Coprime factorization of singular linear systems. A Stein matritial equation approach

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Abstract

In this work immersed in the field of control theory on a Given a singular linear dynamic time invariant represented by $Ex^{+}(t) = Ax(t)Bu(t)$, $y(t) = Cx(t)$. We want to classify singular systems such that by means a feedback and an output injection, the transfer matrix of the system is a polynomial, for that we analyze conditions for obtention of a coprime factorization of transfer matrices of singular linear systems defined over commutative rings R with element unit. The problem presented is related to the existence of solutions of a Stein matritial equation $XE - NXA = Z$.

Key words: Singular systems, feedback, output injection, coprime factorization.

1 Introduction

Let R be a commutative ring with unity and ($Ex^{+}(t) = Ax(t) + Bu(t)$, $y(t) = Cx(t)$) be a singular system over R, that we represent by (E, A, B, C). Then, the transfer function of the system (E, A, B, C) is given by $H(s) = C(sE - A)^{-1}B$ and provides an inputoutput relationship of the system. The matrix $(sE-A)^{-1}$ in the transfer function is called the dynamical state matrix.

This systems appear in literature when, for example, one studies linear systems depending on a parameter or linear systems with delays.

We are interested in classify the singular systems (E, A, B, C) for which there exist feedbacks (proportional and/or derivative) F_E^B, F_A^B, and/or output injections (proportional and/or derivative) F_E^C,F_A^C, such that the state matrix $(s(E+F_E^CC+BF_E^B)-(A+F_A^CC+BF_A^B))^{-1}$ in the transfer function is polynomial. We will call systems with polynomial state matrix by feedback (proportional and/or derivative) and/or output injection (proportional and/or derivative) and we will write simply as pbfoi-systems, the systems verifying this property. In the case where the state matrix $(s(E+BF_E^B)-(A+BF_A^B))^{-1}$ in the transfer function is polynomial we will write pbf and we will write pboi in the case $(s(E+F_E^CC)-(A+F_A^CC))^{-1}$ is polynomial.

Notice that, if this property holds then the system is regularisable, remember that a system (E,A,B,C) is regularisable if and only if there exist feedbacks F_E^B, F_A^B, and output injections F_E^C,F_A^C, such that $\det(s(E+F_E^CC+BF_E^B)-(A+F_A^CC+BF_A^B))\neq 0$ for some $s\in R$. (Someone of the feedbacks and output injections, or all can be zero).

In order to use a simple reduced system preserving these properties, we consider the following equivalence relation deduced of to apply the standard transformations in state, input and output spaces, premultiply the first equation by an invertible matrix, making feedback (proportional and derivative) as well as output injection (proportional and derivative). More concretely. Two systems (E_i,A_i,B_i,C_i), $i=1,2$, are equivalent if and only if there exist matrices $P\in Gl(n;\mathbb{C})$, $Q\in Gl(p;\mathbb{C})$, $R\in Gl(m;\mathbb{C})$, $S\in Gl(q;\mathbb{C})$, $F_E^B,F_A^B\in M_{m\times n}(\mathbb{C})$, $F_E^C,F_A^C\in M_{p\times q}(\mathbb{C})$ such that

$$\begin{array}{ll} E_2 & = QE_1P+QB_1F_E^B+F_E^CC_1P, \\ A_2 & = QA_1P+QB_1F_A^B+F_A^CC_1P, \\ B_2 & = QB_1R, \\ C_2 & = SC_1P. \end{array} \tag{1}$$

Note that, considering this equivalence relation and restricting out to the regularisable systems for $R=\mathbb{C}$, it is possible to reduce the system to (E_c,A_c,B_c,C_c) where

$$E_c=\begin{pmatrix} I_1 &&&& \\ & I_2 &&& \\ && I_3 && \\ &&& I_4 & \\ &&&& N_1 \end{pmatrix},$$

$$A_c=\begin{pmatrix} N_2 &&&& \\ & N_3 &&& \\ && N_4 && \\ &&& J & \\ &&&& I_5 \end{pmatrix},$$

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$$B_c = \begin{pmatrix} B_1 & 0 & 0 \\ 0 & B_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C_c = \begin{pmatrix} C_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & C_2 & 0 & 0 \end{pmatrix}$$

and N_i denotes a nilpotent matrix in its reduced form: $N_i = \operatorname{diag}(N_{i_1}, \ldots, N_{i_t})$, $N_{i_j} = \begin{pmatrix} 0 & I_{n_{i_j}-1} \\ 0 & 0 \end{pmatrix} \in M_{n_{i_j}}(C)$.

The matrix J denotes the Jordan matrix $J = \operatorname{diag}(J_1(\lambda_1), \ldots, J_m(\lambda_m))$, with $J_i(\lambda_i) = \operatorname{diag}(J_{i_1}(\lambda_i, \ldots, J_{i_t}(\lambda_i))$ and $J_{i_j}(\lambda_i) = \lambda_i I + N$.

Notice that not all subsystems must necessarily appear in canonical reduced form.

Remark 1.1 *Canonical reduced form can be obtained easily using the complete set of invariants (see [6]).*

2 Coprime factorization

A quality of the systems pbfoi is that the state matrix associated to the transfer function of the system obtained after applying the corresponding feedback (proportional and/or derivative) and/or output injection (proportional and/or derivative) admits a coprime matrix function description.

Two polynomial matrices $N(s) \in M_{p\times m}(R[s])$ and $D(s) \in M_m(R[s])$ are called (Bézout) right coprime if $\begin{pmatrix} N(s) \\ D(s) \end{pmatrix}$ is left-invertible, that is to say, if there exist $X(s) \in M_{m\times p}(R[s])$, $Y(s) \in M_m(R[s])$ satisfying "Bézout identity"

$$X(s)N(s) + Y(s)D(s) = I_m.$$

The polynomial matrices $X(s)$ and $Y(s)$ are called left Bézout factors for the pair $(N(s), D(s))$.

Let $R(s)$ be a rational matrix admitting a factorization $R(s) = N(s)D^{-1}(s)$, we will call this factorization a rcf (right coprime factorization) of $R(s)$.

Theorem 2.1 *Let (E, A, B, C) be a pbfoi system. Then there exist a right coprime factorization of the state matrix associated to the transfer function of the system.*

Proof. Taking into account that (E, A, B, C) is a pbfoi system $(s(E + F_E^C C + BF_E^B) - (A + F_A^C C + BF_A^B))^{-1} = Q(s)$ is polynomial. First, the matrix pair $(N(s), D(s))$ with $N(s) = Q(s)$ and $D(s) = I - (s(BF_E^B +$

$F_E^C C) + (BF_A^B + F_A^C C))Q(s)$ is coprime: $X(s)N(s) + Y(s)D(s) = I$ with $X(s) = s(BF_E^B + F_E^C C) + (BF_A^B + F_A^C C)$ and $Y(s) = I$.

Second,

$$\begin{array}{l} D(s) = \\ I - X(s)Q(s) + (sE - A)Q(s) - (sE - A)Q(s) = \\ I - (X(s) + (sE - A))Q(s) + (sE - A)Q(s) = \\ (sE - A)Q(s). \end{array}$$

Consequently $\det D(s) = \gamma \det(sE - A)$ for all $s \in R$, $N(s)D^{-1}(s) = Q(s)((sE - A)Q(s))^{-1} = (sE - A)^{-1}$.

Restricting to the subclass of singular systems in the linear variety $(0, 0, 0, C) + [(E, A, B, 0)]$ that we will write as triple of matrices.

Theorem 2.2 *Let (E, A, B) by a pbf system. Then, there exist a right coprime factorization of the transfer function associated to the system.*

Remark 2.1 *For the proof of theorem observe that*

1) $B(I - X(s)Q(s)B)^{-1} = (I - BX(s)Q(s))^{-1}B$
2) *and the well known formula* $\det X \det(Y - ZX^{-1}T) = \det Y \det(X - TY^{-1}Z)$.

Proof. Taking into account that $(s(E + BF_E^B) - (A + BF_A^B))^{-1} = Q(s)$ is polynomial. Consider the matrix pair $(N(s), D(s))$ with $N(s) = Q(s)B$ and $D(s) = I - (sF_E^B - F_A^B)Q(s)B$ is coprime: $X(s)N(s) + Y(s)D(s) = I$ with $X(s) = sF_E^B - F_A^B$ and $Y(s) = I$.

$\det Q(s)^{-1} \det(I - X(s)Q(s)B) = \det I \det(Q(s)^{-1} - BX(s)) = \det(sE - A)$

So $\det D(s) = \dfrac{1}{\det Q(s)} \det(sE - A)$.

$$\begin{array}{l} N(s)D(s)^{-1} = Q(s)B(I - X(s)Q)s)B)^{-1} = \\ Q(s)(I - BX(s)Q(s))^{-1}B = \\ Q(s)(I - B(sF_E^B - F_A^B)Q(s))^{-1}B = \\ (Q(s)^1)^{-1}(I - B(sF_E^B - F_A^B)Q(s))^{-1}B = \\ ((I - B(sF_E^B - F_A^B)Qs)Q(s)^{-1})^{-1}B = \\ (Q(s)^{-1} - B(sF_E^B - F_A^B)^{-1}B = \\ (sE + BF_E^B - A - BF_A^B - sBF_E^B + BF_A^B)^{-1}B = \\ (sE - A)^{-1}B \end{array}$$

By duality, in the linear variety $(0, 0, B, 0) + [(E, A, 0, C)]$, we have the following result.

Corollary 2.1 *Let (E, A, C) be a pboi system. Then, there exist a left coprime factorization of the transfer function associated to the system.*

3 Stein matritial equation

In the case where the polynomial matrix $(s(E+F_E^C C+BF_E^B)-(A+F_A^C C+BF_A^B))^{-1}$ exists, it can be obtained solving a Stein matrix equation.

Proposition 3.1 *Let (E, A, B, C) be a pbfoi linear system, then there exist F_A^B,F_A^C, F_E^B,F_E^C, such that $A + BF_A^B + F_A^C C$ is invertible and $(E + BF_E^B + F_E^C C)(A + BF_A^B + F_A^C C)^{-1}$ is nilpotent.*

Proof. If (E, A, B, C) is a pbfoi linear system, then there exist F_A^B,F_A^C, F_E^B,F_E^C, such that $P(s) = s(E + F_E^C C + BF_E^B) - (A + F_A^C C + BF_A^B)$ is invertible, so there exist $Q(s) = s^\ell Q_\ell + \ldots + sQ_1 + Q_0$ such that $P(s)Q(s) = I_n$.

Consequently:

$$\begin{array}{l}
-(A + BF_A^B + F_A^C C)Q_0 = I_n,\\
(E + BF_E^B + F_E^C C)Q_0 - (A + BF_A^B + F_A^C C)Q_1 = 0,\\
(E + BF_E^B + F_E^C C)Q_1 - (A + BF_A^B + F_A^C C)Q_2 = 0,\\
\vdots\\
(E + BF_E^B + F_E^C C)Q_{\ell-1} - (A + BF_A^B + F_A^C C)Q_\ell = 0,\\
(E + BF_E^B + F_E^C C)Q_\ell = 0.
\end{array}$$

First equality says that $-(A + BF_A^B + F_A^C C)^{-1} = Q_0$.
Hence, since $A + BF_A^B + F_A^C C$ is invertible, we can obtain Q_i, $\ell \geq i \geq 1$.

$$\begin{array}{l}
Q_i = -((A + BF_A^B + F_A^C C)^{-1}(E + BF_E^B + F_E^C C))^i\\
\quad (A + BF_A^B + F_A^C C)^{-1} =\\
\quad -(A + BF_A^B + F_A^C C)^{-1}((E + BF_E^B + F_E^C C)\\
\quad (A + BF_A^B + F_A^C C)^{-1})^i.
\end{array}$$

So, last equation implies

$$\begin{array}{l}
0 = (E + BF_E^B + F_E^C C)Q_\ell =\\
((E + BF_E^B + F_E^C C)(A + BF_A^B + F_A^C C)^{-1})^{\ell+1.}
\end{array}$$

Consequently

$$(E + BF_E^B + F_E^C C)(A + BF_A^B + F_A^C C)^{-1} \tag{2}$$

is a nilpotent matrix and taking into account that $Q_\ell \neq 0$, the nilpotency order is $\ell + 1$.

Corollary 3.1 *If a system (E, A, B, C) is pbfoi then it is repairable.*

Remember that a system (E, A, B, C) is repairable if and only if there exist F_A^B and F_A^C such that $A + BF_A^B + F_A^C C$ is invertible, (for more information about repairable systems see [7]).

Remark 3.1 *Converse is not true as we can see in the following example: let (E, A, B, C) be a system with $E = \begin{pmatrix} 0&0&0\\0&0&0\\0&0&1 \end{pmatrix}$, $A = I_3$, $B = \begin{pmatrix} 1\\0\\0 \end{pmatrix}$, $C = \begin{pmatrix} 0&1&0 \end{pmatrix}$, considering all possible feedbacks F_E^B, F_A^B, and output injections F_E^C, F_A^C matrix $s(E + F_E^C C + BF_E^B) - (A + F_A^C C + BF_A^B)$ is*

$$\begin{pmatrix} -1 - c_1 + sa_1 & -c_2 - d_1 + s(a_2 + b_1) & -c_3 + sa_3 \\ 0 & -1 - d_2 + sb_2 & 0 \\ 0 & -d_3 + sb_3 & -1 + s \end{pmatrix},$$

which inverse is not polynomial because of $\det(s(E + F_E^C C + BF_E^B) - (A + F_A^C C + BF_A^B)) \notin \mathbb{C}_0$.

Proposition 3.2 *Let (E, A, B, C) be a pbfoi system. Then the equation $XE - NXA = Z$ with N a nilpotent has a solution (X, Z) with X invertible.*

Proof. Matrix (2) is equivalent to a nilpotent matrix N in its reduced Jordan form

$$(E + BF_E^B + F_E^C C)(A + BF_A^B + F_A^C C)^{-1} = X^{-1} N X,$$

equivalently

$$X(E + BF_E^B + F_E^C C) = NX(A + BF_A^B + F_A^C C)$$

and

$$XE - NXA = -X(F_E^C C + BF_E^B) + NX(F_A^C C + BF_A^B) = Z. \qquad (3)$$

The existence of F_E^B, F_E^C, F_A^B, F_A^C, verifying proposition 3.1 implies that the equation $XE - NXA = Z$ has a solution with X invertible (matrix of basis change) and $Z = -X(F_E^C C + BF_E^B) + NX(F_A^C C + BF_A^B)$.

Remark 3.2 *Taking into account corollary 3.1, if the system (E, A, B, C) is pbfoi, then it is repairable and there exist F_A^B and F_A^C be such that $A + BF_A^B + F_A^C C$ is invertible. We consider the solution (X, Z) of the equation $XE - NXA = Z$ and the matrix $M = X^{-1}Z - X^{-1}NX(F_A^C C + BF_A^B)$. It is easy to observe that $M = F_E^C C + BF_E^B$.*

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and we can deduce the following corollary which in a some sense can be considered reciprocal of Proposition 3.2.

Corollary 3.2 *Suppose that the system* (E, A, B, C) *is repairable and let* F_A^B *and* F_A^C *be such that* $A + BF_A^B + F_A^C C$ *is invertible.*

If the equation $XE - NXA = Z$*, with* N *a nilpotent matrix, has a solution* (X, Z) *with* X *invertible, and the equation* $-(F_E^C C + BF_E^B) = M$ *with* $M = X^{-1}Z - X^{-1}NX(F_A^C C + BF_A^B)$ *has a solution* (F_E^C, F_E^B)*. Then, the system is pbfoi, and*

$$Q_i = -(A + BF_A^B + F_A^C C)^{-1} X N^i X^{-1}.$$

then, we can obtain $Q(s)$ solving a linear system.

Example 3.1 *Let* (E, A, B) *with* $E = I$, $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Solving

$$\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix}$$

$$\begin{pmatrix} x_1 & x_2 - x_3 \\ x_3 & x_4 \end{pmatrix} = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix}$$

Taking a particular (invertible) solution for X*, for example* $X = Z = I$ *and solving* $M = -BF_E^B = I - NBF_A^B$ *we have that a possible solution is*

$F_E^B = \begin{pmatrix} 0 & 1 \end{pmatrix}$, $F_A^B = \begin{pmatrix} 1 & 0 \end{pmatrix}$

and $Q(s) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} s.$

Notice that we can provide various solutions by choosing different solutions for X*.*

Remark 3.3 *We can solve the equation* $XE - NXA = Z$ *using Kronecker product, vec operator and linearizing the system in the following manner*

$$(E^t \otimes I - A^t \otimes N) vec\,(X) = vec\,(Z). \tag{4}$$

It is easy to observe that if the matrix E *is invertible it is the same for the matrix of the equation (4).*

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4 Characterization of pbfoi-systems

In this section we will try to characterize pbfoi-systems.

Proposition 4.1 *Let (E, A, B, C) and (E_1, A_1, B_1, C_1) be equivalent systems. There exist F_E^B, F_A^B, F_E^C,F_A^C, such that $(s(E + F_E^C C + BF_E^B) - (A + F_A^C C + BF_A^B))^{-1}$ is polynomial if and only if and there exist $F_{E_1}^{B_1}$, $F_{A_1}^{B_1}$, $F_{E_1}^{C_1}$, $F_{A_1}^{C_1}$, such that $(s(E_1 + F_{E_1}^{C_1} C_1 + B_1 F_{E_1}^{B_1}) - (A_1 + F_{A_1}^{C_1} C_1 + B_1 F_{A_1}^{B_1}))^{-1}$ is polynomial.*

Proof. Equivalency of systems implies

$$\begin{array}{l} E_1 = QEP + \bar{F}_E^C CP + QB\bar{F}_E^B, \\ A_1 = QAP + \bar{F}_A^C CP + QB\bar{F}_A^B, \\ B_1 = QBR, \\ C_1 = SCP. \end{array}$$

So,

$$\begin{array}{l} (s(E_1 + F_{E_1}^{C_1} C_1 + B_1 F_{E_1}^{B_1}) - (A_1 + F_{A_1}^{C_1} C_1 + B_1 F_{A_1}^{B_1}))^{-1} = \\ (s(QEP + \bar{F}_E^C CP + QB\bar{F}_E^B + F_{E_1}^{C_1} SCP + QBRF_{E_1}^{B_1}) - \\ (QAP + \bar{F}_A^C CP + QB\bar{F}_A^B + F_{A_1}^{C_1} SCP + QBRF_{A_1}^{B_1})^{-1} = \\ (sQ(E + Q^{-1}\bar{F}_E^C C + B\bar{F}_E^B P^{-1} + Q^{-1}F_{E_1}^{C_1} SC + BRF_{E_1}^{B_1} P^{-1})P - \\ Q(A + Q^{-1}\bar{F}_A^C C + B\bar{F}_A^B P^{-1} + Q^{-1}F_{A_1}^{C_1} SC + BRF_{A_1}^{B_1} P^{-1})P)^{-1} = \\ P^{-1}(s(E + Q^{-1}\bar{F}_E^C C + B\bar{F}_E^B P^{-1} + Q^{-1}F_{E_1}^{C_1} SC + BRF_{E_1}^{B_1} P^{-1}) - \\ (A + Q^{-1}\bar{F}_A^C C + B\bar{F}_A^B P^{-1} + Q^{-1}F_{A_1}^{C_1} SC + BRF_{A_1}^{B_1} P^{-1})^{-1}Q^{-1} = \\ P^{-1}(s(E + (Q^{-1}\bar{F}_E^C + Q^{-1}F_{E_1}^{C_1} S)C + B(\bar{F}_E^B P^{-1} + RF_{E_1}^{B_1} P^{-1})) - \\ (A + (Q^{-1}\bar{F}_A^C + Q^{-1}F_{A_1}^{C_1} S)C + B(\bar{F}_A^B P^{-1} + RF_{A_1}^{B_1} P^{-1})))^{-1}Q^{-1} \end{array}$$

and $F_E^C = Q^{-1}\bar{F}_E^C + Q^{-1}F_{E_1}^{C_1} S$, $F_E^C = \bar{F}_E^B P^{-1} + RF_{E_1}^{B_1} P^{-1}$, $F_A^C = Q^{-1}\bar{F}_A^C + Q^{-1}F_{A_1}^{C_1} S$, $F_A^B = \bar{F}_A^B P^{-1} + RF_{A_1}^{B_1} P^{-1}$

4.1 Case $R = \mathbb{C}$

Firstly, we analyze the case where the ring R is the field of complex numbers, because in this case, there are a canonical reduced form which facilitates the study.

Proposition 4.1 permit us to characterize the pbfoi-systems.

Lemma 4.1 *Let (E, A, B, C) be a system equivalent to (E_r, A_r, B_r, C_r), with $E_r = \begin{pmatrix} I_2 & & \\ & I_3 & \\ & & N_1 \end{pmatrix}$, $A_r = \begin{pmatrix} N_3 & & \\ & N_4 & \\ & & I_5 \end{pmatrix}$, $B = \begin{pmatrix} B_2 \\ 0 \\ 0 \end{pmatrix}$ and $C_r = \begin{pmatrix} 0 & C_2 & 0 \end{pmatrix}$. Then, the system is pbfoi.*

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Proof. It is easy to prove that the system is equivalent (see [7]) to $(\bar{E}, \bar{A}, \bar{B}, \bar{C})$ with $\bar{E} = \begin{pmatrix} N_3 & & \\ & N_4 & \\ & & N_1 \end{pmatrix}$, $\bar{A} = \begin{pmatrix} I_2 & & \\ & I_3 & \\ & & I_5 \end{pmatrix}$ $\bar{B} = B_r$, and $\bar{C} = C_r$. Then, taking $F_{\bar{E}}^{\bar{B}} = F_{\bar{A}}^{\bar{B}} = 0$ and $F_{\bar{E}}^{\bar{C}} = F_{\bar{A}}^{\bar{C}} = 0$ we have that $(s(\bar{E}+F_{\bar{E}}^{\bar{C}}\bar{C}+\bar{B}F_{\bar{E}}^{\bar{B}})-(\bar{A}+F_{\bar{A}}^{\bar{C}}\bar{C}+\bar{B}F_{\bar{A}}^{\bar{B}}))$ is invertible.

Now, it suffices to apply proposition 4.1

Lemma 4.2 *Let (E, A, B, C) be a system equivalent to (E_r, A_r, B_r, C_r) with $E_r = \begin{pmatrix} I_2 & & & \\ & I_3 & & \\ & & I_4 & \\ & & & N_1 \end{pmatrix}$, $A_r = \begin{pmatrix} N_3 & & & \\ & N_4 & & \\ & & J & \\ & & & I_5 \end{pmatrix}$, $B = \begin{pmatrix} B_2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ and $C_r = \begin{pmatrix} 0 & C_2 & 0 & 0 \end{pmatrix}$. Then, the system can be not pbfoi.*

Proof. It is easy to prove that the system is equivalent (see [7]) to $(\bar{E}, \bar{A}, \bar{B}, \bar{C})$ with $\bar{E} = \begin{pmatrix} N_3 & & & \\ & N_4 & & \\ & & I_4 & \\ & & & N_1 \end{pmatrix}$, $\bar{A} = \begin{pmatrix} I_2 & & & \\ & I_3 & & \\ & & J & \\ & & & I_5 \end{pmatrix}$ $\bar{B} = B_r$ and $\bar{C} = C_r$.

Then,

$$\begin{array}{l} \det(s(\bar{E}+F_{\bar{E}}^{\bar{C}}\bar{C}+\bar{B}F_{\bar{E}}^{\bar{B}})-(\bar{A}+F_{\bar{A}}^{\bar{C}}\bar{C}+\bar{B}F_{\bar{A}}^{\bar{B}})) = \\ \det\det(sI_4 - J) \notin \mathbb{C}_0 \end{array}.$$

Now, it suffices to apply the proposition 4.1 and the proof is concluded.

Taking into account corollary 3.1, from now on we consider repairable systems.

Theorem 4.1 *Let (E, A, B, C) be a repairable system verifying one of the following conditions*

1. *the system has not finite zeros.*
2. *the number t of Jordan blocks is less or equal than $r = \operatorname{rank} B_1 = \operatorname{rank} C_1$.*

Then, the systems is pbfoi.

Proof. If the system (E, A, B, C) is pbfoi it is repairable. So, the system is equivalent (see [7]) to (E_1, A_1, B_1, C_1) with

$$E_1 = \begin{pmatrix} \bar{E} & & & \\ & N_1 & & \\ & & N_2 & \\ & & & \bar{J} \end{pmatrix}, A_1 = \begin{pmatrix} \bar{A} & & & \\ & I_1 & & \\ & & I_2 & \\ & & & I \end{pmatrix},$$

$$B_r = \begin{pmatrix} B_1 & 0 & 0 & 0 \\ 0 & B_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, C_1 = \begin{pmatrix} C_1 & 0 & 0 & 0 \\ 0 & 0 & C_2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

with $\bar{E} = \begin{pmatrix} 0 & \\ & I \end{pmatrix}$, $\bar{J} = \begin{pmatrix} J & \\ & N_3 \end{pmatrix}$, $\bar{A} = \begin{pmatrix} 0 & \\ & N \end{pmatrix}$ $B_1 = \begin{pmatrix} I \\ 0 \end{pmatrix}$, $C_1 = \begin{pmatrix} I & 0 \end{pmatrix}$ and $J = \operatorname{diag}(J_1, \ldots, J_t)$, J_i non derogatory with simple non-zero eigenvalue (different J_i may be the same eigenvalue). After lemmas it suffices to consider systems in the form $\left(\begin{pmatrix} 0 & \\ & J \end{pmatrix}, \begin{pmatrix} I & \\ & I \end{pmatrix}, \begin{pmatrix} I \\ 0 \end{pmatrix}, \begin{pmatrix} I & 0 \end{pmatrix} \right)$ which are equivalent to $\left(\begin{pmatrix} 0 & \\ & I \end{pmatrix}, \begin{pmatrix} I & \\ & J^{-1} \end{pmatrix}, \begin{pmatrix} I \\ 0 \end{pmatrix}, \begin{pmatrix} I & 0 \end{pmatrix} \right)$.

Suppose now $t = 1$, that is to say $J^{-1} = \begin{pmatrix} a & 1 & & & \\ & a & & & \\ & & \ddots & & \\ & & & a & 1 \\ & & & & a \end{pmatrix}$, and taking

$$F_A^B = \begin{pmatrix} -1 & 1 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ 0 & -1 & 1 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ \vdots & & & \ddots & \ddots & & & & \vdots \\ 0 & 0 & 0 & \ldots & -1 & 1 & 0 & \ldots & 0 \end{pmatrix}, F_A^C = \begin{pmatrix} 0 & 0 & \ldots & 0 \\ \vdots & & & \vdots \\ 1 & 0 & \ldots & 0 \end{pmatrix}, \text{ and}$$

$F_E^C = 0$, $F_E^B = 0$, we have $\det(s(E + BF_E^B + F_E^C) + A + BF_A^B + F_A^C) =$

$$\det \begin{pmatrix} 0 & 1 & 0 & 0 & & & & & \\ 0 & 0 & 1 & 0 & & & & & \\ 0 & & \ddots & \ddots & & & & & \\ 0 & 0 & & & 1 & 0 & & & \\ 0 & 0 & & & a+s & 1 & & & \\ 0 & 0 & & & & a+s & 1 & & \\ \vdots & & & & & \ddots & & & \\ 0 & 0 & 0 & & & & \ldots & a+s & 1 \\ 1 & 0 & 0 & & & & \ldots & & a+s \end{pmatrix} = 1.$$

For $1 < t \leq r = \operatorname{rank} B_1 = \operatorname{rank} C_1$, the system (E, A, B, C) with $E = \begin{pmatrix} 0 & & & \\ & J_1 & & \\ & & \ddots & \\ & & & J_t \end{pmatrix}$, $A = \begin{pmatrix} 0 & & & \\ & I_1 & & \\ & & \ddots & \\ & & & I_t \end{pmatrix}$ is equivalent to (E_1, A_1, B_1, C_1) with

$$E_1 = \begin{pmatrix} 0 & & & & & \\ & I & & & & \\ & & \ddots & & & \\ & & & 0_1 & & \\ & & & & I & \\ & & & & & 0 \end{pmatrix}, A_1 = \begin{pmatrix} 0 & & & & & \\ & J_1^{-1} & & & & \\ & & \ddots & & & \\ & & & 0 & & \\ & & & & J_t^{-1} & \\ & & & & & I \end{pmatrix}, B_1 =$$

$$\begin{pmatrix} 1 & & \\ 0 & & \\ \vdots & & \\ 0 & & \\ & \vdots & \\ & 1 & \\ & 0 & \\ & \vdots & \\ & 0 & \\ & & 0 \end{pmatrix}, C_1 = \begin{pmatrix} 1 & 0 & \dots & 0 & & & & & \\ & & & & \dots & & & & \\ & & & & & 1 & 0 & \dots & 0 \\ & & & & & & & & & 0 \end{pmatrix}.$$ Then, it suffices to apply the case $t = 1$

For $t > r$ the result is not true, as we can see in the following example.

Example 4.1 *Let*

$$\left(\begin{pmatrix} 0 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} 0 & & \\ & 0 & \\ & & 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \right)$$

be a repairable system,

$$\det \begin{pmatrix} s(a_1 + b_1) - (c_1 + d_1 & sa_2 - c_2) & sa_3 - c_3 \\ sb_2 - d_2 & s & 0 \\ sb_3 - d_3 & 0 & s \end{pmatrix} \notin \mathbb{C}_0.$$

So, the system is not pbfoi.

4.2 Case R a principal ideal domain

On one hand, by proposition 4.1, it is clear that if we have an equivalent system to a system in the previous form, then we can construct a coprime factorization of the transfer matrix of the system. On the other hand, in principal ideal domains, it is no possible to reduce a system to a form like $\mathbb{C}$. So, in order to realize a first study over principal ideal domains, we consider systems $x^+(t) = Ax(t) + Bu(t)$, it is, we consider systems in the linear variety $(0, 0, 0, C) + [(I, A, B, 0)]$. We will write that systems as a pair of matrices (A, B).

In our particular case of the pbfoi systems (A, B) we have that $(I_n + BF_E)(A + BF_A)^{-1}$ is a $\ell + 1$-order nilpotent matrix and in this case the matrix $(A + BF_A)^{-1}(I_n + BF_E)$ is also a $\ell + 1$-order nilpotent matrix. So, we can consider the matrix equation in the form $AXN - X = BY$ with N a r-order nilpotent matrix.

Below we introduce some lemmas useful for the development of the section.

Lemma 4.3 *Let $M \in M_{m\times n}(R)$ be an arbitrary matrix. Then the matrix $X = \sum_{i=0}^{r-1} A^i BMN^i$ is the unique solution of the equation $X - AXN = BM$. $N^t \otimes A$ is a r-order nilpotent matrix de orden r.*

Proof.

$$\sum_{i=0}^{r-1} A^i BMN^i - A(\sum_{i=0}^{r-1} A^i BMN^i)N = BM$$

The uniqueness is due to the matrix $I - N^t \otimes A$ is invertible.

It is obvious to prove the following lemmas.

Lemma 4.4 *Let $X - AXN = BM$ be the Stein equation, and $N_1 = PNP^{-1}$ be a nilpotent matrix equivalent under similarity to N. Then $X_1 = XP^{-1}$ is the solution of Stein equation $AX_1N_1 + X_1 = BM_1$ with $M_1 = MP^{-1}$ if and only if X is solution of the Stein equation $X - AXN = BM$.*

Lemma 4.5 *Let $(A_1, B_1) = (P^{-1}AP + P^{-1}BF_A, P^{-1}BQ)$ be a system with $P \in Gl(n; R)$, $F_A \in M_{m\times n}(R)$ and $Q \in Gl(m; R)$.Then X is the solution of the Stein equation $AXN + X = BM$ if and only if $X_1 = PX$ is solution of the Stein equation $A_1XN + X = B_1M_1$ with $M_1 = QM - F_AXN$.*

Lemma 4.6 *Let (A, B), (A_1, B_1) be two feedback equivalent systems over R. Then, A is invertible modulo B if and only if A_1 is invertible modulo B_1. Furthermore, if $(A_1, B_1) = (P^{-1}AP + P^{-1}BF, P^{-1}BQ)$ and there exists K_1 such that $A_1 + B_1K_1$ is invertible, then the matrix $A + BK$ with $K = (F + QK_1)P^{-1}$ is invertible.*

Lemma 4.7 *Let $A \in M_n(R)$, $B \in M_{n\times m}(R)$ be two matrices and $N \in M_n(R)$ a r-order nilpotent matrix ($r \leq n + 1$). If the Stein equation $X - AXN = BM$ has a solution X invertible for a given matrix M, then the system (A, B) is reachable.*

Proof. By hypothesis we have $X = BM + AXN$ invertible and for lemma 4.3 $X = \sum_{i=0}^{r-1} A^i BMN^i$. So, $X = BM + ABMN + A^2BMN^2 + A^3BMN^3 + \ldots + A^{r-1}BMN^{r-1} + A^rBMN^r$. Then, and taking into account that $r - 1 \leq n$, is is clear that the reachability matrix of the system (A, B) verifies $\mathcal{U}_n(B \mid AB \mid A^2B \mid \ldots \mid A^{n-1}B) = R$.

Remark 4.1 *The existence of feedbacks F_A and F_E that make invertible the transfer matrix of the close-loop system (A, B), is related to a invertible solution of the Stein equation $X - AXN = BM$ and the invertibility of A modulo B. It is known that over polo assignable rings, all reachable system (A, B) verifies that A is feedback invertible modulo B. The fields, local rings, principal ideal domains, Dedekind rings and rings of dimension zero or one are polo assignable rings (ver [4]).*

Suppose now the matrix $M = Y$ is unknown, we have the following result.

Proposition 4.2 *Let (A, B) be a system over a principal ideal domain. Then are equivalent conditions:*

1. *There exist F_E and F_A such that $P(s) = (sI_n - (A + sBF_E + BF_A))$ is an unimodular matrix.*
2. *The system is repairable, it is, there exist F_A such that $A + BF_A$ is invertible. The equation $X - NXA = BY$, with N nilpotent, has a solution (X, Y) with X invertible.*

Proof. First implication is direct by corollary 3.1 and proposition 3.2. Reciprocally, we consider $F_E = (F_AXN - Y)X^{-1} \in M_{m\times n}(R)$, then $(I_n + BF_E)(A + BF_A)^{-1}$ is nilpotent of order r: $((I_n + BF_E)(A + BF_A)^{-1})^r = TN^rT^{-1} = 0$, where $T = ((A + BF_A))X$. Furthermore, since $((I_n + BF_E)(A + BF_A))^{r-1} \neq 0$, we define

$$Q_i = ((A + BF_A)^{-1}(I_n + BF_E))^i(A + BF_A)^{-1},$$

for all $i = 0, 1, \ldots, r - 1$. So, we have $(I_n + BF_E)Q_{r-1} = 0$ and $Q_{r-1} \neq 0$. Finally, we consider polynomial matrix $Q(s) = \sum_{i=0}^{r-1} Q_i s^i$ verifying $P(s)Q(s) = I_n$. Note that $r = \ell + 1$.

Corollary 4.1 *Let (A, B) be a repairable system. If equation $X - AXN = BY$, with N nilpotent, has a solution (X, Y) with X invertible, then there exist a coprime factorization of the transfer matrix associated to the system.*

Proof. By theorem 2.1 and proposition 4.2, $(N(s) = \sum_{i=0}^{l} N_i s^i, D(s) = \sum_{i=0}^{l} N_i s^i)$ with $N_0 = XC$, $N_i = XN^iC$ for all $i = 1, \ldots, \ell$, $D_0 = BF_A(A + BF_A)^{-1} - I_m$, $D_1 = BYC$ and $D_{i+1} = BYN^iC$ for all $i = 1, \ldots, \ell$, where $C = X^{-1}(A + BF_A)^{-1}$, is a coprime factorization of the transfer matrix associated to the system (A, B).

Remark 4.2 *We can write a procedure with Input (A, B) n-dimensional m-input reachable system, and Output $(N(s), D(s))$ coprime matrix fraction description of the transfer matrix of the system. In particular, $H(s) = (sI_n - A + sBF_E + BF_A)^{-1}B$ is a polynomial transfer matrix.*

`Step 1.-` *Give canonical form*
$(A_1, B_1) = (P^{-1}AP + P^{-1}BF, P^{-1}BQ)$.

`Step 2.-` *Find F' such that $A_1 + B_1F'$ is invertible.*

`Step 3.-` *Solve equation $X_1 - A_1X_1N = B_1Y_1$.*

`Step 4.-` *Calculate*

$X = PX_1$ *and* $Y = QY_1 - FX_1N$.

`Step 5.-` *Calculate*

$F_A = (F + QF')P^{-1}$ *and* $F_E = (F_AXN-)X^{-1}$.

`Step 6.-` *Return polynomial coeff. of $N(s)$ and $D(s)$*

$$\begin{aligned} &N_0 = XC, \quad N_i = (-1)^i XN^iC, \\ &C = X^{-1}(-A + BF_A)^{-1} \\ &D_0 = BF_A(-A + BF_A)^{-1} - I_m, \quad D_1 = BYC, \\ &D_{i+1} = BYN^iC \end{aligned}$$

4.2.1 Single input reachable system

Theorem 4.2 *Let (A, B) be a single input reachable system. If N is nilpotent of order n, then there exist Y such that $AXN + X = BY$ equation has a solution (X, Y) with X invertible.*

Proof. First, by proposition 4.1, we can consider an equivalent canonical system.

$$(A_R, B_R) = \left(\begin{pmatrix} \underline{0}^t & 0 \\ \mathrm{I}_{n-1} & \underline{0} \end{pmatrix}, \begin{pmatrix} 1 \\ \underline{0} \end{pmatrix} \right)$$

Second, if N has nilpotent order $r < n$ then X is no invertible: $X = (B \ldots (-1)^{r-1}A^{r-1}B(-1)^rA^rB \ldots (-1)^{n-1}A^{n-1}B)(Y \ldots YN^{r-1}0 \ldots 0)^t =$

$(B \, \dots \, (-1)^{r-1}A^{r-1}B) \, (Y \, \dots YN^{r-1})^t$, so

$$X = \begin{pmatrix} 1 & \dots & 0 \\ & \ddots & \\ 0 & \dots & (-1)^{r-1} \\ \underline{0} & \dots & \underline{0} \end{pmatrix} \begin{pmatrix} Y \\ \vdots \\ YN^{r-1} \end{pmatrix}$$

is no invertible. Hence, we suppose N of order n and reduced triangular form (see [11]), $N = (a_{ij})$ with $a_{ij} = 0 \; \forall j \leq i$. In this case

$$X = X_1 X_2$$

with

$$X_1 = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & -1 & & \\ & & \ddots & \\ & & & (-1)^{n-1} \end{pmatrix}$$

$$X_2 = \begin{pmatrix} y_1 & y_2 & y_3 & \dots & y_n \\ 0 & a_{12}y_1 & a_{13}y_1 + a_{23}y_2 & \dots & \sum_{i=1}^{n-1} a_{in}y_i \\ & \ddots & \ddots & & \vdots \\ 0 & 0 & 0 & \dots & \prod_{i=1}^{n-1} a_{ii+1}y_1 \end{pmatrix}.$$

Since N is of order n, $a_{ii+1} \neq 0$ for all $i = 1, \dots n-1$. so, we can consider Y such that $y_1 \neq 0$.

Corollary 4.2 *Let (A, B) be a single input reachable system. Then (A, B) is a pfboi-system.*

Proof. We suppose (A, B) reduced canonical system. If we consider $F_A = \begin{pmatrix} 0 & \dots & 0 & 1 \end{pmatrix}$ and $F_E = (F_A X N - Y)X^{-1}$, then $A + BF_A$ and $P(s) = (sI_n - A + sBF_E + BF_A)$ are invertible matrices.

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