
&\le a_n\|v-x^*\|^2+b_n\|x_n-x^*\|^2\\
&\quad +(1-a_n-b_n)\big[\|x_n-x^*\|^2\\
&\quad +(2\mu c^{'}+\mu^2-\frac{2\mu d^{'}}{L^2})\|Bu_n-Bx^*\|^2\big]\\
&\le a_n\|v-x^*\|^2+\|x_n-x^*\|^2\\
&\quad +(1-a_n-b_n)(2\mu c^{'}+\mu^2-\frac{2\mu d^{'}}{L^2})\|Bu_n-Bx^*\|^2.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&-(1-a_n-b_n)\lambda(\lambda-2\alpha)\|Ay_n-Ay^*\|^2\\
&\le a_n\|v-x^*\|^2+(\|x_n-x^*\|+\|x_{n+1}-x^*\|)\|x_{n+1}-x_n\|,
\end{aligned}$$

and

$$\begin{aligned}
&-(1-a_n-b_n)(2\mu c^{'}+\mu^2-\frac{2\mu d^{'}}{L^2})\|Bu_n-Bx^*\|^2\\
&\le a_n\|v-x^*\|^2+(\|x_n-x^*\|+\|x_{n+1}-x^*\|)\|x_{n+1}-x_n\|.
\end{aligned}$$

This together with (3.11), (C1) and (C2), we obtain

$$\|Ay_n-Ay^*\|\to 0 \ \text{ and } \ \|Bu_n-Bx^*\|\to 0 \ \text{ as } \ n\to\infty. \tag{3.16}$$

Next, we prove that $\|S_nt_n-t_n\|\to 0$ as $n\to\infty$. From (2.1) and nonexpansiveness

International Journal of Mathematical Engineering and Science
ISSN : 2277-6982 Volume 1 Issue 12 (December 2012)
http://www.ijmes.com/ https://sites.google.com/site/ijmesjournal/

of $I-\mu B$, we get

$$\begin{aligned}\|y_n-y^*\|^2 &= \|P_C(u_n-\mu Bu_n)-P_C(x^*-\mu Bx^*)\|^2\\ &\le \langle (u_n-\mu Bu_n)-(x^*-\mu Bx^*), y_n-y^*\rangle\\ &= \frac{1}{2}\big[\|(u_n-\mu Bu_n)-(x^*-\mu Bx^*)\|^2+\|y_n-y^*\|^2\\ &\quad -\|(u_n-\mu Bu_n)-(x^*-\mu Bx^*)-(y_n-y^*)\|^2\big]\\ &\le \frac{1}{2}\big[\|u_n-x^*\|^2+\|y_n-y^*\|^2-\|(u_n-x^*)-(y_n-y^*)\|^2\\ &\quad +2\mu\langle (u_n-x^*)-(y_n-y^*), Bu_n-Bx^*\rangle-\mu^2\|Bu_n-Bx^*\|^2\big].\end{aligned}$$

By (3.1), we obtain

$$\begin{aligned}\|y_n-y^*\|^2 &\le \|u_n-x^*\|^2-\|(u_n-x^*)-(y_n-y^*)\|^2\\ &\quad +2\mu\langle (u_n-x^*)-(y_n-y^*), Bu_n-Bx^*\rangle-\mu^2\|Bu_n-Bx^*\|^2\\ &\le \|x_n-x^*\|^2-\|(u_n-x^*)-(y_n-y^*)\|^2\\ &\quad +2\mu\langle (u_n-x^*)-(y_n-y^*), Bu_n-Bx^*\rangle-\mu^2\|Bu_n-Bx^*\|^2.\end{aligned}$$

Hence

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\le a_n\|v-x^*\|^2+b_n\|x_n-x^*\|^2+(1-a_n-b_n)\|y_n-y^*\|^2\\ &\le a_n\|v-x^*\|^2+b_n\|x_n-x^*\|^2+(1-a_n-b_n)\big[\|x_n-x^*\|^2\\ &\quad -\|(u_n-x^*)-(y_n-y^*)\|^2\\ &\quad +2\mu\langle (u_n-x^*)-(y_n-y^*), Bu_n-Bx^*\rangle-\mu^2\|Bu_n-Bx^*\|^2\big]\\ &\le a_n\|v-x^*\|^2+\|x_n-x^*\|^2\\ &\quad -(1-a_n-b_n)\|(u_n-x^*)-(y_n-y^*)\|^2\\ &\quad +(1-a_n-b_n)2\mu\|(u_n-x^*)-(y_n-y^*)\|\|Bu_n-Bx^*\|,\end{aligned}$$

which implies that

$$\begin{aligned}&(1-a_n-b_n)\|(u_n-x^*)-(y_n-y^*)\|^2\\ &\le a_n\|v-x^*\|^2+(1-a_n-b_n)2\mu\|(u_n-x^*)-(y_n-y^*)\|\|Bu_n-Bx^*\|\\ &\quad +(\|x_n-x^*\|+\|x_{n+1}-x^*\|)\|x_{n+1}-x_n\|.\end{aligned}$$

This together with (C1), (3.11) and (3.16), we obtain

$$\|(u_n-x^*)-(y_n-y^*)\|\to 0 \quad \text{as} \quad n\to\infty. \tag{3.17}$$

From Lemma 2.6 and (2.2), it follows that

$$\begin{aligned}\|(y_n-t_n)+(x^*-y^*)\|^2 &= \|(y_n-\lambda Ay_n)-(y^*-\lambda Ay^*)\\ &\quad -\big[P_C(y_n-\lambda Ay_n)-P_C(y^*-\lambda Ay^*)\big]+\lambda(Ay_n-Ay^*)\|^2\end{aligned}$$

$$
\begin{aligned}
&\le \|(y_n - \lambda Ay_n) - (y^* - \lambda Ay^*) - \big[P_C(y_n - \lambda Ay_n) - P_C(y^* - \lambda Ay^*)\big]\|^2 \\
&\quad + 2\lambda\langle Ay_n - Ay^*, (y_n - t_n) + (x^* - y^*)\rangle \\
&\le \|(y_n - \lambda Ay_n) - (y^* - \lambda Ay^*)\|^2 - \|P_C(y_n - \lambda Ay_n) - P_C(y^* - \lambda Ay^*)\|^2 \\
&\quad + 2\lambda\|Ay_n - Ay^*\|\|(y_n - t_n) + (x^* - y^*)\| \\
&\le \|(y_n - \lambda Ay_n) - (y^* - \lambda Ay^*)\|^2 - \|S_nP_C(y_n - \lambda Ay_n) - S_nP_C(y^* - \lambda Ay^*)\|^2 \\
&\quad + 2\lambda\|Ay_n - Ay^*\|\|(y_n - t_n) + (x^* - y^*)\| \\
&\le \|(y_n - \lambda Ay_n) - (y^* - \lambda Ay^*) \\
&\quad - (S_nt_n - x^*)\|\big[\|(y_n - \lambda Ay_n) - (y^* - \lambda Ay^*)\| + \|S_nt_n - x^*\|\big] \\
&\quad + 2\lambda\|Ay_n - Ay^*\|\|(y_n - t_n) + (x^* - y^*)\| \\
&= \|u_n - S_nt_n + x^* - y^* - (u_n - y_n) \\
&\quad - \lambda(Ay_n - Ay^*)\|\big[\|(y_n - \lambda Ay_n) - (y^* - \lambda Ay^*)\| + \|S_nt_n - x^*\|\big] \\
&\quad + 2\lambda\|Ay_n - Ay^*\|\|(y_n - t_n) + (x^* - y^*)\|.
\end{aligned}
$$

This together with (3.15), (3.17) and (3.16), we obtain $\|(y_n - t_n) + (x^* - y^*)\| \to 0$ as $n \to \infty$. This together with (3.12), (3.14) and (3.17), we obtain that

$$
\begin{aligned}
\|S_nt_n - t_n\| \le \|S_nt_n - x_n\| + \|x_n - u_n\| + \|(u_n - y_n) - (x^* - y^*)\| \\
+ \|(y_n - t_n) + (x^* - y^*)\| \to 0 \quad \text{as} \quad n \to \infty. \qquad (3.18)
\end{aligned}
$$

Next, we show that

$$\limsup_{n\to\infty}\langle v - \overline{x}, x_n - \overline{x}\rangle \le 0,$$

where $\overline{x} = P_\Omega v$.

Indeed, since $\{t_n\}$ and $\{S_nt_n\}$ are two bounded sequences in C, we can choose a subsequence $\{t_{n_i}\}$ of $\{t_n\}$ such that $t_{n_i} \rightharpoonup z \in C$ and

$$\limsup_{n\to\infty}\langle v - \overline{x}, S_nt_n - \overline{x}\rangle = \lim_{i\to\infty}\langle v - \overline{x}, S_{n_i}t_{n_i} - \overline{x}\rangle.$$

Since $\lim_{n\to\infty}\|S_nt_n - t_n\| = 0$, we obtain that $S_{n_i}t_{n_i} \rightharpoonup z$ as $i \to \infty$.

Next, we show that $z \in \Omega$.

(a) We first show $z \in \bigcap_{i=1}^N F(T_i)$.

We can assume that $\alpha_1^{n,j} \to \alpha_1^j \in (0,1)$ and $\alpha_1^{n,N} \to \alpha_1^N \in (0,1]$ as $n \to \infty$ for all $j \in \{1,2,\dots,N-1\}$ and $\alpha_3^{n,j} \to \alpha_3^j \in [0,1)$ as $n \to \infty$ for $j = 1,2,\dots,N$. Let S be the S-mappings generated by $T_1, T_2, \dots, T_N$ and $\alpha_1, \alpha_2, \dots, \alpha_N$ where $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j)$, for $j = 1,2,\dots,N$. From Lemma 2.8, we have $\|S_nt_n - St_n\| \to 0$ as $n \to \infty$. Since

$$\|St_n - t_n\| \le \|St_n - S_nt_n\| + \|S_nt_n - t_n\|,$$

it follows by (3.18) that $\|St_n - t_n\| \to 0$ as $n \to \infty$.
Since $t_{n_i} \rightharpoonup z$ and $\|St_n - t_n\| \to 0$, we obtain by Lemma 2.5 and Lemma 2.7 that $z \in F(S) = \bigcap_{i=1}^{N} F(T_i)$.

(b) Now, we show that $z \in GSVI(C, A, B)$.
Since

$$\|t_n - x_n\| \leq \|S_n t_n - t_n\| + \|S_n t_n - x_n\|,$$

it follows from (3.18) and (3.12) that $\|t_n - x_n\| \to 0$ as $n \to \infty$. Furthermore, by Lemma 3.1, we have $G : C \to C$ is nonexpansive. Then, we have

$$\begin{aligned}\|t_n - G(t_n)\| &= \|P_C(y_n - \lambda A y_n) - G(t_n)\| \\ &= \|P_C\big[P(u_n - \mu B u_n) - \lambda A P(u_n - \mu B u_n)\big] - G(t_n)\| \\ &= \|G(u_n) - G(t_n)\| \leq \|u_n - t_n\| \\ &\leq \|u_n - x_n\| + \|x_n - t_n\|,\end{aligned}$$

which implies $\|t_n - G(t_n)\| \to 0$ as $n \to \infty$. Again by Lemma 2.5, we have $z \in GSVI(C, A, B)$.

(c) We show that $z \in MEP(F, \varphi)$. Since $t_{n_i} \rightharpoonup z$ and $\|x_n - t_n\| \to 0$, we obtain that $x_{n_i} \rightharpoonup z$. From $\|u_n - x_n\| \to 0$, we also obtain that $u_{n_i} \rightharpoonup z$. By using the same argument as that in the proof of [11, Theorem 3.1, pp. 1825], we can show that $z \in MEP(F, \varphi)$. Therefore there holds $z \in \Omega$.

On the other hand, it follows from (2.3) and $S_{n_i} t_{n_i} \rightharpoonup z$ as $i \to \infty$ that

$$\begin{aligned}\limsup_{n\to\infty}\langle v - \overline{x}, x_n - \overline{x}\rangle &= \limsup_{n\to\infty}\langle v - \overline{x}, S_n t_n - \overline{x}\rangle = \lim_{i\to\infty}\langle v - \overline{x}, S_{n_i} t_{n_i} - \overline{x}\rangle \\ &= \langle v - \overline{x}, z - \overline{x}\rangle \leq 0. \qquad (3.19)\end{aligned}$$

Hence, we have

$$\begin{aligned}\|x_{n+1} - \overline{x}\|^2 &= \langle a_n v + b_n x_n + (1 - a_n - b_n) S_n t_n - \overline{x}, x_{n+1} - \overline{x}\rangle \\ &= a_n\langle v - \overline{x}, x_{n+1} - \overline{x}\rangle + b_n\langle x_n - \overline{x}, x_{n+1} - \overline{x}\rangle \\ &\quad + (1 - a_n - b_n)\langle S_n t_n - \overline{x}, x_{n+1} - \overline{x}\rangle \\ &\leq a_n\langle v - \overline{x}, x_{n+1} - \overline{x}\rangle + \frac{1}{2} b_n(\|x_n - \overline{x}\|^2 + \|x_{n+1} - \overline{x}\|^2) \\ &\quad + \frac{1}{2}(1 - a_n - b_n)(\|t_n - \overline{x}\|^2 + \|x_{n+1} - \overline{x}\|^2) \\ &\leq a_n\langle v - \overline{x}, x_{n+1} - \overline{x}\rangle + \frac{1}{2} b_n(\|x_n - \overline{x}\|^2 + \|x_{n+1} - \overline{x}\|^2) \\ &\quad + \frac{1}{2}(1 - a_n - b_n)(\|x_n - \overline{x}\|^2 + \|x_{n+1} - \overline{x}\|^2) \\ &= a_n\langle v - \overline{x}, x_{n+1} - \overline{x}\rangle + \frac{1}{2}(1 - a_n)(\|x_n - \overline{x}\|^2 + \|x_{n+1} - \overline{x}\|^2),\end{aligned}$$

which implies that

$$\|x_{n+1}-\overline{x}\|^2 \leq (1-a_n)\|x_n-\overline{x}\|^2 + 2a_n\langle v-\overline{x}, x_{n+1}-\overline{x}\rangle.$$

It follows from Lemma 2.2 and (3.19) that $\{x_n\}$ converges strongly to $\overline{x}$. This completes the proof. □

If $\alpha_2^{n,j} = 0$ for all $j \in \{1, 2, \ldots, N\}$ and all $n \in \mathbb{N}$ in Theorem 3.2, then the mapping S_n reduces to the mapping W_n, so the following result is obtained directly from Theorem 3.2.

Corollary 3.3. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let F be a function from $C\times C$ to $\mathbb{R}$ satisfying (A1)-(A5) and $\varphi : C \to \mathbb{R}\bigcup\{+\infty\}$ be a proper lower semicontinuous and convex function. Let $A : C \to H$ be α-inverse-strongly monotone mapping and let $B : C \to H$ be a L-Lipschitzian and relaxed (c', d')-cocoercive mapping. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive self-mappings of C such that $\Omega = \bigcap_{i=1}^N F(T_i)\bigcap GSVI(C, A, B)\bigcap MEP(F, \varphi) \neq \emptyset$. For all $n \in \mathbb{N}$, let $\{\alpha_1^{n,j}\}_{j=1}^N \subset (0, b]$ with $0 < b < 1$. Let W_n be the W-mapping generated by $T_1, T_2, \ldots, T_N$ and $\alpha_1^{n,1}, \alpha_1^{n,2}, \ldots, \alpha_1^{n,N}$. Assume that either (B1) or (B2) holds and that v is an arbitrary point in C. Let $x_1 \in C$ and $\{x_n\}, \{u_n\}, \{y_n\}$ be the sequences generated by*

$$\begin{cases} F(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = P_C(u_n - \mu B u_n), \\ x_{n+1} = a_n v + b_n x_n + (1-a_n-b_n)W_n P_C(y_n - \lambda A y_n), \quad n \geq 1. \end{cases}$$

If $\lambda \in (0, 2\alpha)$, $0 < \mu < \frac{2(d'-c'L^2)}{L^2}$ and the sequences $\{r_n\}$, $\{a_n\}$, $\{b_n\}$, $\{\alpha_1^{n,j}\}_{j=1}^N$ are as in Theorem 3.2, then $\{x_n\}$ converges strongly to $\overline{x} = P_\Omega v$ and $(\overline{x}, \overline{y})$ is a solution of GSVI (1.2), where $\overline{y} = P_C(\overline{x} - \mu B\overline{x})$.

If $N = 1$, $T_1 = S$, $\varphi = 0$ and $\alpha_2^{n,1}$, $\alpha_3^{n,1} = 0$ $\forall n \in \mathbb{N}$ in Theorem 3.2, then we obtain the following result.

Corollary 3.4. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let F be a function from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A5). Let $A : C \to H$ be α-inverse-strongly monotone mapping and let $B : C \to H$ be a L-Lipschitzian and relaxed (c', d')-cocoercive mapping. Let S be a nonexpansive self-mappings of C such that $\Omega = F(S)\bigcap GSVI(C, A, B)\bigcap EP(F) \neq \emptyset$. Assume that v is an arbitrary* *point in C. Let $x_1 \in C$ and $\{x_n\}, \{u_n\}, \{y_n\}$ be the sequences generated by*

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = P_C(u_n - \mu B u_n), \\ x_{n+1} = a_n v + b_n x_n + (1 - a_n - b_n) S P_C(y_n - \lambda A y_n), \quad n \geq 1. \end{cases}$$

If $\lambda \in (0, 2\alpha)$, $0 < \mu < \frac{2(d^{'} - c^{'} L^2)}{L^2}$ and the sequences $\{r_n\}$, $\{a_n\}$, $\{b_n\}$ are as in Theorem 3.2, then $\{x_n\}$ converges strongly to $\overline{x} \in P_\Omega v$ and $(\overline{x}, \overline{y})$ is a solution of GSVI (1.2), where $\overline{y} = P_C(\overline{x} - \mu B \overline{x})$.

Acknowledgement. The author would like to thank Professor Dr. Suthep Suantai and the reviewer for careful reading, valuable comment and suggestions on this paper. The author also thank the Commission on Higher Education, the Thailand Research Fund, Thaksin university, the Centre of Excellence in Mathematics for their financial support.

References

[1] S. Atsushiba, W. Takahashi, Strong convergence theorems for a finite family of nonexpansive mappings and applications, in: B.N. Prasad Birth Centenary Commemoration Volume, Indian J. Math. 41 (3) (1999) 435–453.

[2] L. C. Ceng, C. Y. Wang and J. C. Yao, Strong convergence theorems by a relaxed extragradient method for a general system of variational inequalities, Math Meth Oper Res 67 (2008) 375–390.

[3] L. C. Ceng, M. M. Wong and A. Latif, Generalized extragradient iterative method for systems of variational inequalities, Journal of Inequalities and Applications 2012 (2012) Doi:10.1186/1029-242X-2012-88.

[4] L. C. Ceng, J. C. Yao, A hybrid iterative scheme for mixed equilibrium problems and fixed point problems, J. Comput. App. Math. 214 (2008) 186–201.

[5] Y. J. Cho, I. K. Argyros and N. Petrot, Approximation methods for common solutions of generalized equilibrium, system of nonlinear variational inequalities and fixed point problems, Computer and Mathematics with Applications 60 (2010) 2292–2301.

[6] K. Goebel, W. A. Kirk, Topics on metric fixed-point theory. Cambridge University Press, Cambridge (1990).

[7] A. Kangtunyakarn, S. Suantai, Hybrid iterative scheme for generalized equilibrium problems and fixed point problems of finite family of nonexpansive mappings, Nonlinear Analysis: Hybrid Systems 3 (2009) 296–309.

[8] K. R. Kazmi and S. H. Rizvi, A hybrid extragradient method for approximating the common solutions of a variational inequality, a system of variational inequalities, a mixed equilibrium problem and a fixed point problem, Applied Mathematics and Computation 218 (2012) 5439–5452.

[9] P. Kumam and P. Katchang, A system of mixed equilibrium problems, a general system of variational inequality problems for relaxed cocoercive, and fixed point problems for nonexpansive semigroup and strictly pseudocontractive mappings, Journal of Applied Mathematics 2012 (2012) Doi:10.1155/2012/414831.

[10] M. O. Osilike, D. I. Igbokwe, Weak and strong convergence theorems for fixed points of pseudocontractions and solutions of monotone type operator equations, Comput. Math. Appl. 40 (2000) 559–239.

[11] J. W. Peng, J. C. Yao, Strong convergence theorems of iterative schemes based on extragradient method for mixed equilibrium problems and fixed point problems, Mathematical and Computer Modelling 49 (2009) 1816–1828.

[12] H. Piri, A general iterative method for finding common solutions of system of equilibrium problems, system of variational inequalities and fixed point problems, Mathematical and Computer Modelling 55 (2012) 1622–1638.

[13] X. Qin, M. Shang and Y. Su, Strong convergence of a general iterative algorithm for equilibrium problems and variational inequality problems, Mathematical and Computer Modelling 48 (2008) 1033–1046.

[14] Y. Shehu, Iterative method for fixed point problem, variational inequality and generalized mixed equilibrium problems with applications, J. Glob. Optim 52 (2012) 57–77.

[15] T. Suzuki, Strong convergence of Krasnoselskii and Manns type sequences for one-parameter nonexpansive semigroups without Bochner integrals, J. Math. Anal. Appl. 305 (2005) 227–239.

[16] W. Takahashi and K. Shimoji, Convergence theorems for nonexpansive mappings and feasibility problems, Mathematical and Computer Modelling 32 (2000) 1463–1471.

[17] R. U. Verma, On a new system of nonlinear variational inequalities and associated iterative algorithms, Math Sci Res Hot-Line 3 (1999) 65–68.

[18] R. Wangkeeree and U. Kamraksa, An iterative approximation method for solving a general system of variational inequality problems and mixed equilibrium problems, Nonlinear Analysis: Hybrid systems 3 (2009) 615–630.

[19] R. Wangkeeree and P. Preechasilp, A new iterative scheme for solving the equilibrium problems, variational inequality problems, and fixed point problems in Hilbert spaces, Journal of Applied Mathematics 2012 (2012) Doi:10.1155/2012/154968.

[20] H. K. Xu, Viscosity approximation methods for nonexpansive mappings, J. Math. Anal. Appl. 298 (2004) 279–291.

[21] Y. Yao, Y. J. Cho and Chen, An iterative algorithm for solving fixed point problems, variational inequalities problems and mixed equilibrium problems, Nonlinear Analysis 71 (2009) 3363–3373.

[22] Y. Yao, Y. C. Liou and S. M. Kang, Approach to common elements of variational inequality problems and fixed point problems via a relaxed extragradient method, Computers and Mathematics with Applications 59 (2010) 3472–3480.

[23] Y. Yao, Y. C. Liou, M. M. Wong and J. C. Yao, Strong convergence of a hybrid method for monotone variational inequalities and fixed point problems, Fixed Point Theory and Applications 2011 (2011) Doi: 101186/1687-1812-2011-53.