On Clifford Space Relativity, Black Hole Entropy, Rainbow Metrics, Generalized Dispersion and Uncertainty Relations

Carlos Castro*

April , 2014

Center for Theoretical Studies of Physical Systems, Clark Atlanta University, Atlanta; perelmanc@hotmail.com

Abstract

An analysis of some of the applications of Clifford Space Relativity to the physics behind the modified black hole entropy-area relations, rainbow metrics, generalized dispersion and minimal length stringy uncertainty relations is presented.

Keywords : Clifford algebras; Extended Relativity in Clifford Spaces; String Theory; Doubly Special Relativity; Rainbow metrics; Black Hole Entropy; Noncommutative Geometry; Quantum Clifford-Hopf algebras.

1 Introduction : Novel Consequences of Clifford Space Relativity

In the past years, the Extended Relativity Theory in C-spaces (Clifford spaces) and Clifford-Phase spaces were developed [1], [2]. The Extended Relativity theory in Clifford-spaces (C-spaces) is a natural extension of the ordinary Relativity theory whose generalized coordinates are Clifford polyvector-valued quantities which incorporate the lines, areas, volumes, and hyper-volumes degrees of freedom associated with the collective dynamics of particles, strings, membranes, p-branes (closed p-branes) moving in a D-dimensional target spacetime background. C-space Relativity permits to study the dynamics of all (closed) p-branes, for different values of p, on a unified footing. Our theory has 2 fundamental parameters : the speed of a light c and a length scale which can be set equal

*Dedicated to the memory of Adam Bowers

to the Planck length. The role of “photons” in C-space is played by *tensionless* branes. An extensive review of the Extended Relativity Theory in Clifford spaces can be found in [1]. The polyvector valued coordinates $x^\mu, x^{\mu_1\mu_2}, x^{\mu_1\mu_2\mu_3}, ...$ are now linked to the basis vectors generators γ^μ, bi-vectors generators $\gamma_\mu \wedge \gamma_\nu$, tri-vectors generators, $\gamma_{\mu_1} \wedge \gamma_{\mu_2} \wedge \gamma_{\mu_3}$, ... of the Clifford algebra, $\{ \gamma_a, \gamma_b \} = 2 g_{ab} \mathbf{1}$, including the Clifford algebra unit element (associated to a scalar coordinate). These polyvector valued coordinates can be interpreted as the quenched-degrees of freedom of an ensemble of p-loops associated with the dynamics of closed p-branes, for $p = 0, 1, 2, ..., D-1$, embedded in a target D-dimensional spacetime background.

The C-space polyvector-valued momentum is defined as $\mathbf{P} = d\mathbf{X}/d\Sigma$ where $\mathbf{X}$ is the Clifford-valued coordinate corresponding to the $Cl(1,3)$ algebra in four-dimensions, for example,

$$\mathbf{X} = s \mathbf{1} + x^\mu \gamma_\mu + x^{\mu\nu} \gamma_\mu \wedge \gamma_\nu + x^{\mu\nu\rho} \gamma_\mu \wedge \gamma_\nu \wedge \gamma_\rho + x^{\mu\nu\rho\tau} \gamma_\mu \wedge \gamma_\nu \wedge \gamma_\rho \wedge \gamma_\tau \quad (1.1)$$

where we have omitted combinatorial numerical factors for convenience in the expansion (1). It can be generalized to any dimensions, including $D = 0$. The component s is the Clifford scalar component of the polyvector-valued coordinate and $d\Sigma$ is the infinitesimal C-space proper “time” interval which is *invariant* under $Cl(1,3)$ transformations which are the Clifford-algebra extensions of the $SO(1,3)$ Lorentz transformations [1]. One should emphasize that $d\Sigma$, which is given by the square root of the quadratic interval in C-space

$$(d\Sigma)^2 = (ds)^2 + dx_\mu \, dx^\mu + dx_{\mu\nu} \, dx^{\mu\nu} + \ldots \quad (1.2)$$

is *not* equal to the proper time Lorentz-invariant interval $d\tau$ in ordinary spacetime $(d\tau)^2 = g_{\mu\nu}dx^\mu dx^\nu = dx_\mu dx^\mu$. In order to match units in all terms of eqs-(1.1,1.2) suitable powers of a length scale (say Planck scale) must be introduced. For convenience purposes it is can be set to unity. For extensive details of the generalized Lorentz transformations (poly-rotations) in flat C-spaces and references we refer to [1].

Let us now consider a basis in C-space given by

$$E_A = \gamma, \ \gamma_\mu, \ \gamma_\mu \wedge \gamma_\nu, \ \gamma_\mu \wedge \gamma_\nu \wedge \gamma_\rho, \ ... \quad (1.3)$$

where γ is the unit element of the Clifford algebra that we label as $\mathbf{1}$ from now on. In (3) when one writes an r-vector basis $\gamma_{\mu_1} \wedge \gamma_{\mu_2} \wedge ... \wedge \gamma_{\mu_r}$ we take the indices in "lexicographical" order so that $\mu_1 < \mu_2 < .... < \mu_r$. An element of C-space is a Clifford number, called also *Polyvector* or *Clifford aggregate* which we now write in the form

$$X = X^A E_A = s \mathbf{1} + x^\mu \gamma_\mu + x^{\mu\nu} \gamma_\mu \wedge \gamma_\nu + ... \quad (1.4)$$

A C-space is parametrized not only by 1-vector coordinates x^μ but also by the 2-vector coordinates $x^{\mu\nu}$, 3-vector coordinates $x^{\mu\nu\alpha}$, ..., called also *holographic coordinates*, since they describe the holographic projections of 1-loops, 2-loops, 3-loops,..., onto the coordinate planes . By p-loop we mean a closed p-brane; in particular, a 1-loop is closed string. In order to avoid using the powers of the Planck scale length parameter L_p in the expansion of the polyvector X (in order to match units) we can set it to unity to simplify matters. In a *flat* C-space the basis vectors E^A, E_A are *constants*. In a *curved* C-space this is no longer true. Each E^A, E_A is a function of the C-space coordinates

$$X^A \;=\; \{ \; s, \; x^\mu, \; x^{\mu_1\mu_2}, \; ....., \; x^{\mu_1\mu_2.....\mu_D} \; \} \tag{1.5}$$

which include scalar, vector, bivector,..., p-vector,... coordinates in the underlying D-dim base spacetime and whose corresponding C-space is 2^D-dimensional since the Clifford algebra in D-dim is 2^D-dimensional.

The C-space metric is chosen to be $G^{AB} = 0$ when the *grade* $A \neq$ *grade* B. For the same-grade metric components $g^{[a_1a_2...a_k]\;[b_1b_2...b_k]}$ of G^{AB}, the metric can decomposed into its irreducible factors as antisymmetrized sums of products of η^{ab} given by the following *determinant* [14]

$$G^{AB} \;\equiv\; det \left(\begin{array}{cccc} \eta^{a_1b_1} & \dots & \dots & \eta^{a_1b_k} \\ \eta^{a_2b_1} & \dots & \dots & \eta^{a_2b_k} \\ ------ & ------ & ------ & ------ \\ \eta^{a_kb_1} & \dots & \dots & \eta^{a_kb_k} \end{array} \right) \tag{1.6}$$

The spacetime signature is chosen to be $(-,+,+,....,+)$. One still has the freedom to choose the sign of the scalar-scalar components G_{**} of the C-space metric G_{AB}. In the next section we shall see that $G_{**} = -1 < 0$ is the right choice.

Recently, novel physical consequences of the Extended Relativity Theory in C-spaces (Clifford spaces) were explored in [4]. The latter theory provides a very different physical explanation of the phenomenon of "relativity of locality" than the one described by the Doubly Special Relativity (DSR) framework. Furthermore, an elegant *nonlinear* momentum-addition law was derived in order to tackle the "soccer-ball" problem in DSR. Neither derivation in C-spaces requires a *curved* momentum space nor a deformation of the Lorentz algebra. While the constant (energy-independent) speed of photon propagation is always compatible with the generalized photon dispersion relations in C-spaces, another important consequence was that the generalized C-space photon dispersion relations allowed also for energy-dependent speeds of propagation while still *retaining* the Lorentz symmetry in ordinary spacetimes, while breaking the *extended* Lorentz symmetry in C-spaces. This does *not* occur in DSR nor in other approaches, like the presence of quantum spacetime foam.

We learnt from Special Relativity that the concept of simultaneity is also relative. By the same token, we have shown in [4] that the concept of spacetime locality is *relative* due to the *mixing* of area-bivector coordinates with spacetime vector coordinates under generalized Lorentz transformations in C-space. In the most general case, there will be mixing of all polyvector valued coordinates. This was the motivation to build a unified theory of all extended objects, p-branes, for all values of p subject to the condition $p+1 = D$.

In [5] we explored the many novel physical consequences of Born's Reciprocal Relativity theory [7], [9], [10] in flat phase-space and generalized the theory to the curved phase-space scenario. We provided six specific novel physical results resulting from Born's Reciprocal Relativity and which are *not* present in Special Relativity. These were : momentum-dependent time delay in the emission and detection of photons; energy-dependent notion of locality; superluminal behavior; relative rotation of photon trajectories due to the aberration of light; invariance of areas-cells in phase-space and modified dispersion relations. We finalized by constructing a Born reciprocal general relativity theory in curved phase-spaces which required the introduction of a complex Hermitian metric, torsion and nonmetricity.

We should emphasize that *no* spacetime foam was introduced, nor Lorentz invariance was broken, in order to explain the time delay in the photon emission/arrival. In the conventional approaches of DSR (Double Special Relativity) where there is a Lorentz invariance breakdown [12], a longer wavelength photon (lower energy) experiences a smoother spacetime than a shorter wavelength photon (higher energy) because the higher energy photon experiences more of the graininess/foamy structure of spacetime at shorter scales. Consequently, the less energetic photons will move faster (less impeded) than the higher energetic ones and will arrive at earlier times.

However, in our case above [5] the time delay is entirely due to the very nature of Born's Reciprocal Relativity when one looks at pure acceleration (force) boosts transformations of the phase space coordinates in $flat$ phase-space. No *curved* momentum space is required as it happens in [12]. The time delay condition in Born's Reciprocal Relativity theory implied also that higher momentum (higher energy) photons will take longer to arrive than the lower momentum (lower energy) ones.

Superluminal particles were studied within the framework of the Extended Relativity theory in Clifford spaces (C-spaces) in [6]. In the simplest scenario, it was found that it is the contribution of the Clifford scalar component P of the poly-vector-valued momentum $\mathbf{P}$ which is responsible for the superluminal behavior in ordinary spacetime due to the fact that the effective mass $\sqrt{\mathcal{M}^2 - P^2}$ can be imaginary (tachyonic). However from the point of view of C-space there is no superluminal behaviour (tachyonic) because the true physical mass still obeys $\mathcal{M}^2 > 0$. As discussed in detailed by [1], [3] one can have tachyonic (superluminal) behavior in ordinary spacetime while having non-tachyonic behavior in C-space. Hence from the C-space point of view there is no violation of causality nor the Clifford-extended Lorentz symmetry. The analog of "photons" in C-space are *tensionless* strings and branes [1].

The addition law of areal velocities and a minimal length interpretation L was recently studied in [4]. The argument relied entirely on the physics behind the extended notion of Lorentz transformations in C-space, and *does not* invoke Quantum Gravity arguments nor quantum group deformations of Lorentz/Poincare algebras. The physics of the Extended Relativity theory in C-spaces requires the introduction of the speed of light and a minimal scale. In [2] we have shown how the construction of an Extended Relativity Theory in Clifford *Phase* Spaces requires the introduction of a *maximal* scale which can be identified with the Hubble scale and leads to Modifications of Gravity at the Planck/Hubble scales. Born's Reciprocal Relativity demands that a minimal length corresponds to a minimal momentum that can be set to be $p_{min} = \hbar / R_{Hubble}$. For full details we refer to [2].

Despite the fact that the length parameter L (which must be introduced in the C-

space interval in eq-(1.2) in order to match units) has the physical interpretation of a *minimal* length, this does *not* mean that the spatial separation between two events in C-space cannot be *smaller* than L. The Planck scale minimal length argument is mainly associated with Quantum Mechanics and Black Hole Physics. The energy involved in the physical measurement process to localize a Planck mass particle, within Planck scale resolutions, becomes very large and such that a black hole forms enclosing the particle behind the black hole horizon. Since one does not have physical access to the black hole interior one cannot probe scales beyond the Planck scale. We shall set aside for the moment the current firewall controversy of black holes.

Recently, we improved our earlier work in [16] and derived the minimal length string/membrane uncertainty relations by imposing momentum slices in flat Clifford spaces [24]. The Jacobi identities associated with the modified Weyl-Heisenberg algebra require noncommuting spacetime coordinates, but commuting momenta, and which is compatible with the notion of curved momentum space. The purpose of this work is mainly to follow a different approach than the one taken in [24] by noticing that rainbow metrics [13] are a natural consequence of taking momentum slices in Cspaces. Generalized dispersion and uncertainty relations are found in addition to modified black hole area-entropy relations.

2 On Rainbow Metrics and Generalized Dispersion and Uncertainty Relations from Clifford Spaces

2.1 Clifford Space Relativity induces generalized dispersion and uncertainty relations

In this section we shall provide a $different$ derivation of the generalized uncertainty relations than the one described in [24]. Our derivation in this work is based on the concept of rainbow metrics [13].

The generalization of the Weyl-Heisenberg algebra to C-spaces and involving polyvector-valued coordinates and momenta (in natural units $\hbar = 1$) is [1] $[X_A,\ P_B] = \ i\ G_{AB}$ and does not lead to minimal uncertainty conditions for ΔX_A. To obtain the minimal length stringy uncertainty relations in ordinary spacetimes requires more work. It involves taking polymomentum slices through C-space. This is the subject of this section.

The on-shell mass condition for a massive polyparticle moving in the 2^4-dimensional flat C-space, corresponding to a Clifford algebra in $D = 4$, can be written in terms of the polymomentum (polyvector-valued) components, in natural units $L = L_P = 1, \hbar = c = 1$, as

$$\pi^2 \ + \ p_\mu \ p^\mu \ + \ p_{\mu_1\mu_2} \ p^{\mu_1\mu_2} \ + \ p_{\mu_1\mu_2\mu_3} \ p^{\mu_1\mu_2\mu_3} \ + \ p_{\mu_1\mu_2....\mu_4} p^{\mu_1\mu_2...\mu_4} \ = \ - \ \mathcal{M}^2 \qquad (2.1)$$

Let us *break* the ordinary Lorentz invariance by imposing the non-Lorentz invariant conditions on the poly-momenta in C-space

$$p_{ij}\ p^{ij}\ =\ \beta_1\ |\vec{p}|^4,\quad p_{ijk}\ p^{ijk}\ =\ \beta_2\ |\vec{p}|^6$$
$$p_{0i}\ p^{0i}\ =\ \alpha_1\ (p_0)^2\ |\vec{p}|^2,\quad p_{0ij}\ p^{0ij}\ =\ \alpha_2\ (p_0)^2\ |\vec{p}|^4,\quad p_{0ijk}\ p^{0ijk}\ =\ \alpha_3\ (p_0)^2\ |\vec{p}|^6 \qquad (2.2)$$

where the α's and β's are numerical parameters. The mass-shell condition in C-space $P_A P^A = -\mathcal{M}^2$ becomes after inserting the conditions (2.2) and taking into account the chosen signature $(-,+,+,+)$

$$|\vec{p}|^2\ (\ \frac{\pi^2}{|\vec{p}|^2}\ +1\ +\beta_1\ |\vec{p}|^2\ +\beta_2\ |\vec{p}|^4\ )\ -\ (p_0)^2\ (\ 1\ +\alpha_1\ |\vec{p}|^2\ +\alpha_2\ |\vec{p}|^4\ +\ \alpha_3\ |\vec{p}|^6\ )\ =\ -\ \mathcal{M}^2 \quad (2.3)$$

In [24] we interpreted the terms inside the parenthesis in (2.3) as if one had a metric in momentum space as follows

$$g_{ij}(\pi^2,|\vec{p}|^2)\ p^i\ p^j\ +\ g_{00}(|\vec{p}|^2)\ p^0\ p^0\ =\ g^{ij}(\pi^2,|\vec{p}|^2)\ p_i\ p_j\ +\ g^{00}(|\vec{p}|^2)\ p_0\ p_0\ =\ -\ \mathcal{M}^2 \quad (2.4)$$

choosing a flat metric

$$g_{ij}(\pi^2,|\vec{p}|^2)\ =\ \delta_{ij}\ \Rightarrow\ \frac{\pi^2}{|\vec{p}|^2}\ +1\ +\beta_1\ |\vec{p}|^2\ +\beta_2\ |\vec{p}|^4\ =\ 1 \qquad (2.5)$$

leads to a non-Lorentz invariant constraint among π^2 and $|\vec{p}|^2$. The former π^2 is a Lorentz scalar but not the latter. The flat metric condition for g_{00} gives

$$g_{00}(|\vec{p}|^2)\ =\ -1\ \Rightarrow\ -\ (\ 1\ +\alpha_1\ |\vec{p}|^2\ +\alpha_2\ |\vec{p}|^4\ +\ \alpha_3\ |\vec{p}|^6\ )\ =\ -1 \qquad (2.6)$$

from which one infers that the parameters $\alpha_1 = \alpha_2 = \alpha_3 = 0$ are zero because one should not impose constraints of the values of $|\vec{p}|^2$. Hence, having $\alpha_1 = \alpha_2 = \alpha_3 = 0$ in (2.6) implies that the polymomentum slice (2.2) in C-space will set the following values to zero : $p_{0i} = p_{0ij} = p_{0ijk} = 0$.

We fixed the choice for the sign of the scalar-scalar components G_{**} of the C-space metric G_{AB} by having $G_{**} = -1 < 0$ so that when one evaluates the mass-shell condition $P_A P^A = -\mathcal{M}^2$ one will have the π^2 term with the required negative sign $G_{**}\pi^2 = -\pi^2 < 0$ so that eq-(2.5) should read instead

$$-\ \frac{\pi^2}{|\vec{p}|^2}\ +1\ +\beta_1\ |\vec{p}|^2\ +\beta_2\ |\vec{p}|^4\ =\ 1 \qquad (2.7)$$

leading to the key inequality $1\ \leq\ 1\ +\beta_1\ |\vec{p}|^2\ +\beta_2\ |\vec{p}|^4$ and consistent with the fact that $\beta_1, \beta_2 > 0$. The flat metric $g_{ij}(\pi^2,|\vec{p}|^2) = \delta_{ij}$ and $g_{00}(|\vec{p}|^2) = -1$ does not lead to modifications of the Weyl-Heisenberg algebra. However, it is upon using the latter proper key *inequality*, and treating the coordinates and momenta as self-adjoint quantum *operators*, which leads to the following generalized uncertainty relations

$$\Delta x_i \; \Delta p_j \;\geq\; \frac{\hbar}{2} \; | < (\; 1 \; + \beta_1 \; |\vec{p}|^2 \; + \beta_2 \; |\vec{p}|^4 \;) > | \; \delta_{ij} \;\geq\; \frac{\hbar}{2} \; \delta_{ij} \tag{2.8}$$

In this work we will take a very different approach than the one described above [24] based in having a flat metric and yielding a key inequality, by noticing that the terms inside the parenthesis behave as if one had a *rainbow* metric as follows

$$g^{ij}(\pi^2, \; |\vec{p}|^2) \; p_i \; p_j \; + \; g^{00}(|\vec{p}|^2) \; p_0 \; p_0 \; = \; g^2(\pi^2, \; |\vec{p}|^2) \; |\vec{p}|^2 \; - \; f^2(|\vec{p}|^2) \; E^2 \; = \; - \; \mathcal{M}^2 \tag{2.9}$$

A rainbow metric [13] is a one-parameter family of metrics which depends on the energy (momentum) of the test particles moving in a given spacetime background, and forming a rainbow of metrics (rainbow geometry). Setting $\pi^2 = 0$ in (2.3) one has then that the squared rainbow functions are given by

$$g^2(\pi^2 = 0, \; |\vec{p}|^2) \; \equiv \; 1 \; + \beta_1 \; |\vec{p}|^2 \; + \beta_2 \; |\vec{p}|^4, \quad \beta_1, \beta_2 > 0 \tag{2.10a}$$

$$f^2(|\vec{p}|^2) \; \equiv \; 1 \; + \alpha_1 \; |\vec{p}|^2 \; + \alpha_2 \; |\vec{p}|^4 \; + \; \alpha_3 \; |\vec{p}|^6, \quad \alpha_1, \alpha_2, \alpha_3 > 0 \tag{2.10b}$$

Given

$$g^{ij} \; = \; g^2(\pi^2 = 0, \; |\vec{p}|^2) \; \delta^{ij} \; = \; (\; 1 \; + \beta_1 \; |\vec{p}|^2 \; + \beta_2 \; |\vec{p}|^4 \;) \; \delta^{ij} \tag{2.11a}$$

$$g^{00} \; = \; - \; f^2(|\vec{p}|^2) \; \delta^{00} \; = \; - \; (\; 1 \; + \alpha_1 \; |\vec{p}|^2 \; + \alpha_2 \; |\vec{p}|^4 \; + \; \alpha_3 \; |\vec{p}|^6 \;) \tag{2.11b}$$

the *rainbow* metric is then *defined* as

$$ds^2 \; = \; g_{\mu\nu} \; dx^\mu \; dx^\nu \; =$$

$$- \; (\; 1 \; + \alpha_1 \; |\vec{p}|^2 \; + \alpha_2 \; |\vec{p}|^4 \; + \; \alpha_3 \; |\vec{p}|^6 \;)^{-1} \; (dt)^2 \; + \; (\; 1 \; + \beta_1 \; |\vec{p}|^2 \; + \beta_2 \; |\vec{p}|^4 \;)^{-1} \; (dx^i)^2 \tag{2.12}$$

One may notice also that from eqs-(2.9, 2.10, 2.11) one arrives at the modified dispersion relations

$$(1 \; + \alpha_1 \; |\vec{p}|^2 \; + \alpha_2 \; |\vec{p}|^4 \; + \; \alpha_3 \; |\vec{p}|^6) \; E^2 \; - \; (\; 1 \; + \beta_1 \; |\vec{p}|^2 \; + \beta_2 \; |\vec{p}|^4 \;) \; |\vec{p}|^2 \; = \; \mathcal{M}^2 \; \Rightarrow$$

$$E^2 \; - \; \frac{1 \; + \beta_1 \; |\vec{p}|^2 \; + \beta_2 \; |\vec{p}|^4}{1 \; + \alpha_1 \; |\vec{p}|^2 \; + \alpha_2 \; |\vec{p}|^4 \; + \; \alpha_3 \; |\vec{p}|^6} \; |\vec{p}|^2 \; = \; \frac{\mathcal{M}^2}{1 \; + \alpha_1 \; |\vec{p}|^2 \; + \alpha_2 \; |\vec{p}|^4 \; + \; \alpha_3 \; |\vec{p}|^6} \tag{2.13}$$

setting $\alpha_3 = 0; \alpha_1 = \beta_1; \alpha_2 = \beta_2$, and after performing a Taylor series expansion, eq-(2.13) simplifies to

$$E^2 \; - \; |\vec{p}|^2 \; = \; \frac{\mathcal{M}^2}{1 \; + \alpha_1 \; |\vec{p}|^2 \; + \alpha_2 \; |\vec{p}|^4} \; =$$

$$\mathcal{M}^2 \; \Big(\; 1 \; - \; (\alpha_1 \; |\vec{p}|^2 \; + \alpha_2 \; |\vec{p}|^4) \; + \; (\alpha_1 \; |\vec{p}|^2 \; + \alpha_2 \; |\vec{p}|^4)^2 \; - \; (\alpha_1 \; |\vec{p}|^2 \; + \alpha_2 \; |\vec{p}|^4)^3 \; + \; .... \; \Big) \tag{2.14}$$

since the constants α_1, α_2 are proportional to powers of the Planck length as $(L_P/\hbar)^2 \sim E_P^{-2}, (L_P/\hbar)^4 \sim E_P^{-4}$, respectively, after equating $\mathcal{M} = m$ one can infer that the dispersion relation obtained in eq-(2.14) has the form

$$E^2 \ - \ |\vec{p}|^2 \ = \ m^2 \ + \ \sum_{n=1}^{\infty} c_n(m^2, E_P^2) \ |\vec{p}|^{2n} \tag{2.15}$$

where the coefficients c_n are explicitly given in terms of the Planck energy E_P and the mass m as follows

$$c_1 \ = \ -\alpha_1 \ m^2; \ \ c_2 \ = \ (\ (\alpha_1)^2 - \alpha_2 \ ) \ m^2; \ \ c_3 \ = \ (\ 2 \ \alpha_1 \ \alpha_2 \ - \ (\alpha_1)^3 \ ) \ m^2; \ ...... \tag{2.16}$$

The advantage in using Clifford Space Relativity is that one is able to derive the explicit expression for *all* the coefficients c_n in terms of the two parameters α_1, α_2 which, in turn, are proportional to E_P^{-2}, E_P^{-4}, respectively. In the low energy limit, one recovers the standard dispersion relation $E^2 - |\vec{p}|^2 = m^2$. Another salient feature of the generalized dispersion relation (2.15) , with coefficients given by (2.16), is that in the massless case $m = 0$ one still retains the condition $E^2 - |\vec{p}|^2 = 0$ (massless particles still propagate at the speed of light). This is not case with the generalized dispersion relation proposed in the literature [25] because these numerical coefficients depend on a running mass scale and E_P.

Another physical consequence is that the rainbow metric (2.11) when $\alpha_3 = 0; \alpha_1 = \beta_1; \alpha_2 = \beta_2$ yields *modifications* of the Weyl-Heisenberg algebra

$$[x^\mu, \ p^\nu] \ = \ i \ \hbar \ g^{\mu\nu}(|\vec{p}|^2) \tag{2.17}$$

resulting from the momentum-dependent metric (2.11), and which in turn leads to the following uncertainty relations

$$\Delta x^\mu \ \Delta p^\nu \ \geq \ \frac{\hbar}{2} \ | \ < (\ 1 \ + \alpha_1 \ |\vec{p}|^2 \ + \alpha_2 \ |\vec{p}|^4 \ ) > \ \eta^{\mu\nu} \ | \tag{2.18}$$

where $< .... >$ denote the QM expectation values $< \Psi|......|\Psi >$. See [23] for rigorous mathematical details. However, there is a *caveat* : if one recurs to the rainbow metrics and begins with the modified Weyl-Heisenberg algebra $[x^\mu, p^\nu] = i\hbar g^{\mu\nu}(\vec{p})$, a lowering of indices with the rainbow metric leads to

$$[x_\mu, \ p_\nu] \ = \ i\hbar \ g_{\mu\nu}(\vec{p}) \ + \ .... \ \neq \ i\hbar \ g_{\mu\nu}(\vec{p}) \ = \ i\hbar \ (\ 1 \ + \alpha_1 \ |\vec{p}|^2 \ + \alpha_2 \ |\vec{p}|^4 \ )^{-1} \ \eta_{\mu\nu} \tag{2.19}$$

thus there is an *asymmetry* in the functional forms of the modified Weyl-Heisenberg algebra due to the commutators

$$[x_\mu, \ p_\nu] \ = \ [g_{\mu\rho}(\vec{p}) \ x^\rho, \ g_{\nu\eta} \ (\vec{p}) \ p^\eta] \ = \ i\hbar \ g_{\mu\nu}(\vec{p}) \ + \ g_{\mu\rho}(\vec{p}) \ [x^\rho, \ g_{\nu\eta}(\vec{p})] \ p^\eta \tag{2.20}$$

and

$$[p_\mu, \ p_\nu] \ = \ 0; \ \ [x_\mu, \ x_\nu] \ \neq 0; \ \ [x^\rho, \ g_{\nu\eta}(\vec{p})] \ = \ i\hbar \ g^{\rho\tau}(\vec{p}) \ \frac{\partial g_{\nu\eta}(\vec{p})}{\partial p^\tau} \tag{2.21}$$

In order to have $[x^\mu, p^\nu] = i\hbar g^{\mu\nu}(\vec{p})$ and $[x_\mu,\ p_\nu] = i\hbar g_{\mu\nu}(\vec{p})$, *simultaneously*, one must restrict the rainbow metric to obey the conditions

$$g_{\mu\rho}(\vec{p})\ [x^\rho,\ g_{\nu\eta}(\vec{p})]\ p^\eta\ =\ i\hbar\ g_{\mu\rho}(\vec{p})\ g^{\rho\tau}(\vec{p})\ \frac{\partial g_{\nu\eta}(\vec{p})}{\partial p^\tau}\ p^\eta\ =\ i\hbar\ \frac{\partial g_{\nu\eta}(\vec{p})}{\partial p^\mu}\ p^\eta\ =\ 0 \qquad (2.22)$$

besides the trivial solutions $g_{\nu\eta} = \eta_{\mu\nu}$ so that $\frac{\partial g_{\nu\eta}(\vec{p})}{\partial p^\mu} = 0$, there is the nontrivial solution to eq-(2.22) when

$$g_{00}\ =\ -1,\quad g_{0k} = g_{k0} = 0 \qquad (2.23a)$$

and

$$\frac{\partial g_{kl}(\vec{p})}{\partial p^i}\ =\ p_i\ \delta_{kl}\ -\ \frac{p_i\ p_k\ p_l}{|\vec{p}|^2} \qquad (2.23b)$$

leading to a *zero* contraction with p^l : $\frac{\partial g_{kl}(\vec{p})}{\partial p^i}\ p^l = 0$. After integrating (2.23b) and introducing the parameter α whose units are $(L_P)^2$, gives for the spatial metric components

$$g_{kl}\ =\ \alpha \int\ (\ p_i\ \delta_{kl}\ -\ \frac{p_i\ p_k\ p_l}{|\vec{p}|^2}\ )\ dp^i\ =\ \frac{\alpha}{2}\int\ (\ \delta_{kl}\ -\ \frac{p_k\ p_l}{|\vec{p}|^2}\ )\ d|\vec{p}|^2;\quad p^i \equiv \delta^{ij} p_j \qquad (2.23c)$$

Therefore, a metric given by eqs-(2.23) yields the generalized uncertainty relations for the spatial components of the coordinates and momenta

$$\Delta x_k\ \Delta p_l\ \ \geq\ \frac{\hbar}{2}\ |\ < g_{kl} >\ |,\quad \Delta x^k\ \Delta p^l\ \ \geq\ \frac{\hbar}{2}\ |\ < g^{kl} >\ |. \qquad (2.24)$$

and which is a *modification* of the stringy uncertainty relations

$$\Delta x_k\ \Delta p_l\ \geq\ \frac{\hbar}{2}\ |\ <\ (\ 1\ +\beta\ |\vec{p}|^2)\ >\ \delta_{kl}\ |\ \geq\ \frac{\hbar}{2}\ (\ 1\ +\beta\ |\Delta\vec{p}|^2)\ \delta_{kl} \qquad (2.25)$$

due to the presence of the second terms of the nontrivial integral in (2.23c). The constant of integration in (2.23c) can be chosen to generate the constant term δ_{kl} of (2.25). Keeping the leading terms in powers of L_P in eqs-(2.25), gives for example

$$\Delta x\ \Delta p_x\ \geq\ \frac{\hbar}{2}\,|<\ (\ 1+\beta\ |\vec{p}|^2)\ >|\ \geq\ \frac{\hbar}{2}\,|<\ (\ 1+\beta\ p_x^2)\ \ >|\ \geq\ \frac{\hbar}{2}\ (\ 1+\beta\ (\Delta p_x)^2) \qquad (2.26)$$

where we have used the identities $< p_x^2 >= (\Delta p_x)^2 + < p_x >^2$ in last inequality of (2.26), and taken $\beta > 0$ which allows to remove the absolute sign since all quantities are now positive definite.

From (2.26) one arrives at the minimal length stringy uncertainty relations

$$\Delta x\ \Delta p_x\ \geq\ \frac{\hbar}{2}\ (\ 1\ +\beta\ (\Delta p_x)^2)\ \Rightarrow\ \Delta x\ \geq\ \frac{\hbar}{2\Delta p_x}\ +\ (\frac{\hbar\beta}{2})\ \Delta p_x \qquad (2.27)$$

Minimizing the expression in (2.27) and inserting the Planck scale L_P which was set to unity one has for the minimum position uncertainty a quantity of the order of the Planck scale

$$(\Delta x)_{min} \;=\; L_P\,\sqrt{\beta},\quad \beta > 0 \tag{2.28}$$

In [24] we remarked that the *higher* order corrections to the stringy uncertainty relations in eq-(2.8) stem from the higher grade polymomentum variables in C-space and correspond, physically, to the membrane contributions to the modified uncertainty relations. Hence, the stringy and membrane corrections to the uncertainty relations in $D = 4$ are of the form (similar equations follow for the other spatial coordinates)

$$\Delta x\ \Delta p_x \;\geq\; \frac{\hbar}{2}\,[\ 1\ + \beta_1\ (\Delta p_x)^2\ +\ \beta_2\ (\Delta p_x)^4\ ] \tag{2.29}$$

leading to

$$\Delta x \;\geq\; \frac{\hbar}{2}\,[\ \frac{1}{\Delta p_x}\ + \beta_1\ (\Delta p_x)\ +\ \beta_2\ (\Delta p_x)^3\ ] \tag{2.30}$$

the extremization problem of (2.30) is more complicated but there is a local minimum when $\beta_1 > 0, \beta_2 > 0$. The value of Δp_x which yields the local minimum for Δx is

$$(\Delta p_x)_o \;=\; \left(\frac{-\ \beta_1\ +\ \sqrt{(\beta_1)^2 + 12\beta_2}}{6\beta_2}\right)^{\frac{1}{2}},\quad \beta_1 > 0,\ \beta_2 > 0 \tag{2.31}$$

If one sets the above value of $(\Delta p_x)_o$ and minimal length uncertainty to coincide with the Planck momentum and Planck scale, respectively, one can fix the numerical values of β_1, β_2. In higher dimensions than $D = 4$ one will capture the p-brane contributions beyond the membrane case due to the contributions of the higher grade polymomenta components. The dimensions (units) of the parameters in eqs-(2.29, 2.30) are $[\beta_1] = (L/\hbar)^2,\ [\beta_2] = (L/\hbar)^4$.

2.2 Jacobi Identities and Noncommutative Spacetime

To continue we study the Jacobi identities that are linked to noncommuting spacetime coordinates. Let us start with a *modified* Weyl-Heisenberg algebra in a flat spacetime given by

$$[x^i,\ p_j] \;=\; i\,\hbar\,\Theta(|\vec{p}|^2)\ \delta^i_j,\ \ [x^i,\ p^j] \;=\; i\,\hbar\,\Theta(|\vec{p}|^2)\ \delta^{ij},\ \ [x_i,\ p_j] \;=\; i\,\hbar\,\Theta(|\vec{p}|^2)\ \delta_{ij}, \tag{2.32}$$

The Jacobi identities are

$$[\ x^i,\ [x^j,\ p_k]\ ]\ +\ [\ x^j,\ [p_k,\ x^i]\ ]\ +\ [\ p_k,\ [x^i,\ x^j]\ ]\ =\ 0 \tag{2.33a}$$

$$[\ x^i,\ [x^j,\ x^k]\ ]\ +\ [\ x^j,\ [x^k,\ x^i]\ ]\ +\ [\ x^k,\ [x^i,\ x^j]\ ]\ =\ 0, \tag{2.33b}$$

etc, ... Let us try the ansatz

$$[x^i,\ x^j] \ = \ i\ \hbar\ f^{ij}{}_l(\vec{p})\ x^l, \quad [p_j,\ p_k] \ = \ 0 \tag{2.34}$$

due to the noncommutativity of x^i, p^j one could have written instead of (2.34) the following more symmetric form for the commutators

$$[x^i,\ x^j] \ = \ \frac{i\hbar}{2}\ \{f^{ij}{}_l(\vec{p}),\ x^l\} \ = \ \frac{i\hbar}{2}\ f^{ij}{}_l(\vec{p})\ x^l \ + \frac{i\hbar}{2}\ x^l\ f^{ij}{}_l(\vec{p}) \tag{2.35}$$

For simplicity, we will just use the commutators displayed in eq-(2.34) instead of those in eq-(2.35). It will not affect the final results. After some straightforward algebra one learns from the Jacobi identities (2.33) that the structure functions $f^{ij}{}_l(\vec{p})$ are given in terms of the function $\Theta(|\vec{p}|^2)$ as follows

$$\delta^j_k\ \frac{\partial\Theta(|\vec{p}|^2)}{\partial p_i}\ -\ \delta^i_k\ \frac{\partial\Theta(|\vec{p}|^2)}{\partial p_j}\ =\ f^{ij}{}_k(\vec{p}) \tag{2.36}$$

it is explicitly antisymmetric in ij as expected. Using the second set of Jacobi identities for the noncommutative spacetime coordinates, the relations $[x^i, F(\vec{p})] = i\hbar_{eff}(|\vec{p}|^2)\ (\partial F(\vec{p})/\partial p_i)$, the Liebnitz law $[x^i, AB] = A[x^i, B] + [x^i, A]B$, and the solutions obtained for $f^{ij}{}_k(\vec{p})$ given in (2.36), one can verify, after some algebra, that indeed one has

$$(\ f^{jk}{}_l\ f^{il}{}_m\ +\ f^{ki}{}_l\ f^{jl}{}_m\ +\ f^{ij}{}_l\ f^{kl}{}_m\ )\ x^m\ =\ 0 \tag{2.37}$$

$$(\ \frac{\partial f^{jk}{}_l}{\partial p_i}\ +\ \frac{\partial f^{ki}{}_l}{\partial p_j}\ +\ \frac{\partial f^{ij}{}_l}{\partial p_k}\ )\ x^l\ =\ 0 \tag{2.38}$$

and the Jacobi identities (2.33) are satisfied. It is important to emphasize that the terms inside the parenthesis in eqs-(2.37, 2.38) are *not* zero. What *is* zero is the net summation after the full contraction with the x^m, x^l terms is performed.

Therefore, to satisfy the Jacobi identities one must have a Noncommutative spacetime. Kempf and Mangano [23] used the commutator $[x_i, p_j] = i\hbar\Theta_{ij}(\vec{p})$, where Θ_{ij} is a more general rotationally invariant function of the momenta coordinates, and for commutator $[x_i, x_j]$ they have the more symmetric expression described by (2.35). After studying the Jacobi identities they arrived at

$$[x_i, x_j]\ =\ i\hbar\ \{\ x_a,\ \Theta^{-1}_{ar}\ \Theta_{s[i}\ \Theta_{j]r,s}\ \},\quad \Theta_{jr,s}\ \equiv\ \frac{\partial\Theta_{jr}}{\partial p^s} \tag{2.39}$$

where $\{,\}$ denotes the anti-commutator. See [23] for further details.

2.3 Lorentz Invariant Case

Let us write again the on-shell mass condition for a massive polyparticle moving in the 2^4-dimensional flat C-space, corresponding to a Clifford algebra in $D = 4$, in terms of the polymomentum (polyvector-valued) components, in natural units $L = L_P = 1, \hbar = c = 1$, as

$$\pi^2 \ + \ p_\mu \ p^\mu \ + \ p_{\mu_1\mu_2} \ p^{\mu_1\mu_2} \ + \ p_{\mu_1\mu_2\mu_3} \ p^{\mu_1\mu_2\mu_3} \ + \ p_{\mu_1\mu_2....\mu_4} p^{\mu_1\mu_2...\mu_4} \ = \ - \ \mathcal{M}^2 \quad (2.40)$$

A particular Lorentz invariant *slice* through the flat C-space can be taken by imposing the set of algebraic conditions on the polymomenta coordinates

$$p_{\mu_1\mu_2} \ p^{\mu_1\mu_2} \ = \ \lambda_1 \ (p_\mu p^\mu)^2 \ = \ \lambda_1 \ p^4, \ \ p_{\mu_1\mu_2\mu_3} \ p^{\mu_1\mu_2\mu_3} \ = \ \lambda_2 \ (p_\mu p^\mu)^3 \ = \ \lambda_2 \ p^6$$

$$p_{\mu_1\mu_2\mu_3\mu_4} \ p^{\mu_1\mu_2\mu_3\mu_4} \ = \ \lambda_3 \ (p_\mu p^\mu)^4 \ = \ \lambda_3 \ p^8$$

$$p^2 \ \equiv \ p_\mu \ p^\mu \ = \ |\vec{p}|^2 \ - \ (p_0)^2 \ = \ (p_x)^2 + (p_y)^2 + (p_z)^2 \ - \ E^2 \quad (2.41)$$

where the λ's are numerical parameters. π is the Clifford scalar part of the momentum polyvector and is invariant under C-space transformations. The slice conditions in eqs-(2.41) will *break* the generalized (extended) Lorentz symmetry in C-space because these conditions are *not* preserved under the most general C-space transformations as described in [4]. Nevertheless, the residual standard Lorentz symmetry (in ordinary spacetime) will still remain intact because the conditions/constraints in eqs-(2.41) are explicitly Lorentz invariant.

Inserting the conditions of eqs-(2.41) into eq-(2.40) yields

$$\pi^2 \ + \ p^2 \ (\ 1 \ + \lambda_1 \ p^2 \ + \lambda_2 \ p^4 \ + \lambda_3 \ p^6 \ ) \ = \ \pi^2 \ + \ f^2(p^2) \ p^2 \ = \ - \ \mathcal{M}^2 \quad (2.42)$$

The last expression (2.42) is a generalization of the "gravity rainbow" metric where the function $f^2(p^2)$ in our case is the analog of the rainbow function squared. It is important to emphasize that in C-space one has $P_A P^A = -\mathcal{M}^2$ but $p^2 = |\vec{p}|^2 - E^2$ is no longer a *constant* equal to $-m^2$. This fact is also consistent with the generalized dispersion relation (2.15). What is a constant is the polymomentum squared involving all the polyvector components in addition to the vector components p_μ.

Therefore, the rainbow metric corresponding to (2.42) involves now the Clifford scalar coordinate σ, which is the canonical conjugate variable to the Clifford scalar momentum π, and the x^μ coordinates (canonical to the momentum p_μ). The rainbow metric is now given by an effective $D + 1$-dim metric

$$d\sigma^2 \ + \ g_{\mu\nu} \ dx^\mu \ dx^\nu \ = \ d\sigma^2 \ + \ (\ 1 \ + \lambda_1 \ p^2 \ + \lambda_2 \ p^4 \ + \lambda_3 \ p^6 \ )^{-1} \ \eta_{\mu\nu} \quad (2.43)$$

a modified Weyl-Heisenberg algebra can defined as follows

$$[\sigma,\ \pi] \ = \ i\hbar, \ \ [x^\mu,\ p^\nu] \ = \ i\ \hbar\ g^{\mu\nu}(p^2) \ = \ i\ \hbar\ f^2(p^2)\ \eta^{\mu\nu} \qquad (2.45)$$

and which *emerged* from taking a slice in C-space displayed by eqs-(2.41).

If one were to set $\pi = 0$ in (2.40) it leads to a quartic algebraic equation for p^2 and that will fix the numerical values of p^2 given by the four roots of the algebraic equation. The four roots are themselves functions of $\mathcal{M}^2$ and the parameters $\lambda_1, \lambda_2, \lambda_3$. The rainbow function squared $f^2(p^2)$ will have $fixed$ numerical values instead of being a variable function and hence the rainbow metric $g_{\mu\nu}$ will be just trivially proportional to the Minkowski metric $\eta_{\mu\nu}$ and will not modify the Weyl-Heisenberg algebra since one could reabsorb the constant of proportionality into $\hbar$. For this reason one must retain π and σ in eqs-(2.40, 2.42).

However, as it occurred in the previous section there is a caveat : by lowering indices one ends up with $[x_\mu,\ p_\nu] = i\hbar\ g_{\mu\nu}(p^2)\ +\ .....\neq i\hbar g_{\mu\nu}(p^2)$ leading to an asymmetry in the functional form of the modified Weyl-Heisenberg algebra unless the metric is restricted to obey similar conditions to eq-(2.2) . In this Lorentz covariant case the nontrivial metric must be restricted to be of the form

$$g_{\mu\nu} \ = \ \alpha \int (\ p_\rho\ \eta_{\mu\nu} - \frac{p_\rho\ p_\mu\ p_\nu}{p^2}\ )\ dp^\rho \ = \ \frac{\alpha}{2} \int (\ \eta_{\mu\nu} - \frac{p_\mu\ p_\nu}{p^2}\ )\ dp^2;\ \ p^2 \equiv \eta_{\rho\sigma}\ p^\rho\ p^\sigma \quad (2.46)$$

The constant of integration (which can be set to $\frac{2}{\alpha}$) in (2.46) is what generates the constant terms in the expression for $g_{\mu\nu} = \eta_{\mu\nu}(\ 1\ +\ \frac{\alpha}{2}\ p^2)\ +\ ....$, so that the modified Weyl-Heisenberg algebra will have a symmetric functional form $[x^\mu, p^\nu] = i\hbar g^{\mu\nu}$; $[x_\mu, p_\nu] = i\hbar g_{\mu\nu}$ and furnish the generalized uncertainty relations for all the components of the coordinates and momenta

$$\Delta x_\mu\ \Delta p_\nu \ \ \geq\ \frac{\hbar}{2}\ |\ < g_{\mu\nu} >\ |, \quad \Delta x^\mu\ \Delta p^\nu \ \ \geq\ \frac{\hbar}{2}\ |\ < g^{\mu\nu} >\ |. \qquad (2.47)$$

To sum up, one may argue that because (i) one should not restrict the form of the metric like one does in eqs-(2.46), and (ii) since the metric in eqs-(2.5, 2.6) is flat there are no problems with raising and lowering indices inside commutators, therefore we find it more appealing to follow the approach taken in [24] rather than the rainbow metric approach in order to derive the generalized stringy uncertainty relations. A curved momentum space was also studied within the context of DSR by [12]. Finsler geometry is the proper arena to study metrics which depend on both coordinates and velocities/momenta. The role of Clifford algebras in Finsler geometry and Noncommutative geometry has been studied by [15].

To finalize, we may add that in the most general case one must recur to a (curved) phase space and a *matrix*-valued Planck "constant" $\hbar_{\mu\nu}(x_\rho, p_\rho)$ which is a function of both coordinates and momenta. The most general Weyl-Heisenberg algebra is then given by

$$[x_\mu,\ p_\nu] \ = \ i\ \hbar_{\mu\nu}(x_\rho, p_\rho) \ = \ i\hbar\ \Theta_{\mu\nu}(x_\rho, p_\rho) \qquad (2.48)$$

However, since one must obey the Jacobi identities among the commutators, one must have in the most general case that the coordinates and momenta must be noncommutative

$$[x_\mu,\ x_\nu]\ \neq\ 0,\ \ [p_\mu,\ p_\nu]\ \neq\ 0 \tag{2.49}$$

To simplify matters we may chose $\Theta_{\mu\nu} = g(p^2)\eta_{\mu\nu}$; $[p_\mu, p_\nu] = 0$ but $[x_\mu, x_\nu] \neq 0$ and whose physical motivation lies in the fact that the tangent space to a curved-momentum space can be identified with spacetime. A flat spacetime (zero curvature) is compatible with commuting momentum $[p_\mu, p_\nu] = [i\hbar\nabla_{x^\mu}, i\hbar\nabla_{x^\nu}] = 0$. Whereas $[x_\mu, x_\nu] = [i\hbar\nabla_{p^\mu}, i\hbar\nabla_{p^\nu}] \neq 0$ is consistent with a non-zero curvature in momentum space.

3 Modified Black Hole Entropy-Area Relation

Let us begin with the first law of black hole thermodynamics

$$dS\ =\ \frac{dM}{T}\ \Rightarrow\ S\ =\ \int\ \frac{dM}{T}\ =\ \int\ \frac{dM}{dE}\ \frac{dE}{T} \tag{3.1}$$

and write the rainbow metric modifications of the Schwarzchild metric as follows

$$ds^2\ =\ -\ \frac{1}{f^2(E/E_P)}\ (1 - \frac{2GM}{r})\ dt^2\ +\ (1 - \frac{2GM}{r})^{-1}\ \frac{dr^2}{g^2(E/E_P)}\ +\ \frac{r^2 d\Omega^2}{g^2(E/E_P)} \tag{3.2}$$

such that the modified Hawking temperature becomes [26]

$$T\ =\ \frac{1}{2\pi}\ \sqrt{\ -\ \frac{1}{4}\ g^{rr}\ g^{tt}\ (\frac{dg_{tt}}{dr})^2}\ (r = 2GM)\ =\ \frac{g(E/E_P)}{f(E/E_P)\ 8\pi GM} \tag{3.3}$$

In natural units $\hbar = c = k_B = 1$, the standard uncertainty relation is $\Delta x\ \Delta p\ \geq\ \frac{1}{2}$. We shall follow now the arguments of [27]. If one sets the position uncertainty Δx of a massless particle (photon) to be given by a quantity of the order of the Schwarzschild horizon radius $\Delta x = \lambda(2GM)$, and equates the momentum uncertainty Δp of the massless quantum particle near the horizon of a black hole to be equal to the black hole temperature $T = E$ (since at thermodynamic equilibrium the temperature of the particle is equal to the black hole temperature) one has then the mass-temperature-energy relation $\lambda(2GM) = (1/2T) = (1/2E)$, so that $dM/dE = -(1/4\lambda GE^2)$ and the integral (3.1) becomes, after writing $8\pi GM = (4\pi/2\lambda E)$,

$$S\ =\ -\ \int\ \frac{4\pi}{(2\lambda)^2}\ \frac{f(E/E_P)}{2G\ g(E/E_P)}\ \frac{dE}{E^3} \tag{3.4}$$

The above expression is the modified black-hole entropy relation due to the contribution of the rainbow functions to the Schwarzschild metric. When the rainbow functions obey the condition $f(E/E_P) = g(E/E_P)$, after recurring to the mass-temperature-energy relation $\lambda(2GM) = (1/2T) = (1/2E)$, the integral (3.4) reduces to

$$S = \frac{4\pi}{4G}\ \frac{1}{(2\lambda)^2 E^2}\ =\ \frac{4\pi}{4G}\ (2GM)^2\ =\ \frac{A}{4L_P^2} \tag{3.5}$$

and which is the Bekenstein-Hawking entropy given by one quarter of the area of the spherical horizon in Planck scale units $G = L_P^2 = (1/E_P^2)$ (when $\hbar = c = 1$). If, and only if, the proportionality constant is $\lambda = 2\pi$ the relation $\lambda(2GM) = 4\pi GM = (1/2T) \Rightarrow 8\pi GM = 1/T$, coincides with the Hawking temperature expression.

For example, the author [26] chose the rainbow functions

$$f(E/E_P) \;=\; 1, \;\; g(E/E_P) \;=\; \sqrt{1 \;-\; \xi \;(\frac{E}{E_P})^n}, \;\; \xi = constant \tag{3.6}$$

and in the case $n = 4$, the rainbow modified entropy is [26]

$$S \;=\; \pi \; \frac{\sqrt{16M^4 \;-\; \xi \; E_P^4}}{E_P^2} \tag{3.7}$$

which reduces to the Bekenstein-Hawking entropy (3.5) when the numerical parameter $\xi = 0$. The physical relevance of the rainbow modified entropy is that it is *zero* at a *nonzero* value of M and which represents the information contained in the black hole remnant. There is *no* total and catastrophic evaporation of the black hole since the black hole reaches a zero temperature at a nonzero mass value (the remnant), and a maximum finite temperature at a finite value of the mass, see [26] for details.

The stringy uncertainty relation approach to the modifications of the black hole area-entropy relation [27] starts from the modified uncertainty relation (in natural units $\hbar = c = k_B = 1$)

$$\Delta x \; \Delta p \;\geq\; \frac{1}{2} \; (\; 1 \;+\; \beta \; (\Delta p)^2 \;) \tag{3.8}$$

after equating $\Delta p = T = E$ and setting $\Delta x = \lambda(2GM)$, it leads to the mass-energy relation

$$M \;\geq\; \frac{1}{4\lambda G} \; (\; \frac{1}{E} \; + \beta \; E \;) \tag{3.9a}$$

so that

$$\frac{dM}{dE} \;=\; \frac{1}{4\lambda G} \; (\beta \;-\; \frac{1}{E^2}) \tag{3.9b}$$

the entropy integral (3.1) becomes, after setting $T = E$, $\lambda = 2\pi$ and including the integration constant as a term proportional to the logarithm of E_P,

$$S \;=\; \int \; \frac{1}{4\lambda G} \; (\beta E^2 \;-\; 1) \; \frac{dE}{E^3} \;=\; \frac{1}{8\pi G} \; (\; \beta \; |ln(\frac{E}{E_P})| \;+\; \frac{1}{2E^2} \;) \tag{3.10}$$

After expressing E in terms of M obtained from solving the quadratic equation in (3.9), and inserting $E = E(M)$ into eq-(3.10) , one recovers the logarithmic *corrections* to the entropy-area relation [27] $S(A) = \frac{A}{4L_P^2} + ...$, upon equating $A = 4\pi(2GM)^2$.

One may notice that the functional expression (3.4) for the rainbow modified black hole entropy is *not* the same as the stringy-uncertainty-inspired black hole entropy (3.10).

If one equates the integrands (3.4, 3.10), and sets $2\lambda = 4\pi$, it gives

$$\frac{f(E/E_p)}{g(E/E_P)} \ = \ 1 \ - \ \beta E^2 \tag{3.11}$$

as mentioned earlier, the parameter β is proportional to $(E_P)^{-2}$: $\beta = b(E_P)^{-2}$ so that the equality (3.11) becomes

$$\frac{f(E/E_p)}{g(E/E_P)} \ = \ 1 \ - \ \beta E^2 \ = \ 1 \ - b \ (\frac{E}{E_P})^2 \tag{3.12}$$

and it will constrain the functional form of the *ratio* of the two rainbow functions. This will clearly restrict considerably the infinity of choices for the two rainbow functions. In the low energy limit the rainbow functions obey $f(E/E_P) \to 1; g(E/E_P) \to 1$, and combined with $(\frac{E}{E_P})^2 \to 0$, one does find a clear agreement among the left and right hand side of eq-(3.12). More general uncertainty relations like those given by eqs-(2.29, 2.30) will lead to more complicated integrals than (3.10) and more complicated expressions for the ratios $\frac{f(E/E_p)}{g(E/E_P)}$ in (3.12).

We conclude with some final remarks. The theory of Scale Relativity proposed by Nottale [11] is based on a minimal observational length-scale, the Planck scale, as there is in Special Relativity a maximum speed, the speed of light, and deserves to be looked within the Clifford algebraic perspective. In the quantization program of gravity a key role must be played by quantum Clifford-Hopf algebras since the latter q-Clifford algebras naturally contain the κ-deformed Poincare algebras [18], [19], which are essential ingredients in the formulation of DSR within the context of Noncommutative spaces. The Minkowski spacetime quantum Clifford algebra structure associated with the conformal group and the Clifford-Hopf alternative κ-deformed quantum Poincare algebra was investigated [17]. The resulting algebra is equivalent to the deformed anti-de Sitter algebra $U_q(so(3,2))$, when the associated Clifford-Hopf algebra is taken into account, together with the associated quantum Clifford algebra and a (not braided) deformation of the periodicity Atiyah-Bott-Shapiro theorem [21].

Acknowledgments

We are indebted to M. Bowers for assistance.

References

[1] C. Castro and M. Pavsic, "The Extended Relativity Theory in Clifford-spaces", Progress in Physics, **vol. 1** (2005) 31. Phys. Letts **B 559** (2003) 74. Int. J. Theor. Phys **42** (2003) 1693.

[2] C. Castro, "The Extended Relativity Theory in Clifford Phase Spaces and Modifications of Gravity at the Planck/Hubble scales" , to appear in Advances in Applied Clifford Algebras.

[3] M. Pavsic, Found. of Phys. 33 (2003) 1277.

M. Pavsic, *"The Landscape of Theoretical Physics : A Global View, from point particles to the brane world and beyond, in search of a Unifying Principle'"* , (Fundamental Theories of Physics, vol. 19, Kluwer Academic Publishers, Dordrecht, Boston, London, 2001).

[4] C. Castro, "Novel Physical Consequences of the Extended Relativity in Clifford Spaces", submitted to Advances in Applied Clifford Algebras. [viXra.org/abs/1401.0237].

C.Castro, "Why a Minimal Length follows from the Extended Relativity Principle in Clifford Spaces" submitted to Modern. Phys. **A** Letters.

[5] C. Castro, "The many novel physical consequences of Born's Reciprocal Relativity in Phase Spaces", Int. Journal of Mod. Phys. A 26, no. 21 (2011) 3653-3678

C. Castro, On Born's Deformed Reciprocal Complex Gravitational Theory and Noncommutative Gravity Phys Letts **B 668** (2008) 442.

C. Castro, "Born's Reciprocal Gravity in Curved Phase-Spaces and the Cosmological Constant Found. Phys. **42**, issue 8 (2012) 1031.

[6] C. Castro, "Superluminal particles and the Extended Relativity Theories", Foundations of Physics **vol 42**, issue 9 (2012) 1135.

[7] M. Born, Proc. Royal Society **A 165** (1938) 291. Rev. Mod. Physics **21** (1949) 463.

[8] E. Caianiello, "Is there a maximal acceleration?", Lett. Nuovo Cimento 32 (1981) 65.

[9] S. Low, Jour. Phys **A** Math. Gen **35** (2002) 5711. Il Nuovo Cimento **B 108** (1993) 841; Found. Phys. **36** (2007) 1036. J. Math. Phys **38** (1997) 2197.

[10] P. Jarvis and S. Morgan, Found. Phys. Lett **19** (2006) 501.

[11] L. Nottale, *Scale Relativity And Fractal Space-Time: A New Approach to Unifying Relativity and Quantum Mechanics* (2011 World Scientific Publishing Company).

L. Nottale, *Fractal Space-Time and Micro-physics* (World Scientific, May 1993).

Scale Relativity, http://en.wikipedia.org/wiki/Laurent_Nottale

[12] G. Amelino-Camelia, Int. J. Mod. Phys **D 11** (2002) 35. Int. J. Mod. Phys **D 11** (2002) 1643.

G. Amelino-Camelia, L. Freidel, J. Kowalski-Glikman and L. Smolin, "The principle of relative locality" arXiv.org : 1101.0931.

L. Freidel and L. Smolin, "Gamma ray burst delay times probe the geom- etry of momentum space" arXiv.org : 1103.5626

[13] J. Magueijo and L. Smolin, Class. Quant. Grav. 21, 1725 (2004)

G. Amelino-Camelia, J. R. Ellis, N. E. Mavromatos and D. V. Nanopoulos, Int. J. Mod. Phys. A 12, 607 (1997)

[14] C. Castro, "Progress in Clifford Space Gravity", Advances in Applied Clifford Algebras **vol 23**, Issue 1 (2013),

[15] S. Vacaru, "Finsler Branes and Quantum Gravity Phenomenology with Lorentz Symmetry Violations", Class. Quant. Grav. **28** : 215991 (2011).

S. Vacaru, "Spectral Functionals, Nonholonomic Dirac Operators, and Noncommutative Ricci Flows" J. Math. Phys. **50** : 073503 (2009).

S. Vacaru, P. Stavrinos, E. Gaburov, and D. Gonta, *Clifford and Riemann-Finsler Structures in Geometric Mechanics and Gravity*, Differential Geometry and Dynamical Systems, Monograph no. **7** (Geometry Balkan Press, 2006).

[16] C. Castro, Foundations of Physics. **8** (2000) 1301.

[17] R. da Rocha, A. E. Bernardini and J. Vaz Jr, Int. J. Geom. Meth. Mod. Phys. **7** (2010) 821.

[18] J. Lukierski, A. Nowicki, H. Ruegg and V. Tolstoy, Phys. Letts **B 264** (1991) 331.

J. Lukierski, H. Ruegg and W. Zakrzewski, Ann. Phys. **243** (1995) 90.

[19] S. Majid and H. Ruegg, Phys. Letts **B 334** (1994) 348.

[20] P. Achieri, M. Dimitrijevic, F. Meyer and J. Wess, "Noncommutative Geometry and Gravity" [arXiv : hep-th/0510059] and references therein.

[21] B. Fauser, "A Treatise on Quantum Clifford Algebras" arXiv : math/0202059.

[22] C. Castro, J. Phys **A** : Math. Theor. **43** (2010) 365201

[23] A. Kempf and G. Mangano, Phys. Rev. **D 55** (1997) 7909.

A. Kempf, "Minimal Length Uncertainty Relation and Ultraviolet Regularisation", hep-th/9612084.

[24] C. Castro, "The Minimal Length Stringy Uncertainty Relations follow from Clifford Space Relativity" submitted to Phys. Letts. **B** , March 2014.

[25] F. Girelli, S. Liberati and L. Sindoni, "Planck-scale Modfied Dispersion Relations and Finsler Geometry" gr-qc/0611024.

[26] A. Ali, "Black Hole Remnant from Gravity's Rainbow" arXiv : 1402.5320.

[27] R. Adler, P. Chen and D. Santiago, Gen. Rela. Grav. **33** (2001) 2101.

S. Das and E. Vagenas, Phys. Rev. Lett. **101** (2008) 221301.

A. Dutta and S. Gangopadhyay, "Remnant mass and entropy of black holes and modified uncertainty principle" arXiv : 1402.2133.