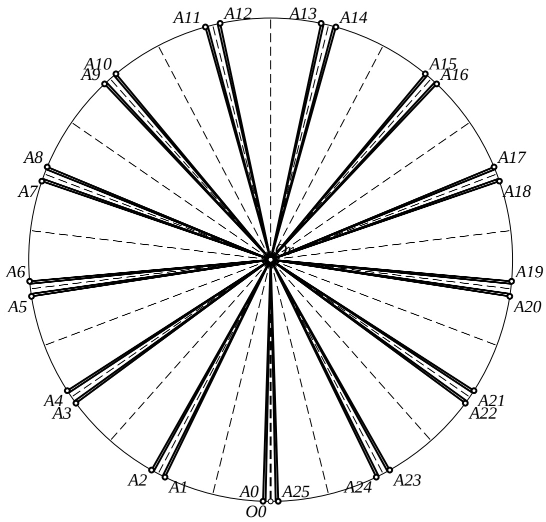
Goldbach, Legendre, Brocard, Elementary considerations on few conjectures.



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Used notations.

0.1 Reminder about the notations used.

For the purpose of this study, we utilise the usual mathematical notations and symbols. However it is suitable to define precisely some of them.

In propositional calculus, a **proposition** P is either true or false by definition. As the purpose of mathematics is to logically link propositions from one to another to derive a conclusion, itself fromulated as a proposition, we will need the logical connectors

- **negation** symbol $\neg$
- **conjonction** symbol "and" $\wedge$
- **disjonction** "inclusive or" symbol $\vee$

as well as the relation symbols

- **implication** symbol $\Longrightarrow$
- **equivalence** symbol $\Longleftrightarrow$

We will also resort to utilise the following logical quantifiers

- **universal** "For all..." $\forall$
- **existential** "There exists at least one..." $\exists$
- **existential** "There exists one and only one..." $\exists!$

Usual notations used in Set Theory will be utilised. The **membership** symbol, and its negation, of an element a contained in a set $\mathbb{A}$ are respectively denoted $\in$ and $\notin$. Also, the **inclusion** symbol of a set $\mathbb{A}$ in a set $\mathbb{B}$ and its negation are respectively denoted $\subset$ and $\not\subset$. Lastly, depending on our needs, we denote the **intersection** and **union** operators of sets respectively

- $\bigcap$ or $\cap$
- $\bigcup$ or $\cup$.

Let $\mathbb{A}$ and $\mathbb{B}$ be two sets, not necessarily distinct, and let $a \in \mathbb{A}$ and $b \in \mathbb{B}$ any two elements of these two sets, the ordered pair (a, b) belongs to the set $\mathbb{A} \times \mathbb{B}$, usually called the **Cartesian product** of the set $\mathbb{A}$ by the set $\mathbb{B}$. This notion of Cartesian product can be of course extended to a product of more than two sets.

In a subset $\mathbb{A}_j \times \mathbb{B}_k$ of the Cartesian product $\mathbb{A} \times \mathbb{B}$, we can define the **binary relation** $\mathcal{R}$

$$(\forall a \in \mathbb{A})\,(\forall b \in \mathbb{B}) \quad ((a\mathcal{R}b) \Longleftrightarrow ((a, b) \in \mathbb{A}_j \times \mathbb{B}_k))$$

This definition leads rather naturally to the notion of **equivalence relation**. A binary relation $\mathcal{R}$ on a set $\mathbb{A}$ is an equivalence relation if and only if

$$\begin{aligned} (\forall a \in \mathbb{A}) \quad & (a\mathcal{R}a) \\ \left(\forall (a, b) \in \mathbb{A}^2\right) \quad & ((a\mathcal{R}b) \Longleftrightarrow (b\mathcal{R}a)) \\ \left(\forall (a, b, c) \in \mathbb{A}^3\right) \quad & ((a\mathcal{R}b) \wedge (b\mathcal{R}c) \Longrightarrow (a\mathcal{R}c)) \end{aligned}$$

The definition of the equivalence relations leads in its turn to the one of **equivalence class**. The equivalence class of an element $a \in \mathbb{A}$ generated by the equivalence relation $\mathcal{R}$ is the set, which we denote $\mathcal{R}\,(a)$

$$((\forall b \in \mathbb{A}) \quad (b \in \mathcal{R}\,(a))) \Longleftrightarrow (a\mathcal{R}b)$$

and we have

$$\mathcal{R}\,(a) \subset \mathbb{A}$$

The set of equivalence classes $\mathcal{R}\,(a_j)$ generated by the equivalence relation $\mathcal{R}$ on the set $\mathbb{A}$ est son **quotient set**, which is denoted $\mathbb{A}/\mathcal{R}$.
The set $\mathbb{A}$ has a number of elements, finite or infinite, and in this last case, countable or uncountable. This number is defined as the **cardinal** of the set and denoted $|\mathbb{A}|$.
We will be interested more specifically in the following sets

- $\mathbb{N}$ Set of the **natural integers**.
- $\mathbb{Z}$ Set of the **rational integers**.
- $\mathbb{Q}$ Set of the **rationnal numbers**.
- $\mathbb{R}$ Set of the **real numbers**.

In the sets $\mathbb{Z}$, $\mathbb{Q}$ et $\mathbb{R}$, the elements, in other word numbers, other than the null element can be positive or negative. Each set $\mathbb{A}$ chosen among these sets contains the subset of its negative numbers, which we denote $\mathbb{A}^-$, the null element, which we denote 0 and the subset of its positive numbers, which we denote $\mathbb{A}^+$. We have

$$\mathbb{A} = \mathbb{A}^- \cup \{0\} \cup \mathbb{A}^+$$

The notion of **absolute value** follows naturally

$$\left(\forall a \in \mathbb{A}^-\right) \quad (|a| = -a) \left(\forall a \in \mathbb{A}^+\right) \quad (|a| = a)$$

As well, for each set $\mathbb{A}$, chosen among any of the here-above mentioned sets, we will denote the set of its non zero $\mathbb{A}^*$

$$(a \in \mathbb{A}^*) \Longleftrightarrow (a \neq 0)$$

and

$$\mathbb{A} = \mathbb{A}^* \cup \{0\}$$

We will use the internal binary operations usually applied to the elements of these sets, the numbers. These operations are denoted

- $+$ for the **addition**
- $\times$ for the **multiplication**.

However, we will most of the time omit this symbol, as is customary.
We will also use the notations

- $-$ for the **soustraction**
- $/$ for the **division**.

After reminding the definition of the **Euclidean division** in the set $\mathbb{Z}$

$$(\forall a \in \mathbb{Z}) (\forall b \in \mathbb{Z}) (\exists q \in \mathbb{Z}) (\exists r \in \mathbb{Z}) \quad (a = bq + r)$$

we are using, whenever $r = 0$, the symbol $|$ for the **exact division** in this same set and we denote

$$((\forall a \in \mathbb{Z}^*) (\forall b \in \mathbb{Z}^*) \quad (b|a)) \Longleftrightarrow ((\exists! c \in \mathbb{N}^*) \quad (a = bc))$$

The Euclidean division by a given prime number p_n in $\mathbb{Z}$ leads to the definition of the equivalence relation, which we denote $\mathcal{R} = p_n$

$$(\forall a \in \mathbb{Z}) (\forall b \in \mathbb{Z}) \quad (a p_n b) \Longleftrightarrow p_n | (a - b)$$

This equivalence relation generates in its turn p_n equivalence classes, as the remainder r of Euclidean division by the prime number p_n can take p_n values among the integers 0, 1, 2, $\cdots$, p_{n-2} et p_{n-1}. These p_n equivalence classes are the elements of the quotient set, which we denote

$$\mathbb{Z}/p_n\mathbb{Z} = \{0, 1, 2, \cdots, p_{n-2}, p_{n-1}\}$$

We utilise the usual notations of the **congruence theory**

$$(a \in \mathbb{Z}) (b \in \mathbb{Z}) (c \in \mathbb{Z}^*) \quad (a \equiv b \quad [c] \Longleftrightarrow c | a - b)$$

The interval, with the two elements a et b of a set $\mathbb{K}$ as endpoints are denoted

- $]a, b[$ for an **open**
- $[a, b]$ for a **closed**
- $]a, b]$ et $[a, b[$ for the **semi-open**.

We will make use of **functions** in their usual definition. Let $\mathbb{K}$ and $\mathbb{K}'$ be two sets and F the set of functions f, which map an element k of $\mathbb{K}$ to an element k' of $\mathbb{K}'$. We denote

$$\begin{aligned} f\mathbb{K} &\longrightarrow \mathbb{K}' \\ k &\longmapsto k' = f(k) \end{aligned}$$

In what follows, the sets $\mathbb{K}$ and $\mathbb{K}'$ will be most of the time the set $\mathbb{R}$ itself, or one of its subsets.

Introduction and preliminary remarks.

Prime numbers appear to be distributed randomly within the set of natural numbers. It was proved long ago that, given an interval $[0, p_k[$ in the set of real numbers $\mathbb{R}$, where p_k and p_{k+1} are two consecutive prime numbers, every natural integer belonging to the interval $[p_k, p_{k+1}^2[$ taken in $\mathbb{R}$ is either prime or a multiple of at least one of the prime numbers belonging to the interval $[0, p_k[$. Besides, a theorem, postulated by Joseph Bertrand and proved by Pafnuty Tchebychev [1] [2] states that

Theorem 1 ***of Bertrand Tchebychev*** *For all natural integer $n > 1$, there exists at least a prime integer that belongs to the interval $]n, 2n]$.*

Also, the definition of the congruence between two numbers a and b, the two of them being non zero, modulo a third natural integer c, non zero itself, which we usually write as follows

$$(a \in \mathbb{N}^*)\,(b \in \mathbb{N}^*)\,(c \in \mathbb{N}^*) \quad ((a \equiv b \quad [c]) \Longleftrightarrow (c|a-b))$$

leads us to consider that a function F_c could exist such as

$$\begin{aligned} F_c : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto F_c(x) \end{aligned}$$

for which

$$(a \in \mathbb{N}^*)\,(b \in \mathbb{N}^*)\,(c \in \mathbb{N}^*) \quad (F_c(a) = F_c(b) \Longleftrightarrow c|a-b)$$

Such a function is evidently periodic, with period C. We endeavour in the following pages to create one possible of these functions F_c and to study some of its property, emphasizing on **symetry** properties in particular.
Then, in the following chapters, we will first consider the strong Goldbach conjecture

Conjecture 1 ***strong of Goldbach*** *Every even natural integer $n \geqslant 4$ is the sum of two prime numbers.*

We will also try and prove the following theorem, by utilising some properties of the periodic functions $S_{p_{n-1}}$ and S_{p_n}, which we will introduce later and the periods of which will be respectively denoted TS_{p_n} and $TS_{p_{n-1}}$

Theorem 2 *For all prime integer p_n and its associated function S_{p_n}, let the set of the intervals*

$$[kp_n, (k+1)\, p_n[$$

where k is any natural integer, and let M_1 be the natural integer

$$M_1 = \frac{1}{4} TS_{p_{n-1}}$$

then, for all $k < M_1$, there exists at least one natural integer

$$a \in [kp_n, (k+1)\, p_n[$$

such that

$$S_{p_n}(a) \neq 0$$

which can be otherwise formulated

$$(\forall k \in \mathbb{N})\, (k < M_1)\, (\exists a \in ([kp_n, (k+1)\, p_n[\cap \mathbb{N})) \quad (S_{p_n}(a) \neq 0)$$

One consequence of this theorem is another theorem that we enunciate hereunder

Theorem 3 ***of Bertrand-Tchebychev extended*** *Given a prime number p_n, there exists at least one prime number in each interval*

$$[kp_n, (k+1)\, p_n[$$

for each non zero natural integer k such that

$$(k+1)\, p_n < p_{n+1}^2$$

This theorem is somewhat similar to the Bertrand-Chebyshev theorem.

These results will enable us, to finish with, to draw some conclusions on two conjectures, one due to Adrien-Marie Legendre [3].

Conjecture 2 ***of Legendre*** *For all natural integer $n \geqslant 2$, there exists at least a prime integer that belongs to the interval $[n^2, (n+1)^2]$.*

the other to Henri Brocard [4].

Conjecture 3 ***of Brocard*** *For all prime integer $p_n \geqslant 2$, there exists at least four prime integers that belong to the interval $[p_n^2, p_{(n+1)}^2]$.*

Definitions.

0.2 Definitions.

We define some sets and some functions that we will have to use.

0.2.1 Finite sets π_{p_n} of prime numbers

let π_{p_n} be the set that contains all the prime numbers p_j (distinct from 1) and less than or equal to a given prime number p_n

$$\pi_{p_n} = \{p_j | (c|p_j \Longleftrightarrow c \in \{1, p_j\}) \wedge (p_j \leqslant p_n)\}$$

The set π_{p_n} is totally ordered, within the definition of the relation $<$.We note that it is also a well ordered set, as it has a least element denoted $p_1 = 2$. So we have

$$\begin{aligned} p_1 &= 2 \\ p_2 &= 3 \\ p_3 &= 5 \\ p_4 &= 7 \\ &\ldots \\ p_n &= \sup \pi_{p_n} \end{aligned}$$

We pose $|(\pi_{p_n})| = n$

0.2.2 The elementary functions

We need to define some functions, some properties of which will be put forward in our study.

The functions s_{a,p_j} et $\overline{s_{a,p_j}}$.

For each prime number $p_j \in \pi_{p_n}$, we define here-under the functions s_{a,p_j} and $\overline{s_{a,p_j}}$, where $a \in \mathbb{N}$

$$\begin{aligned} s_{a,p_j} : \mathbb{R} &\longrightarrow [-1, 1] \\ x &\longmapsto s_{a,p_j}(x) \end{aligned}$$

with

$$s_{a,p_j}(x) = \sin\frac{\pi}{p_j}(a+x)$$

This function vanishes for each and every $(a+x)$ multiple of p_j.

$$\begin{aligned}\overline{s_{a,p_j}} : \mathbb{R} &\longrightarrow [-1,1] \\ x &\longmapsto \overline{s_{a,p_j}}(x)\end{aligned}$$

with

$$\overline{s_{a,p_j}}(x) = \sin\frac{\pi}{p_j}(a-x)$$

This function vanishes for each and every $(a-x)$ multiple of p_j.

The periods of these two functions, which we respectively denote Ts_{a,p_j} and $T\overline{s_{a,p_j}}$ are both equal to $2p_j$.

We will denote for $a=0$

$$s_{0,p_j}(x) = s_{p_j}(x) = \sin\frac{\pi}{p_j}(x)$$

and for $a=2m$

$$\overline{s_{2m,p_j}}(x) = \sin\frac{\pi}{p_j}(2m-x)$$

The functions c_{a,p_j} and $\overline{c_{a,p_j}}$.

Similarly, we define the functions c_{a,p_j} et $\overline{c_{a,p_j}}$ respectively as

$$\begin{aligned}c_{a,p_j} : \mathbb{R} &\longrightarrow [-1,1] \\ x &\longmapsto c_{a,p_j}(x)\end{aligned}$$

with

$$c_{a,p_j}(x) = \cos\frac{\pi}{p_j}(a+x)$$

This function vanishes for each and every $(a+x)$ odd multiple of $\frac{1}{2}p_j$.

$$\begin{aligned}\overline{c_{a,p_j}} : \mathbb{R} &\longrightarrow [-1,1] \\ x &\longmapsto \overline{c_{a,p_j}}(x)\end{aligned}$$

with

$$\overline{c_{a,p_j}}(x) = \cos\frac{\pi}{p_j}(a+x)$$

This function vanishes for each and every $(a-x)$ odd multiple of $\frac{1}{2}p_j$.

The periods of these two functions, which we respectively denote Tc_{a,p_j} and $T\overline{c_{a,p_j}}$ are both equal to $2p_j$.

We will denote for $a = 0$

$$c_{0,p_j}(x) = c_{p_j}(x) = \cos\frac{\pi}{p_j}(x)$$

and for $a = 2m$

$$\overline{c_{2m,p_j}}(x) = \cos\frac{\pi}{p_j}(2m - x)$$

It might be useful to recall that the **sin** and **cos** functions are respectively odd and even.

0.2.3 The product functions.

We need to define the product functions of a finite number of functions s_{a,p_j}. We so define

$$\begin{aligned} S_{p_n} : \mathbb{R} &\longrightarrow [-1, 1] \\ x &\longmapsto S_{p_n}(x) \end{aligned} \tag{1}$$

with

$$S_{p_n}(x) = \prod_{j=1}^{j=n} \sin\frac{\pi}{p_j}(x) = \prod_{j=1}^{j=n} s_{p_j}(x)$$

where the prime number p_j belongs to the set π_{p_n}, which we define as as the **reference set** of the function S_{p_n}.

Similarly, let $\overline{S_{2m,p_n}}$ be the function

$$\begin{aligned} \overline{S_{2m,p_n}} : \mathbb{R} &\longrightarrow [-1, 1] \\ x &\longmapsto \overline{S_{2m,p_n}}(x) \end{aligned}$$

with

$$\overline{S_{2m,p_n}}(x) = \prod_{j=1}^{j=n} \sin\frac{\pi}{p_j}(2m - x) = \prod_{j=1}^{j=n} \overline{s_{2m,p_j}}(x)$$

We note that

$$(2m - x = X) \Longleftrightarrow \left(\overline{S_{2m,p_n}}(x) = \prod_{j=1}^{j=n} \sin\frac{\pi}{p_j}(2m - x) = \prod_{j=1}^{j=n} \sin\frac{\pi}{p_j}(x) = S_{p_n}(X)\right)$$

and hence

$$T\overline{S_{2m,p_n}} = TS_{p_n}$$

These two functions are sharing interesting properties of symmetry.

Finally, we construct a third function G_{m,p_n}

$$\begin{aligned} G_{m,p_n} &: \mathbb{R} \longrightarrow [-1,1] \\ x &\longmapsto G_{m,p_n}(x) \end{aligned}$$

with

$$\begin{aligned} G_{m,p_n}(x) &= S_{p_n}(x) \times \overline{S_{2m,p_n}}(x) \\ &= \left(\prod_{j=1}^{j=n} s_{p_j}(x)\right) \left(\prod_{j=1}^{j=n} \sin\frac{\pi}{p_j}(2m-x)\right) \\ &= \prod_{j=1}^{j=n} s_{p_j}(x)\,\overline{s_{2m,p_j}}(x) \end{aligned}$$

We will utilise as well the product functions of a finite number of functions c_{a,p_j}. We so define

$$\begin{aligned} C_{p_n} &: \mathbb{R} \longrightarrow [-1,1] \\ x &\longmapsto C_{p_n}(x) \end{aligned}$$

with

$$C_{p_n}(x) = \prod_{j=1}^{j=n} \cos\frac{\pi}{p_j}(x) = \prod_{j=1}^{j=n} c_{p_j}(x)$$

where the prime number p_j belongs to the set π_{p_n}, which we define as as the **reference set** of the function C_{p_n}.

We are now going to study these various functions.

Chapter 1

Some properties of the function S_{p_n}.

1.1 Purpose of the chapter

Study of some properties of the function S_{p_n}. A special property of functions S_{p_n} when $n \leq 5$. A simple explanation of the distribution of some prime numbers less than 49.

1.2 Some properties of the function S_{p_n}

We recall the definition of the fumction S_{p_n}

$$\begin{aligned} S_{p_n} : \mathbb{R} &\longrightarrow [-1, 1] \\ x &\longmapsto S_{p_n}(x) \end{aligned}$$

with

$$S_{p_n}(x) = \prod_{j=1}^{j=n} s_{p_j}(x)$$

and

$$s_{p_j}(x) = \sin \frac{\pi}{p_j}(x)$$

1.2.1 Period and parity

The period of the function S_{p_n}, which we denote TS_{p_n}, is two times the product of the periods Ts_{p_j}, where p_j are all the elements of the set π_{p_n}. We then have

$$TS_{p_n} = 2 \times \prod_{j=1}^{j=n} p_j$$

The function S_{p_n} is the product of functions sin and is odd when n is odd and even when n is even. Inside the interval $[0, TS_{p_n}[$, we note that the function S_{p_n} vanishes when x equals all the non-prime integer, as well as all the elements of π_{p_n}. In particular

$$\begin{aligned} S_{p_n}(0) &= S_{p_n}(\frac{TS_{p_n}}{4}) \\ &= S_{p_n}(\frac{TS_{p_n}}{2}) \\ &= S_{p_n}(\frac{3TS_{p_n}}{4}) \\ &= S_{p_n}(TS_{p_n}) \\ &= 0 \end{aligned}$$

For instance, we show the respective graphs of the functions S_3 (see figure-1.1 page-9)

$$S_3(x) = \sin\left(\frac{\pi}{2}x\right)\sin\left(\frac{\pi}{3}x\right)$$

which is an even function, and the function S_5 (see figure-1.2 page-9),

$$S_5(x) = \sin\left(\frac{\pi}{2}x\right)\sin\left(\frac{\pi}{3}x\right)\sin\left(\frac{\pi}{5}x\right)$$

which is an odd function.

1.2.2 Some symmetry properties

We now propose to study some properties some simple symmetry properties of the function $S(p_n)$ in the interval $[0, TS_{p_n}[$. We will limit ourselves to study these properties in the neighbourhood of the natural integers $\frac{TS_{p_n}}{4}$ and $\frac{TS_{p_n}}{2}$. Let x_p and x_q be two real numbers such that

$$\left(\frac{1}{2}(x_p + x_q) = lTS_{p_n}\right)\left(l \in \{\frac{1}{4}, \frac{1}{2}\}\right) \Longleftrightarrow (x_p + x_q = kTS_{p_n})\left(k \in \{\frac{1}{2}, 1\}\right)$$

We have

$$\begin{aligned} S_{p_n}(x_q) &= S_{p_n}(kTS_{p_n} - x_p) \\ &= \prod_{j=1}^{j=n}\left(\sin\frac{\pi}{p_j}(kTS_{p_n} - x_p)\right) \\ &= \prod_{j=1}^{j=n}\left(\sin\left(k\frac{\pi}{p_j}TS_{p_n} - \frac{\pi}{p_j}x_p\right)\right) \end{aligned}$$

Let us pose for all $p_j > 2$

$$2h_j + 1 = \frac{1}{4p_j}TS_{p_n}$$

with

$$h_j \in \mathbb{N}^*$$

then

$$\sin\left(k\frac{\pi}{p_j}TS_{p_n} - \frac{\pi}{p_j}x_p\right) = \sin\left(4k\left(2h_j+1\right)\pi - \frac{\pi}{p_j}x_p\right)$$

Besides, when $p_j = 2$

$$\begin{aligned}\sin\left(k\frac{\pi}{p_j}TS_{p_n} - \frac{\pi}{p_j}x_p\right) &= \sin\left(k\frac{\pi}{2}TS_{p_n} - \frac{\pi}{2}x_p\right) \\ &= \sin\left(2k\left(2h+1\right)\pi - \frac{\pi}{2}x_p\right)\end{aligned}$$

with $h \in \mathbb{N}^*$ We then obtain the following results

Cas $k = \frac{1}{2}$

$$\begin{aligned}\sin\left(4k\left(2h_j+1\right)\pi - \frac{\pi}{p_j}x_p\right) &= \sin\left(2\left(2h_j+1\right)\pi\frac{\pi}{p_j} - x_p\right) \\ &= \sin\left(-\frac{\pi}{p_j}x_p\right)\end{aligned}$$

$$\begin{aligned}\sin\left(2k\left(2h+1\right)\pi - \frac{\pi}{2}x_p\right) &= \sin\left(\left(2h+1\right)\pi - \frac{\pi}{2}x_p\right) \\ &= \sin\left(\frac{\pi}{2}x_p\right)\end{aligned}$$

hence

$$\begin{aligned}S_{p_n}\left(x_q\right) &= S_{p_n}\left(kTS_{p_n} - x_p\right) \\ &= \sin\left(x_p\frac{\pi}{2}\right)\prod_{j=2}^{j=n}\left(\sin\left(-\frac{\pi}{p_j}x_p\right)\right) \\ &= \left(-1\right)^{n-1}\prod_{j=1}^{j=n}\left(\sin\frac{\pi}{p_j}x_p\right)\end{aligned}$$

Cas $k = 1$

$$\begin{aligned}\sin\left(4k\left(2h_j+1\right)\pi - \frac{\pi}{p_j}x_p\right) &= \sin\left(4\left(2h_j+1\right)\pi\frac{\pi}{p_j} - x_p\right) \\ &= \sin\left(-\frac{\pi}{p_j}x_p\right)\end{aligned}$$

$$\begin{aligned}\sin\left(2k\left(2h+1\right)\pi-\frac{\pi}{2}x_p\right) &= \sin\left(2\left(2h+1\right)\pi-\frac{\pi}{2}x_p\right)\\ &= \sin\left(-\frac{\pi}{2}x_p\right)\end{aligned}$$

hence

$$\begin{aligned}S_{p_n}\left(x_q\right) &= S_{p_n}\left(kTS_{p_n}-x_p\right)\\ &= \sin\left(-x_p\frac{\pi}{2}\right)\prod_{j=2}^{j=n}\left(\sin\left(-\frac{\pi}{p_j}x_p\right)\right)\\ &= \left(-1\right)^n\prod_{j=1}^{j=n}\left(\sin\frac{\pi}{p_j}x_p\right)\end{aligned}$$

Conclusion

Inside the interval $[0,TS_{p_n}[$, we can write

$$\left(x_p+x_q=\frac{1}{4}TS_{p_n}\right)\Longrightarrow\left(S_{p_n}\left(x_q\right)=\left(-1\right)^{n-1}\prod_{j=1}^{j=n}\left(\sin\frac{\pi}{p_j}x_p\right)\right)$$

or, formulated otherwise

$$\left(x_p+x_q=\frac{1}{4}TS_{p_n}\right)\Longrightarrow\left(S_{p_n}\left(x_q\right)=\left(-1\right)^{n-1}S_{p_n}\left(x_p\right)\right)\qquad(1.1)$$

and likewise

$$\left(x_p+x_q=\frac{1}{2}TS_{p_n}\right)\Longrightarrow\left(S_{p_n}\left(x_q\right)=\left(-1\right)^{n}\prod_{j=1}^{j=n}\left(\sin\frac{\pi}{p_j}x_p\right)\right)$$

which we can also write

$$\left(x_p+x_q=\frac{1}{2}TS_{p_n}\right)\Longrightarrow\left(S_{p_n}\left(x_q\right)=\left(-1\right)^{n}S_{p_n}\left(x_p\right)\right)\qquad(1.2)$$

1.2.3 A special property of the fonction S_{p_n} when $n\leq 5$.

Let s_{α_j,p_j} be a function such that

$$s_{\alpha_j,p_j}\left(x\right)=\sin\left(\frac{\pi}{p_j}\left(x-\alpha_j\right)\right)=s_{p_j}\left(x-\alpha_j\right)$$

where α_j is a natural integer that belongs to the interval $[0,2p_j[$. We now define the functions U_{p_n}

$$\begin{aligned}U_{p_n}:\mathbb{R}&\longrightarrow[-1,1]\\ x&\longmapsto U_{p_n}\left(x\right)\end{aligned}$$

where

$$U_{p_n}(x) = s_2(x)\, s_{p_n}(x) \prod_{j=2}^{j=n-1} s_{\alpha_j, p_j}(x)$$

Let us first consider the case where $n = 5 \Longleftrightarrow p_n = 11$. Let us look for a function U_{11} that vanishes for each natural integer in the interval $[0, 11[$ and let us write

$$(\forall x \in \{0, 1, 2, \cdots, 9, 10\}) \left(U_{11} = s_2(x)\, s_{11}(x) \prod_{j=2}^{j=4} s_{\alpha_j, p_j}(x) = 0 \right)$$

We note that

$$\begin{aligned} s_{11}(0) &= 0 \\ (\forall x \in \{0, 2, 4, 6, 8, 10\})\, (s_2(x) &= 0) \\ (\forall x \in \{1, 3, 5, 7, 9\})\, (s_2(x) &\neq 0) \\ (\forall x \in \{1, 3, 5, 7, 9\})\, ((s_{11}(x) &\neq 0)) \end{aligned}$$

At least one function s_{α_j, p_j} must vanish when x is equal to one of the odd natural integer in the interval $[0, 11[$. There are three such functions, with $p_j \in \{3, 5, 7\}$. We must have

$$(\forall x \in \{1, 3, 5, 7, 9\})\, (\exists! j \in \{2, 3, 4\})\, \left(s_{\alpha_j, p_j}(x) = s_{p_j}(x - \alpha_j) = 0 \right)$$

We then have a product of three functions s_{α_j, p_j},which must vanish for five distinct natural integer. But the difference between any two of these natural integers is a power of 2, with the exception of the pairs $(1, 7)$ et $(3, 9)$, for which only the functions $s_{1,3}$ et s_3 are respectively cancelled out. The functions $s_{\alpha_3, 5}$ et $s_{\alpha_4, 7}$, as for them, are only cancelled out respectively by one and only one natural integer remaining in the set $\{1, 3, 5, 7, 9\}$.

Such function U_{11}, which must vanish for every integer in the interval $[0, 11[$, therefore cannot exist.

Consequently, there exists necessarily in each interval $[11k, 11(k+1)[$, $k \in N$, at least one natural integer for which the function S_{11} does not vanish. These integers are prime number for each interval, the upper endpoint $11(k+1)$ of which is $\leq 13^2$.

We show in the same manner that for each and every $p_n < 11$, there is at least one natural integer in each interval $[kp_n, (k+1)\, p_n[$, $k \in \mathbb{N}$, for which the function S_{p_n} does not vanish. These natural integers are prime integers for each interval the upper endpoint $(k+1)\, p_n$ of which is $\leq p_{n+1}^2$.

When $p_n \leq 5$, we have

$$\prod_{j=1}^{j=n} p_j < p_{n+1}^2$$

In the special case where $n = 3$, $p_n = 5$, then

$$TS_5 = 2\,(2 \times 3 \times 5)$$

and

$$\frac{TS_5}{2} < 7^2 \Longleftrightarrow (2 \times 3 \times 5) < 7^2$$

In the interval $[0, \frac{TS_5}{2}[$

$$\left(x_p + x_q = \frac{TS_5}{2}\right) \Longleftrightarrow \left(S_5\left(x_q\right) = \left(-1^3\right) \prod_{j=1}^{j=3} \left(\sin \frac{\pi}{p_j} x_p\right)\right)$$

which implies

$$\left(\left(x_p + x_q = \frac{TS_5}{2}\right) \wedge (x_p \neq 0)\right) \Longleftrightarrow (x_q \neq 0)$$

but x_p and x_q are necessarily prime numbers, as they are no multiple of 2, 3 and 5, and at the same time less than 7^2. In this simple case, if x_p is prime number strictly greater than 5, then $x_q = 30 - x_p$ is also a prime number.

1.2.4 Number of natural integers for which the function S_{p_n} does not vanish in the interval $[0, TS_{p_n}[$

Let us consider an odd prime integer p_n and its associated function S_{p_n}. Let in the interval

$$[0, TSp_n[$$

be the set $\mathbb{B}_{p_n}$ of the natural integers, the least divisor of which is greater than p_n. In this manner, $\mathbb{B}_{p_4} = \mathbb{B}_7$ is the set of the natural integers less than $TS_{p_4} = 420$ that are not divisible by any of the prime integers that are strictly less than p_4, to name them 2, 3 and 5.

Let us consider the set $\mathbb{B}_2$ of the natural integers non multiple of 2 (i.e. all the odd numbers), including 1, in the interval $[0, TS_{p_n}[$; Its cardinal $|\mathbb{B}_2|$ is equal to

$$|\mathbb{B}_2| = \left(1 - \frac{1}{2}\right) TS_{p_n}$$

In the same way, the set $\mathbb{B}_3$ of the integers non multiple of 3, including 1, subset of the set $\mathbb{B}_2$, has his cardinal equal to

$$\begin{aligned} |\mathbb{B}_3| &= \left(1 - \frac{1}{3}\right) |\mathbb{B}_2| \\ &= \left(1 - \frac{1}{3}\right) \left[\left(1 - \frac{1}{2}\right) TS_{p_n}\right] \\ &= \left(1 - \frac{1}{2}\right) \left(1 - \frac{1}{3}\right) TS_{p_n} \end{aligned}$$

Step by step, we can calculate the number $|\mathbb{B}_{p_n}|$ of natural integers non multiple of p_n, including 1
- chosen in the set of natural integers non multiple of p_{n-1}, p_{n-1} being the larger prime number less than p_n
- themselves chosen in the set of the natural integers non multiple of p_{n-2}, p_{n-2} being the larger prime number less than p_{n-1}
- ...
- themselves chosen in the set of the natural integers non multiple of $p_{n-(j-1)}$, $p_{n-(j-1)}$ being the larger prime number less than p_{n-j}
- themselves chosen in the set of the natural integers non multiple of 2 that is

$$\begin{aligned} |\mathbb{B}_{p_n}| &= \left(1 - \frac{1}{p_n}\right) |\mathbb{B}_{p_{n-1}}| TS_{p_n} \\ &= \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right) .... \left(1 - \frac{1}{p_n}\right) TS_{p_n} \\ &= \prod_{j=1}^{j=n} \left(1 - \frac{1}{p_j}\right) TS_{p_n} \end{aligned}$$

Now, let us recall that

$$TS_{p_n} = 2\prod_{j=1}^{j=n} p_j$$

we find

$$\begin{aligned} |\mathbb{B}_{p_n}| &= \left[\prod_{j=1}^{j=n} \left(1 - \frac{1}{p_j}\right)\right] \left[2\prod_{j=1}^{j=n} p_j\right] \\ &= 2\prod_{j=1}^{j=n} (p_j - 1) \end{aligned}$$

By analogy with the usual definition of the Euler product, we define the **finite Euler product of rank** n

$$\prod_{j=1}^{j=n} \left(1 - \frac{1}{p_j}\right)$$

Remark

The proportion of natural integers, which we denote δ_n, for which the function S_{p_n} does not vanish in the interval $[0, TS_{p_n}[$ is naturally

$$\begin{aligned} \delta_n &= \frac{TS_{p_n}}{|\mathbb{B}_{p_n}|} \\ &= \prod_{j=1}^{j=n} \frac{1}{\left(1 - \frac{1}{p_j}\right)} \end{aligned}$$

but

$$\lim_{l \to +\infty} \sum_{k=0}^{k=l} \left(\frac{1}{p_j^k} \right) = \frac{1}{\left(1 - \frac{1}{p_j}\right)}$$

and therefore

$$\delta_n = \prod_{j=1}^{j=n} \frac{1}{\left(1 - \frac{1}{p_j}\right)} = \prod_{j=1}^{j=n} \lim_{l \to +\infty} \sum_{k=0}^{k=l} \left(\frac{1}{p_j^k} \right)$$

If now n approaches ∞, then

$$\left(\lim_{n \to +\infty} \delta_n = \lim_{n \to +\infty} \prod_{j=1}^{j=n} \lim_{l \to +\infty} \sum_{k=0}^{k=l} \left(\frac{1}{p_j^k} \right) \right) \Longleftrightarrow \left(\lim_{n \to +\infty} \delta_n = \sum_{j=1}^{+\infty} \left(\frac{1}{j} \right) = \infty \right)$$

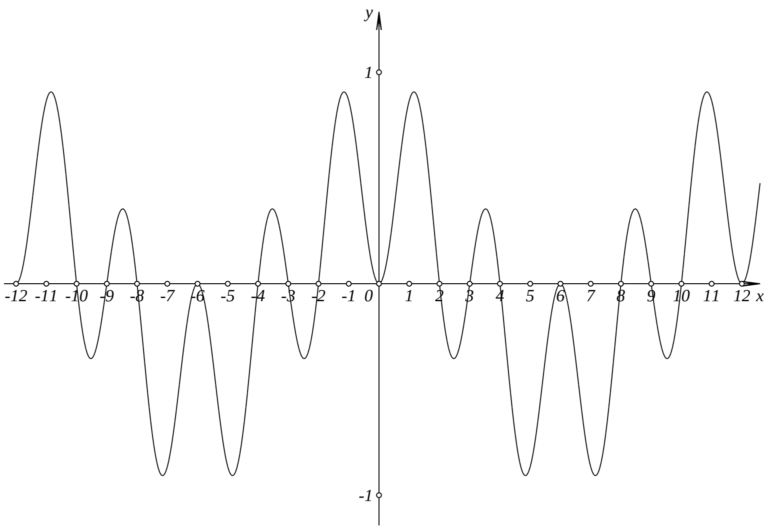

Figure 1.1: Graph of the function S_3

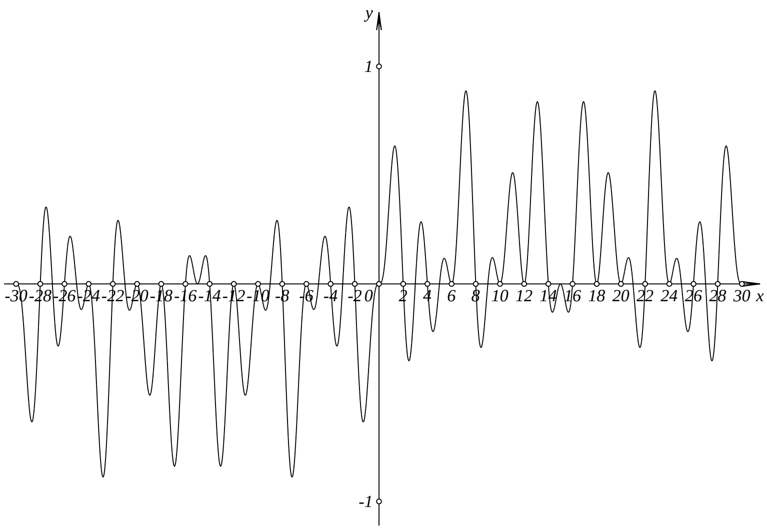

Figure 1.2: Graph of the function S_5

Chapter 2

Some properties of the function G_{m,p_n}.

It is acknowledged that Christian Goldbach stated the following conjecture

Conjecture 4 ***strong of Goldbach*** *For all natural integer $m \geqslant 2$, the even natural integer $2m$ is the sum of two prime numbers.*

For this conjecture, we develop an approach in the two following chapters that could lead to a rigorous proof. The chosen path for our study is based on the idea that it is possible to construct a function defined on $\mathbb{R}$, which would be symmetric with respect to a given natural integer m, the properties of which should enable us to better understand the reasons why this conjecture is likely to be true. Once we have built this function, we will study some of its properties. In particular, we will try to show that this function does not vanish at some natural and relative integers in its domain.

Let

$$\pi_{p_n} = \{p_j | \left(c|p_j \Longleftrightarrow c \in \{1, p_j\}\right) \wedge (p_j \leq p_n)\}$$

be the set that contains all the prime numbers p_j less than or equal to p_n and the function S_{p_n}, which we already defined (see formula 1 page-xi). S_{p_n} is a periodic function with period TS_{p_n} (see formula 1.2.1 page-1). In a way similar to the one used to construct the function S_{p_n}, we will construct the new functions g_{m,p_j} and G_{m,p_n}. Let us begin with the function g_{m,p_j}

$$\begin{aligned} g_{m,p_j} : \mathbb{R} &\longrightarrow [-1, 1] \\ x &\longmapsto g_{m,p_j}(x) \end{aligned}$$

with

$$g_{m,p_j}(x) = \sin\left(\frac{\pi}{p_j} x\right) \sin\left(\frac{\pi}{p_j}(2m - x)\right)$$

where $m \in \mathbb{N}^*$ Using the notations already introduced, this function can also be written

$$\begin{aligned} g_{m,p_j}(x) &= s_{p_j}(x)\, s_{p_j}(2m-x) \\ &= s_{p_j}(x)\, \overline{s_{2m,p_j}}(x) \end{aligned}$$

Then, let us define the function G_{m,p_n}

$$\begin{aligned} G_{m,p_n} &: \mathbb{R} \longrightarrow [-1,1] \\ x &\longmapsto G_{m,p_n}(x) \end{aligned}$$

with

$$G_{m,p_n}(x) = \prod_{j=1}^{j=n} \sin\left(\frac{\pi}{p_j}x\right) \sin\left(\frac{\pi}{p_j}(2m-x)\right)$$

where $m \in \mathbb{N}^*$ This function can also be written

$$\begin{aligned} G_{m,p_n}(x) &= S_{p_n}(x)\, S_{p_n}(2m-x) \\ &= S_{p_n}(x)\, \overline{S_{2m,p_n}}(x) \end{aligned}$$

and also

$$G_{m,p_n}(x) = \prod_{j=1}^{j=n} g_{m,p_j}(x)$$

We expect that the study of this function will provide us with some insight on the strong Goldbach conjecture and its likelihood.

2.1 About some properties of functions g_{m,p_j} et G_{m,p_n}

Functions g_{m,p_j} and G_{m,p_n} display properties of symmetry and periodicity that we will look into here-under.

2.1.1 The functions g_{m,p_j}

Periodicity

Let us recall that

$$T_{s_{p_j}} = 2p_j$$

We have

$$\begin{aligned} s_{p_j}(x) &= (-1)\ s_{p_j}\left(x + \frac{1}{2}Ts_{p_j}\right) \\ &= (-1)\ s_{p_j}\left(x - \frac{1}{2}Ts_{p_j}\right) \end{aligned}$$

and so

$$s_{p_j}\left(2m-x\right)=\left(-1\right)\,s_{p_j}\left(\left(2m-x\right)+\frac{1}{2}Ts_{p_j}\right)$$

and

$$s_{p_j}\left(2m-x\right)=\left(-1\right)\,s_{p_j}\left(\left(2m-x\right)-\frac{1}{2}Ts_{p_j}\right)$$

Let us consider the function g_{m,p_j}

$$g_{m,p_j}\left(x\right)=s_{p_j}\left(x\right)s_{p_j}\left(2m-x\right)$$

then

$$g_{m,p_j}\left(x\right)=\left(-1\right)^2\,s_{p_j}\left(x+\frac{1}{2}Ts_{p_j}\right)s_{p_j}\left(\left(2m-x\right)+\frac{1}{2}Ts_{p_j}\right)$$

and

$$g_{m,p_j}\left(x+\frac{1}{2}T\ s_{p_j}\right)=s_{p_j}\left(x+\frac{1}{2}Ts_{p_j}\right)s_{p_j}\left(2m-\left(x+\frac{1}{2}Ts_{p_j}\right)\right)$$

We have then established that

$$g_{m,p_j}\left(x\right)=g_{m,p_j}\left(x+\frac{1}{2}Ts_{p_j}\right)$$

and therefore, the function g_{m,p_j} is periodic with period

$$\frac{1}{2}Ts_{p_j}=Tg_{p_j,m}=p_j$$

Symmetry

Let us begin with the definition of the function g_{m,p_j}

$$g_{m,p_j}\left(x\right)=s_{p_j}\left(x\right)s_{p_j}\left(2m-x\right)$$

we write

$$g_{m,p_j}\left(2m-x\right)=s_{p_j}\left(2m-x\right)s_{p_j}\left(2m-\left(2m-x\right)\right)$$

hence

$$gm,p_j\left(2m-x\right)=s_{p_j}\left(2m-x\right)s_{p_j}\left(x\right)$$

Commutativity of the product of functions $s_{p_j}\left(2m-x\right)$ and $s_{p_j}\left(x\right)$ allows us to write

$$g_{m,p_j}\left(x\right)=g_{m,p_j}\left(2m-x\right)$$

In particular, when $x=2m$

$$g_{m,p_j}\left(2m-2m\right)=g_{m,p_j}\left(2m\right)=g_{m,p_j}\left(0\right)$$

and

$$s_{p_j}\left(2m-2m\right)=s_{p_j}\left(0\right)=0$$

Zeros

For each and every number x that cancels out the function g_{m,p_j}, we verify

$$\left(g_{m,p_j}(x)=0\right)\Longleftrightarrow\left(s_{p_j}(x)\,s_{p_j}(2m-x)=0\right)$$

and then, these numbers are either of the form hp_j or of the form $2m-lp_j$, where h et l are natural integers. If the two functions s_{p_j} et $\overline{s_{2m,p_j}}$ vanish simultaneously at the same natural integer, then m is necessarily a multiple of p_j. These two functions are then non-distinct. In particular, we note that these two functions vanish when $x=0$, $x=m$ and $x=2m$ in the interval $[0,2m]$.
If, on the other hand, only x is multiple of p_j, then, only the function s_{p_j} vanishes. This function is distinct from the function $\overline{s_{2m,p_j}}$. In particular, in the interval $[0,2m]$, the function $\overline{s_{2m,p_j}}$ does not vanish when $x=0$, $x=m$ and $x=2m$.
Let us now consider the function g_{m,p_j} on one of the intervals

$$[kp_j, kp_j+Tg_{m,p_j}[$$

It vanishes when $x=hp_j$. Also, assuming

$$m\equiv m_j \quad [p_j]$$

we get

$$\left(\sin\left(\frac{\pi}{p_j}(2m-x)\right)=\sin(h\pi)=0\right)\Longleftrightarrow(x=2m_j-lp_j)$$

and then, on the considered interval

$$[kp_j, kp_j+Tg_{m,p_j}[=[kp_j,(k+1)\,p_j[$$

we have two natural integers, kp_j et $(k+1)\,p_j-2m_j$, for which the function g_{m,p_j} vanishes.

Example

We present, as an example for $p_j=5$ and $m=13$,the graph of the function $g_{5,13}$ with period $Tg_{5,13}=5$ in the interval $[0,26[$ (see figure-2.1 page-15) In particular, this graph shows the property of symmetry of this function in the interval $[0,26[$ et $[-2,28[$, as already established in the previous pages.

2.1.2 The function G_{m,p_n}

Periodicity

We already showed

$$S_{p_n}(x)=(-1)S_{p_n}\left(x+\frac{1}{2}TS_{p_n}\right)=(-1)S_{p_n}\left(x-\frac{1}{2}TS_{p_n}\right)$$

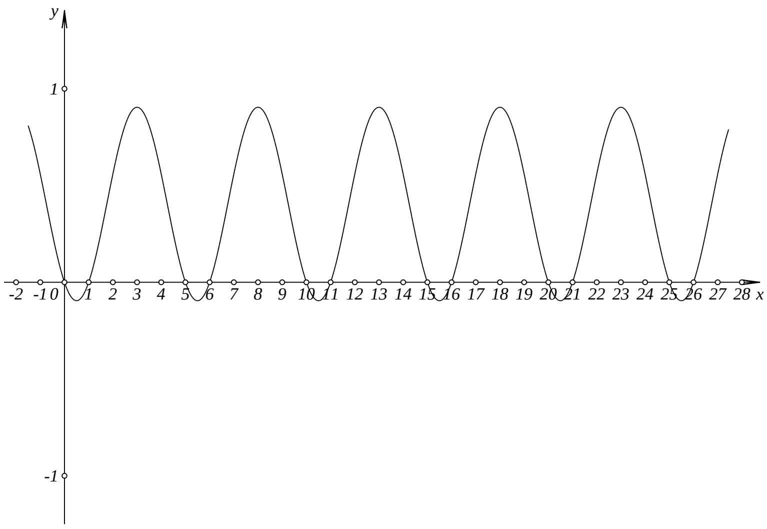

Figure 2.1: Graph of the function $g_{5,13}$

and thus

$$S_{p_n}(2m-x) = (-1)S_{p_n}\left((2m-x)+\frac{1}{2}TS_{p_n}\right)$$

and also

$$S_{p_n}(2m-x) = (-1)S_{p_n}\left((2m-x)-\frac{1}{2}TS_{p_n}\right)$$

Therefore, we can write

$$G_{m,p_n}(x) = S_{p_n}(x)\,S_{p_n}(2m-x)$$

and

$$G_{m,p_n}(x) = (-1)^2 S_{p_n}\left(x+\frac{1}{2}TS_{p_n}\right) S_{p_n}\left((2m-x)+\frac{1}{2}TS_{p_n})\right)$$

and also

$$G_{m,p_n}\left(x+\frac{1}{2}TS_{p_n}\right) = S_{p_n}\left(x+\frac{1}{2}TS_{p_n}\right) S_{p_n}\left(2m-(x+\frac{1}{2}TS_{p_n})\right)$$

and lastly

$$G_{m,p_n}(x) = G_{m,p_n}(x+\frac{1}{2}TS_{p_n})$$

We note that the function G_{m,p_n} is periodic, with period $\frac{1}{2}TS_{p_n}$ and we write

$$TG_{m,p_n} = \frac{1}{2}TS_{p_n}$$

This period is always even for all n.

Symmetry

We can also verify that in the interval $[0, 2m[$

$$G_{m,p_n}(x) = \prod_{j=1}^{j=n} \sin\left(\frac{\pi}{p_j}x\right) \sin\left(\frac{\pi}{p_j}(2m-x)\right)$$

which can also be expressed

$$G_{m,p_n}(x) = \prod_{j=1}^{j=n} \sin\left(\frac{\pi}{p_j}(2m-x)\right) \sin\left(\frac{\pi}{p_j}x\right)$$

and thus

$$(G_{m,p_n}(x) = G_{m,p_n}(2m-x)) \Longleftrightarrow (G_{m,p_n}(m-x) = G_{m,p_n}(m+x))$$

In particular

$$(G_{m,p_n}(m-x) \neq 0) \Longleftrightarrow ((S_{p_n}(m-x) \neq 0) \wedge (S_{p_n}(m+x) \neq 0))$$

Likewise

$$(G_{m,p_n}(m-x) = 0) \Longleftrightarrow ((S_{p_n}(m-x) = 0) \wedge (S_{p_n}(m+x) = 0))$$

By construction, the natural integer m is the centre of symmetry for the function G_{m,p_n} in the interval $[0, 2m[$. In addition, we have

$$G_{m,p_n}\left(m - \frac{1}{2}TG_{m,p_n}\right) = G_{m,p_n}\left(m + \frac{1}{2}TG_{m,p_n}\right)$$

and so, m is also the centre of symmetry for the function G_{m,p_n} in the interval

$$[m - \frac{1}{2}TG_{m,p_n}, m + \frac{1}{2}TG_{m,p_n}[$$

We finally note that

$$\begin{aligned} G_{m,p_n}(-x) &= \prod_{j=1}^{j=n} \sin\left(\frac{\pi}{p_j}(-x)\right) \sin\left(\frac{\pi}{p_j}(2m+x)\right) \\ &= (-1)^n \prod_{j=1}^{j=n} \sin\left(\frac{\pi}{p_j}x\right) \sin\left(\frac{\pi}{p_j}(2m+x)\right) \end{aligned}$$

and

$$G_{m,p_n}(x) = \prod_{j=1}^{j=n} \sin\left(\frac{\pi}{p_j}(x)\right) \sin\left(\frac{\pi}{p_j}(2m-x)\right)$$

Should there exist non zero natural integers as values taken on by x

$$|G_{m,p_n}(-x)| = |G_{m,p_n}(x)|$$

then, we should have

$$(\forall p_j \in \pi_{p_n}) \left(\sin\left(\frac{\pi}{p_j}(2m+x)\right) = \sin\left(\frac{\pi}{p_j}(2m-x)\right) \right)$$

but

$$\sin\left(\frac{\pi}{p_j}(2m+x)\right) = \sin\left(\frac{\pi}{p_j}(2m-x)\right)\cos\left(\frac{\pi}{p_j}(2x)\right) + \cos\left(\frac{\pi}{p_j}(2m-x)\right)\sin\left(\frac{\pi}{p_j}(2x)\right)$$

and so

$$\left(\sin\left(\frac{\pi}{p_j}(2m+x)\right) = \sin\left(\frac{\pi}{p_j}(2m-x)\right) \right) \Longleftrightarrow \left(\cos\left(\frac{\pi}{p_j}(2x)\right) = 1 \Longleftrightarrow \sin\left(\frac{\pi}{p_j}(2x)\right) = 0 \right)$$

This necessarily implies

$$(\exists h_0 \in \mathbb{Z}^*) \left(x = h_0 \prod_{j=1}^{j=n} p_j \right)$$

and we verify

$$(\forall p_k \in \pi_{p_n}) (\exists h_1 \in \mathbb{Z}^*) \left(\sin\left(\frac{\pi}{p_k} h_0 \prod_{j=1}^{j=n} p_j \right) = \sin(h_1 \pi) = 0 \right)$$

which implies

$$G_{m,p_n}\left(h_0 \prod_{j=1}^{j=n} p_j \right) = G_{m,p_n}\left(-h_0 \prod_{j=1}^{j=n} p_j \right) = 0$$

On the other hand, when

$$(h_0 \in \mathbb{Z}^*) \left(x \neq h_0 \prod_{j=1}^{j=n} p_j \right)$$

then

$$G_{m,p_n}(x) \neq G_{m,p_n}(-x)$$

Hence, 0 is not a centre of symmmetry for the function G_{m,p_n}.

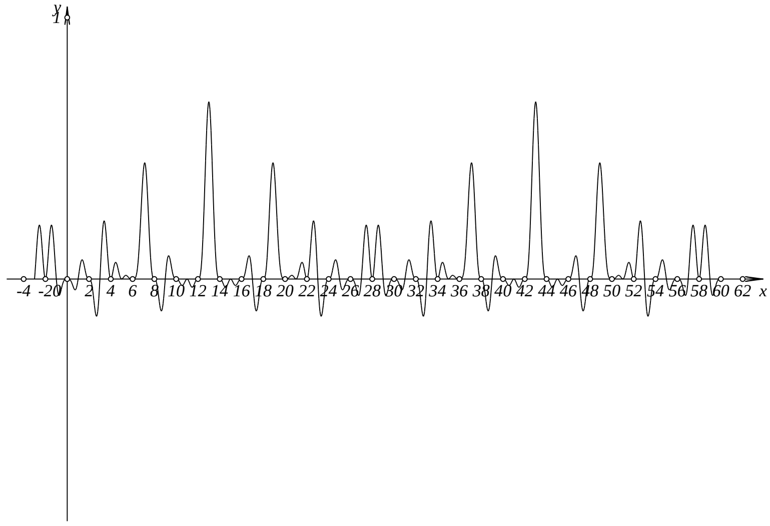

Figure 2.2: Graph of the function $G_{5,13}$ on the interval $[-2, 58[$

Examples

We present, as an example for $p_j = 5$ and $m = 13$,the graph of the function $G_{5,13}$ with period $TG_{5,13} = 5$ in the interval $[-2, 28[$ (see figure-2.1 page-15) In particular, this graph shows the property of symmetry of this function in the interval $[0, 26[$ et $[-2, 28[$, as already established in the previous pages.

Other properties

Up to now, we have not made any hypothesis as regards the parameter m, the value of which has evidently some influence in the behaviour of the function G_{m,p_n} and specially in the way this function vanishes in its domain. By construction, the function vanishes at x when

$$S_{p_n}(x) = 0$$

or else

$$S_{p_n}(2m - x) = \overline{S_{2m,p_n}}(x) = 0$$

Case 1: $m \leq p_n$ The interval $[0, m[$ is included in the interval $[0, p_n[$. We know that the function S_{p_n} vanishes at all the natural integers in the interval $[0, p_n[$, save for 1. Therefore, by symmetry, the function G_{m,p_n} a priori vanishes at all the natural integers in the interval $[0, 2m[$, save for 1 and $2m - 1$, which this function does not necessarily vanishes at. However, if $2m - 1$ is divisible by at least one of the prime integers less than or equal to p_n, then the function G_{m,p_n} vanishes at all the natural integers in the interval $[0, 2m[$. We illustrate

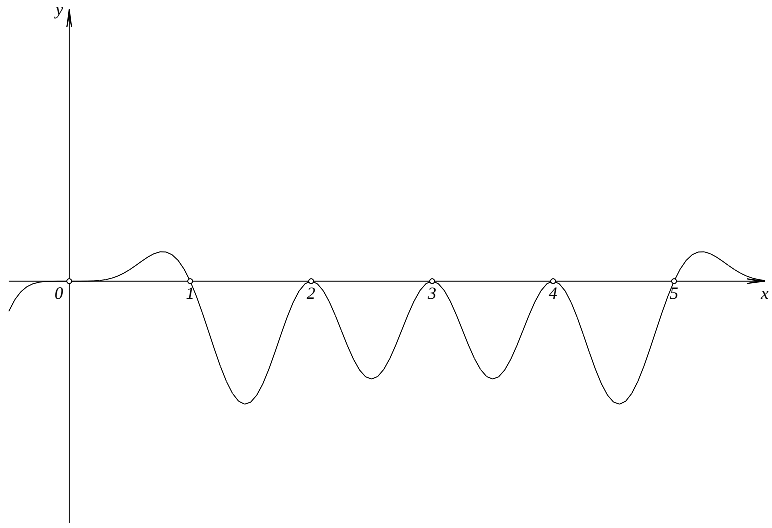

Figure 2.3: Graph of the function $G_{5,3}$ on the interval $[-2, 30[$

this case with the graphs of the functions $G_{5,3}$ et $G_{5,4}$ on the respective intervals $[0, 6[$, $[0, 8[$ and $[0, 10[$ (see figures 2.3 and 2.4 pages 19 and 20).

Case 2: $m > p_n$ The interval $[0, p_n[$ is included in the interval $[0, m[$. Therefore, the function G_{m,p_n} a priori may not vanish at all the natural integers in the interval $[0, 2m[$. We illustrate this case with the graphs of the functions $G_{7,6}$ et $G_{7,7}$ on the respective intervals $[0, 12[$ and $[0, 14[$ (see figures 2.5 and 2.6 pages 21 and 22). This latter case, where the natural integer m is strictly greater than the prime integer p_n, will be the object of the deeper study that follows.

We will show that for all prime integer $P_n > 11$, there exists at least one natural integer in each interval

$$[kp_n, (k+1)\,p_n[$$

which the function S_{p_n} does not vanish at, when k is less than some integers, the value of which depends on p_n. Moreover, when

$$(k+1)\,p_n < p_{n+1}^2$$

such integer is prime. We also note that every natural integer which the function S_{p_n} vanishes at, cancels out the function G_{m,p_n}. The converse is not true.

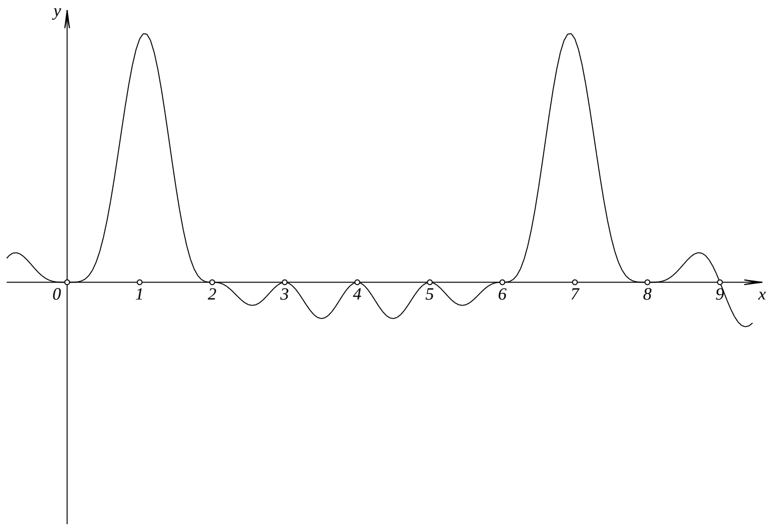

Figure 2.4: Graph of the function $G_{5,4}$ on the interval $[-2, 30[$

Indeed, this function also vanishes when we have

$$\sin\left(\frac{\pi}{p_j}(2m-x)\right)=0$$

for at least one of the prime integers p_j.

The natural integers which do not cancel out the function G_{m,p_n}.

We pair each natural integer m with the function

$$G_{m,p_n}(x)=\prod_{j=1}^{j=\mu}\sin\left(\frac{\pi}{p_j}x\right)\sin\left(\frac{\pi}{p_j}(2m-x)\right)$$

and we choose the prime integers p_n and p_{n+1}, consecutive in the set of the prime numbers, such that

$$p_n^2<2m<p_{n+1}^2$$

then we look at the way the function G_{m,p_n} vanishes in the interval

$$[-\frac{1}{2}TG_{m,p_n}+m,\frac{1}{2}TG_{m,p_n}+m[$$

This interval is centred on the natural integer m and contains TG_{m,p_n} natural integers, with

$$TG_{m,p_n}=\prod_{j=1}^{j=n}p_j$$

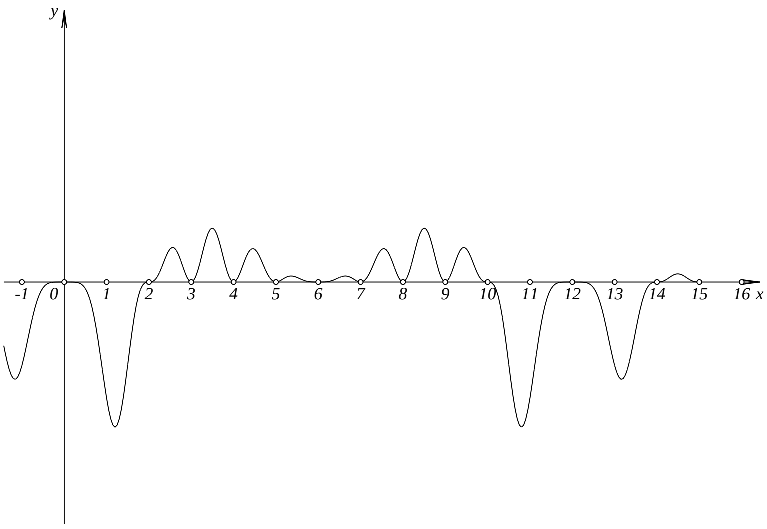

Figure 2.5: Graph of the function $G_{7,6}$ on the interval $[-2, 30[$

Let us consider the natural integers a_k in this interval, and for all these natural integers, their respective remainders $\alpha_{k,j}$ modulo each of the prime integers p_j in the set π_{p_n}. For each of these natural integers, we have for each index j

$$a_k \equiv \alpha_{k,j} \quad [p_j]$$

with

$$\alpha_{k,j} \in \mathbb{Z}/p_j\mathbb{Z}$$

Let us write down each of these natural integers a_k in the interval

$$[-\frac{1}{2}TG_{m,p_n} + m, \frac{1}{2}TG_{m,p_n} + m[$$

and there respective remainders modulo p_j in each of the $\prod_{j=1}^{j=n} p_j$ rows of the following table

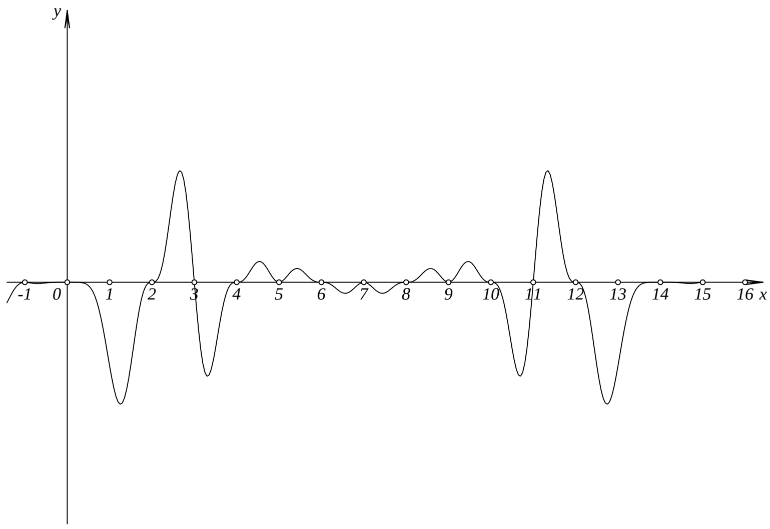

Figure 2.6: Graph of the function $G_{7,7}$ on the interval $[-2, 30[$

$\equiv \quad [p_1]$	$\equiv \quad [p_2]$		$\equiv \quad [p_j]$		$\equiv \quad [p_n]$
$\alpha_{1,1}$	$\alpha_{1,2}$	$\cdots$	$\alpha_{1,j}$	$\cdots$	$\alpha_{1,n}$
$\alpha_{2,1}$	$\alpha_{2,2}$	$\cdots$	$\alpha_{2,j}$	$\cdots$	$\alpha_{2,n}$
$\alpha_{3,1}$	$\alpha_{3,2}$	$\cdots$	$\alpha_{3,j}$	$\cdots$	$\alpha_{3,n}$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\alpha_{k,1}$	$\alpha_{k,2}$	$\cdots$	$\alpha_{k,j}$	$\cdots$	$\alpha_{k,n}$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\alpha_{\prod_{j=1}^{j=n} p_j - 2,1}$	$\alpha_{\prod_{j=1}^{j=n} p_j - 2,2}$	$\cdots$	$\alpha_{\prod_{j=1}^{j=n} p_j - 2,i}$	$\cdots$	$\alpha_{\prod_{j=1}^{j=n} p_j - 2,n}$
$\alpha_{\prod_{j=1}^{j=n} p_j - 1,1}$	$\alpha_{\prod_{j=1}^{j=n} p_j - 1,2}$	$\cdots$	$\alpha_{\prod_{j=1}^{j=n} p_j - 1,i}$	$\cdots$	$\alpha_{\prod_{j=1}^{j=n} p_j - 1,n}$
$\alpha_{\prod_{j=1}^{j=n} p_j,1}$	$\alpha_{\prod_{j=1}^{j=n} p_j,2}$	$\cdots$	$\alpha_{\prod_{j=1}^{j=n} p_j,i}$	$\cdots$	$\alpha_{\prod_{j=1}^{j=n} p_j,n}$

Each of the remainders $\alpha_{k,j}$ can take p_j distinct values in the set

$$\{0, 1, 2, \cdots, j, \cdots, p_j - 1\}$$

Hence, each row of the table can be written in $\prod_{j=1}^{j=n} p_j$ different ways. In addition, we note that two distinct rows containing exactly the same remainders $\alpha_{k,j}$, for each value taken by the index j, necessarily correspond to two distinct natural integers a_{k_1} and a_{k_2} that are such that

$$(\forall p_j \in \pi_{p_n}) \left[(a_{k_1} \equiv a_{k_2} \quad [p_j]) \Longleftrightarrow ((a_{k_1} - a_{k_2}) \equiv 0 \quad [p_j])\right]$$

We then conclude that there can only be one of such numbers in the interval

$$[-\frac{1}{2}TG_{m,p_n}+m,\frac{1}{2}TG_{m,p_n}+m[$$

Consequently, in this interval, two rows taken among the $\prod_{j=1}^{j=n} p_j$ possible rows of the table cannot be identical and the set of these rows contain all the possible rows that can be constructed with the remainders $\alpha_{k,j}$. Let us consider now the natural integers a_κ in the interval

$$[-\frac{1}{2}TG_{m,p_n}+m,\frac{1}{2}TG_{m,p_n}+m[$$

which the function G_{m,p_n} does not vanish at. For each of them, none of the remainders $\alpha_{k,j}$ modulo p_j is zero and each of them cannot take more than p_j-1 different values. The number of natural integers a_k contained in the interval

$$[-\frac{1}{2}TG_{m,p_n}+m,\frac{1}{2}TG_{m,p_n}+m[$$

is therefore equl to $\prod_{j=1}^{j=n}(p_j-1)$. Besides, it is clear that we must verify

$$(\forall a_\kappa)\,(\forall p_j\in\pi_{p_n})\,(a_\kappa-(2m-a_\kappa)\equiv 2m\quad[p_j])$$

Let $\left\{p_j^{(m)}\right\}$ et $\left\{p_j^{\neg(m)}\right\}$ be the sets of the odd prime numbers that respectively divide and do not divide m, and then $2m$. We have

$$\left\{p_j^{(m)}\right\}\cup\left\{p_j^{\neg(m)}\right\}=\pi_{p_n}-\{2\}$$

The set $\left\{p_j^{(m)}\right\}$ is empty if m is itself a prime number or a multiple of prime numbers that do not belong to π_{p_n}. We have

$$\left(\forall p_j^{(m)}\in\left\{p_j^{(m)}\right\}\right)\left(2m\equiv 0\quad[p_j^{(m)}]\right)$$

Similarly

$$\left(\forall p_j^{\neg(m)}\in\left\{p_j^{\neg(m)}\right\}\right)\left(\exists\mu_j\in\mathbb{Z}^*/p_j^{\neg(m)}\right)\left(2m\equiv\mu_j\quad[p_j^{\neg(m)}]\right)$$

We pose

$$\left|\left\{p_j^{(m)}\right\}\right|=\rho$$

which implies

$$\left|\left\{p_j^{\neg(m)}\right\}\right|=(n-1)-\rho$$

Let us assume that there exists at least one prime integer $p_k\in\pi_{p_n}$ that divides $2m-a_\kappa$. Then

$$(\exists p_k\in\pi_{p_n})\,((a_\kappa\equiv 2m\quad[p_k])\Leftrightarrow((2m-a_\kappa)\equiv 0\quad[p_k]))$$

and in this case

$$G_{m,p_n}(a_k) = G_{m,p_n}(2m - a_k) = 0$$

Conversely, the natural integers a_k such that

$$(\forall p_j \in \pi_{p_n})\,(a_\kappa \not\equiv 2m \quad [p_j])$$

satisfy

$$G_{m,p_n}(a_k) = G_{m,p_n}(2m - a_k) \neq 0$$

For each of these natural integers a_k, none of its remainders $\alpha_{k,j}$ modulo p_j is zero. Two cases then present themselves

Case 1

$$\left(\left\{p_j^{(m)}\right\} = \emptyset\right) \Leftrightarrow \left(\left|\left\{p_j^{\neg(m)}\right\}\right| = |\pi_{p_n} - \{2\}| = n - 1\right)$$

Besides, none of its remainders $\alpha_{k,j}$ is equal to the remainder μ_j modulo p_j of $2m$. There are therefore only $p_j - 2$ possible values for each of its remainders $\alpha_{k,j}$. The number of such natural integers a_k contained in the interval

$$[-\frac{1}{2}TG_{m,p_n} + m, \frac{1}{2}TG_{m,p_n} + m[$$

which the function G_{m,p_n} does not vanish at in the same interval is then equal to

$$\Gamma_{G_{m,p_n}} = \prod_{j=2}^{j=n}\left(p_j^{\neg(m)} - 2\right) \tag{2.1}$$

As an illustration, the prime number p_n and the parameter m being respectively chosen equal to 7 an 31, the period of the function $G_{31,7}$ is equal to

$$TG_{31,7} = 210$$

We verify that $7^2 < 62 < 11^2$. As well, $31 \notin \pi_7$. The contemplated interval is

$$[-\frac{1}{2}210 + 31 = -74, \frac{1}{2}210 + 31 = 136[$$

This interval contains

$$\left\{p_j^{(m)}\right\} = \emptyset$$

and

$$\left\{p_j^{\neg(m)}\right\} = \pi_7 - \{2\} = \{3, 5, 7\}$$

Therefore, $\left|\left\{p_j^{(m)}\right\}\right| = 0$ et $\left|\left\{p_j^{\neg(m)}\right\}\right| = 3$. The set of the natural integers that do not cancel out the function $G_{31,7}$ in the interval $[-74, 136[$ is the set

$$\{-59, -47, -41, -17, -11, 1, 19, 31, 43, 61, 73, 79, 103, 109, 121\}$$

It contains 15 natural integers and one can verify that

$$\Gamma_{G_{31,7}} = \prod_{j=2}^{j=3}\left(p_j^{(m)} - 2\right) = (3-2)(5-2)(7-2) = 15$$

Case 2

$$\left(\left\{p_j^{(m)}\right\} \neq \emptyset\right) \Leftrightarrow \left(\left|\left\{p_j^{(m)}\right\}\right| = \rho\right) \Leftrightarrow \left(\left|\left\{p_j^{\urcorner(m)}\right\}\right| = (n-1) - \rho\right)$$

Besides, none of its remainders $\alpha_{k,j}$ is equal to the remainder μ_j modulo $p_j^{\urcorner(m)}$ of $2m$. Each of its remainders $\alpha_{k,j}$ can only take one value among $p_j - 1$ natural integers for each prime integer $p_j \in \left\{p_j^{(m)}\right\}$.

Likewise, None of its remainders $\alpha_{k,j}$ is equal to the remainder μ_j modulo $p_j^{\urcorner(m)}$ of $2m$. Each of its remainders $\alpha_{k,j}$ can only take one value among $p_j - 2$ natural integers for each prime integer $p_j \in \left\{p_j^{\urcorner(m)}\right\}$.

The number of natural integers a_k contained in the interval

$$[-\frac{1}{2}TG_{m,p_n} + m, \frac{1}{2}TG_{m,p_n} + m[$$

which the function G_{m,p_n} does not vanish at in the same interval is then equal to

$$\Gamma_{G_{m,p_n}} = \prod_{k=1}^{k=\rho} \left(p_k^{(m)} - 1\right) \prod_{l=2}^{l=n-\rho} \left(p_l^{\urcorner(m)} - 2\right) \tag{2.2}$$

It is clear that the preceding case is in fact a particular case of this present case where $\rho = 0$, and we can write

$$(\forall n \in \mathbb{N}^*) \left(\prod_{j=2}^{j=n} (p_j - 2) \leq \Gamma_{G_{m,p_n}} < \prod_{j=2}^{j=n} (p_j - 1)\right)$$

the sets $\left\{p_j^{(m)}\right\}$ and $\left\{p_j^{\urcorner(m)}\right\}$ being the sets of the odd prime integers that respectively divide and do not divide m. As an illustration, the prime number p_n and the parameter m being respectively chosen equal to 7 an 30, the period of the function $G_{30,7}$ is equal to

$$TG_{30,7} = 210$$

We verify that $7^2 < 62 < 11^2$. Besides

$$30 \equiv 0 \quad [3]$$

and

$$30 \equiv 0 \quad [5]$$

The contemplated interval is

$$[-\frac{1}{2}210 + 30 = -75, \frac{1}{2}210 + 30 = 135[$$

This interval contains 210 natural integers. We have

$$\left\{p_j^{(m)}\right\} = \{3, 5\}$$

and

$$\left\{p_j^{\neg(m)}\right\} = \pi_5 - \{2,3,5\} = \{7\}$$

Therefore, $\left|\left\{p_j^{(m)}\right\}\right| = 2$ and $\left|\left\{p_j^{\neg(m)}\right\}\right| = 1$. The set of natural integers that cancel out the function $G_{30,7}$ in the interval $[-75, 135[$ is the set

$$\begin{aligned}&\{-71,-67,-61,-53,-47,107,113,121,127,131\}\\&\quad\cup\{-43,-41,-37,-29,-23,83,89,97,101,103\}\\&\quad\cup\{-19,-13,-11,-1,1,59,61,71,73,79\}\\&\quad\cup\{13,17,19,23,29,31,37,41,43,47\}\end{aligned}$$

We purposely divided this set into four subsets containg each 10 natural integers for the sake of clarity. This set then contains 40 natural integers and we verify that

$$\Gamma_{G_{30,7}} = \prod_{k=1}^{k=2}\left(p_k^{(m)} - 1\right)\prod_{l=1}^{l=1}\left(p_l^{\neg(m)} - 2\right) = (3-1)(5-1)(7-2) = 40$$

2.2 Study on the interval $[0, 2m[$

The result we just obtained shows that the function G_{m,p_n} does not vanish at a significant number of natural integers in the interval

$$[-\frac{1}{2}TG_{m,p_n} + m, \frac{1}{2}TG_{m,p_n} + m[$$

These natural integers are necessarily either prime integers that do not belong to π_{p_n}, or natural integers that are multiple of prime integers that do not belong to π_{p_n}. There exists as well two prime integers p_ν and $p_{\nu+1}$, with $\nu \in \mathbb{N}^*$, such that for the corresponding functions G_{m,p_ν} and $G_{m,p_{\nu+1}}$, we should have

$$TG_{m,p_\nu} < 2m < TG_{m,p_{\nu+1}}$$

The function G_{m,p_ν} does not vanish either at a significant number of natural integers in the interval

$$[-\frac{1}{2}TG_{m,p_\nu} + m, \frac{1}{2}TG_{m,p_\nu} + m[$$

2.2.1 Zeros

let us now consider these two functions G_{m,p_n} and G_{m,p_ν} in the closed interval

$$[-\frac{1}{2}TG_{m,p_\nu} + m, \frac{1}{2}TG_{m,p_\nu} + m]$$

where p_ν is such that

$$[-\frac{1}{2}TG_{m,p_\nu} + m, \frac{1}{2}TG_{m,p_\nu} + m] \subset [0, 2m[$$

We already showed that

$$TG_{m,p_\nu} = \prod_{j=1}^{j=\nu} p_j$$

One can notice that the endpoints of the interval

$$[-\frac{1}{2}TG_{m,p_\nu} + m, \frac{1}{2}TG_{m,p_\nu} + m]$$

which we denote respectively A_ν et B_ν are of same parity. For these two endpoints, we have

$$(\forall p_j \leq p_\nu)\,(A_\nu \equiv B_\nu \quad [p_j])$$

We will assume also that the natural integer m is not prime. Let us now recall

$$G_{m,p_n}(x) = S_{p_n}(x)\,\overline{S_{2m,p_n}}(x)$$

with

$$S_{p_n}(x) = \prod_{j=1}^{j=n} \sin\frac{\pi}{p_j}(2m - x) = \prod_{j=1}^{j=n} s_{p_j}(x)$$

$$\overline{S_{2m,p_n}}(x) = \prod_{j=1}^{j=n} \sin\frac{\pi}{p_j}(2m - x) = \prod_{j=1}^{j=n} \overline{s_{2m,p_j}}(x)$$

The function S_{p_n} vanishes at a_n natural integers belonging to the interval

$$[-\frac{1}{2}TG_{m,p_\nu} + m, m[$$

and at b_n natural integers belonging to the interval

$$]m, \frac{1}{2}TG_{m,p_\nu} + m]$$

Symmetrically, the function $\overline{S_{2m,p_n}}$ vanishes at $\overline{a_n} = b_n$ natural integers belonging to the interval

$$[-\frac{1}{2}TG_{m,p_\nu} + m, m[$$

and at $\overline{b_n} = a_n$ natural integers belonging to the interval

$$]m, \frac{1}{2}TG_{m,p_\nu} + m]$$

Therefore, the number of natural integers which the function G_{m,p_n} vanishes at in the interval

$$[-\frac{1}{2}TG_{m,p_\nu} + m, m[$$

is less than or equal to $a_n + b_n$,, when the number of natural integers which the function S_{p_n} vanishes at in the interval

$$[-\frac{1}{2}TG_{m,p_\nu} + m, \frac{1}{2}TG_{m,p_\nu} + m]$$

is itself equal to $a_n + b_n + 1$.
The set of natural integers which the function S_{p_n} vanishes at in the interval

$$[-\frac{1}{2}TG_{m,p_\nu} + m, \frac{1}{2}TG_{m,p_\nu} + m]$$

is also the set of natural integers the least prime divisor is less than or equal to p_n. We denote this set $\mathbb{C}_{p_n}$ and we have

$$|\mathbb{C}_{p_n}| = a_n + b_n + 1 \tag{2.3}$$

From the foregoing, it follows that

- the number of natural integers which the function G_{m,p_n} vanishes at in the interval
$$[-\frac{1}{2}TG_{m,p_\nu} + m, m[$$
is less than or equal to $(a_n + b_n)$. These natural integers are the elements of the set which we denote $\mathbb{D}_{p_n}$ and we have
$$|\mathbb{D}_{p_n}| \leq a_n + b_n \tag{2.4}$$

- the number of natural integers which the function G_{m,p_n} does not vanish at in the interval
$$[-\frac{1}{2}TG_{m,p_\nu} + m, m[$$
is greater than $\frac{1}{2}TG_{m,p_\nu} - (a_n + b_n)$. These integers are the elements of the set which we denote $\mathbb{E}_{p_n}$ and we have
$$|\mathbb{E}_{p_n}| > \frac{1}{2}TG_{m,p_\nu} - (a_n + b_n) \tag{2.5}$$

We now define in the interval $[-\frac{1}{2}TG_{m,p_\nu} + m, \frac{1}{2}TG_{m,p_\nu} + m]$

- the set $\mathbb{A}_2$ of the natural integers the least prime divisor of which is 2, and its complement $\mathbb{B}_2$ in this interval. The cardinals of these two sets are respectively denoted $|\mathbb{A}_2|$ and $|\mathbb{B}_2|$. We have the strict equalities

$$|\mathbb{A}_2| = \frac{1}{2}TG_{m,p_\nu}$$

$$|\mathbb{B}_2| = \left(1 - \frac{1}{2}\right)TG_{m,p_\nu}$$

$\mathbb{B}_2$ is the set of the natural integers the least prime divisor is greater than 2.
- the set $\mathbb{A}_3$ of the natural integers the least prime divisor of which is 3, and its

complement $\mathbb{B}_3$ in the set $\mathbb{B}_2$. The cardinals of these two sets are respectively denoted $|\mathbb{A}_3|$ and $|\mathbb{B}_3|$ and we have yet again the strict equalities

$$|\mathbb{A}_3| = \frac{1}{3}\left(1 - \frac{1}{2}\right) TG_{m,p_\nu}$$

$$|\mathbb{B}_3| = \left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{2}\right) TG_{m,p_\nu}$$

$\mathbb{B}_3$ is the set of the natural integers the least prime divisor is greater than 3. For the sets of natural integers the least prime divisor of which is $5 \leq p_j \leq p_\nu$, there are no longer strict equalities, except when

$$m \equiv 0 \quad [p_j]$$

In that manner, the set $\mathbb{A}_5$ is the set of the natural integers the least prime divisor of which is 5, and its complement $\mathbb{B}_5$ in the set $\mathbb{B}_3$. The cardinals of these two sets are respectively denoted $|\mathbb{A}_5|$ and $|\mathbb{B}_5|$ and we have the inequalities

$$|\mathbb{A}_5| \leq \frac{1}{5}\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right) TG_{m,p_\nu}$$

$$|\mathbb{B}_5| \geq \left(1 - \frac{1}{5}\right)\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right) TG_{m,p_\nu}$$

$\mathbb{B}_5$ is the set of the natural integers the least prime divisor is greater than 5. In general, the set $\mathbb{A}_{p_j}$ is the set of the natural integers the least prime divisor of which is p_j, and its complement $\mathbb{B}_{p_j}$ in the set $\mathbb{B}_{p_{j-1}}$. The cardinals of these two sets are respectively denoted $\left|\mathbb{A}_{p_j}\right|$ and $\left|\mathbb{B}_{p_j}\right|$ and we have the inequalities

$$\left|\mathbb{A}_{p_j}\right| \leq \frac{1}{p_j} \prod_{k=1}^{k=j-1} \left(1 - \frac{1}{p_k}\right) TG_{m,p_\nu} \tag{2.6}$$

$$\left|\mathbb{B}_{p_j}\right| \geq \prod_{k=1}^{k=j} \left(1 - \frac{1}{p_k}\right) TG_{m,p_\nu} \tag{2.7}$$

For all j, $\mathbb{B}_{p_j}$ is the set of the natural integers the least prime divisor is greater than p_j. Moreover, we have

$$\begin{aligned} TG_{m,p_\nu} &= \mathbb{A}_{p_1} \cup \mathbb{B}_{p_1} \\ \mathbb{B}_{p_1} &= \mathbb{A}_{p_2} \cup \mathbb{B}_{p_2} \\ \mathbb{B}_{p_2} &= \mathbb{A}_{p_3} \cup \mathbb{B}_{p_3} \\ &\cdots \\ \mathbb{B}_{p_{j-2}} &= \mathbb{A}_{p_{j-1}} \cup \mathbb{B}_{p_{j-1}} \\ \mathbb{B}_{p_{j-1}} &= \mathbb{A}_{p_j} \cup \mathbb{B}_{p_j} \end{aligned}$$

$$\dots$$

$$\mathbb{B}_{p_{n-1}} = \mathbb{A}_{p_j} \cup \mathbb{B}_{p_n}$$

and thus

$$\mathbb{B}_{p_1} = \mathbb{A}_{p_2} \cup \mathbb{A}_{p_3} \cup \mathbb{B}_{p_3}$$

and following this path from one value of j to the next

$$(\forall j \in \mathbb{N}^*)\,(j \leq n)\left(\mathbb{B}_{p_1} = \bigcup_{k=2}^{k=j-1} \mathbb{A}_{p_k} \cup \mathbb{B}_{p_j}\right)$$

Furthermore, it is clear that the sets $\mathbb{A}_{p_j}$ are pairwise distinct and disjoint and that the set $\mathbb{C}_{p_n}$ of the natural integers the least prime divisor of which is less than or equal to p_n, with $1 < j \leq n$, in the interval

$$[-\frac{1}{2}TG_{m,p_\nu} + m, \frac{1}{2}TG_{m,p_\nu} + m[$$

is equal to

$$\mathbb{C}_{p_n} = \bigcup_{j=1}^{j=n} \mathbb{A}_{p_j}$$

with its cardinal equal to

$$|\mathbb{C}_{p_n}| = \sum_{j=1}^{j=n} |\mathbb{A}_{p_j}| \tag{2.8}$$

Lastly, the set $\mathbb{B}_{p_n}$ of the natural integers the least prime divisor of which is greater than p_n is the complement of the set $\mathbb{C}_{p_n}$ of the natural the integers the least prime divisor of which is less than or equal to p_n in the set of the natural integers belonging in the interval

$$[-\frac{1}{2}TG_{m,p_\nu} + m, \frac{1}{2}TG_{m,p_\nu} + m]$$

and hence

$$|\mathbb{B}_{p_n}| = TG_{m,p_\nu} - (a_n + b_n) \tag{2.9}$$

let us now pose

$$u_1 = \frac{1}{2}$$

$$v_1 = \left(1 - \frac{1}{2}\right)$$

$$u_2 = \frac{1}{3}\left(1 - \frac{1}{2}\right)$$

$$v_2 = \left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{2}\right)$$

$$u_3 = \frac{1}{5}\left(1-\frac{1}{3}\right)\left(1-\frac{1}{2}\right) = \frac{1}{p_3}\prod_{k=1}^{k=2}\left(1-\frac{1}{p_k}\right)$$

$$v_3 = \left(1-\frac{1}{5}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{2}\right) = \prod_{k=1}^{k=3}\left(1-\frac{1}{p_k}\right)$$

$$\cdots$$

$$u_j = \frac{1}{p_j}\prod_{k=1}^{k=j-1}\left(1-\frac{1}{p_k}\right)$$

$$v_j = \prod_{k=1}^{k=j}\left(1-\frac{1}{p_k}\right)$$

The quantities u_j et v_j are the terms of two sequences

$$u_j = \frac{1}{p_j}\prod_{k=1}^{k=j-1}\left(1-\frac{1}{p_k}\right) \tag{2.10}$$

et

$$v_j = \prod_{k=1}^{k=j}\left(1-\frac{1}{p_k}\right) \tag{2.11}$$

and we have

$$(\forall j \in \mathbb{N}^*) \quad (u_j + v_j = v_{j-1})$$

and

$$(\forall j \in \mathbb{N}^*) \quad \left(u_j = \frac{1}{p_j}v_{j-1}\right)$$

We also pose $u_0 = 0$ et $v_0 = 1$ by convention. Moreover

$$u_{j+1} = \frac{1}{p_{j+1}}\prod_{k=1}^{k=j}\left(1-\frac{1}{p_k}\right) = \frac{1}{p_{j+1}}\left(1-\frac{1}{p_j}\right)\prod_{k=1}^{k=j-1}\left(1-\frac{1}{p_k}\right)$$

hence

$$\left(u_{j+1} = \frac{p_j}{p_{j+1}}\left(1-\frac{1}{p_j}\right)u_j\right) \Longleftrightarrow \left(\frac{u_{j+1}}{u_j} = \frac{p_j - 1}{p_{j+1}} < \frac{p_j}{p_{j+1}} < 1\right)$$

which shows that the sequence u_j is decreasing. Now

$$(\forall j \in \mathbb{N}) \quad \left(u_j = \frac{1}{p_j}v_{j-1}\right)$$

and thus

$$\sum_{j=1}^{j=n} u_j = \sum_{j=1}^{j=n}\frac{1}{p_j}v_{j-1}$$

We can now proceed to the next chapter where we will present a path that could lead to a proof of the Goldbach's strong conjecture [5]. We will make use of results already widely known.

Chapter 3

About the Goldbach's strong conjecture

As already hinted at the end of chapter 2, let us begin with establishing some results with the help of Franz Mertens's works [6]

3.1 A lower bound of the sum of the inverses of the first n prime numbers

Let us consider the sum S of the inverses of the prime numbers. We have

$$S = \sum_{j=1}^{\infty} \frac{1}{p_j}$$

and for each prime number p_j

$$\frac{1}{1 - \frac{1}{p_j}} = \sum_{k=1}^{\infty} \frac{1}{p_j^k}$$

Let p_n be the nth prime number and let us choose the integer P such that

$$p_n \leq P < p_{n+1}$$

then

$$\prod_{j=1}^{j=n} \sum_{k=0}^{\infty} \left(\frac{1}{p_j}\right)^k = \sum_{n \in N_{p_n}} \frac{1}{n}$$

where N_{p_n} is the set of the natural integers the greatest prime divisor is p_n. Clearly

$$\sum_{j=1}^{j=P} \frac{1}{j} < \prod_{j=1}^{j=n} \sum_{k=0}^{\infty} \left(\frac{1}{p_j}\right)^k$$

yet

$$\frac{1}{1-\frac{1}{p_j}} = \sum_{k=1}^{\infty} \frac{1}{p_j^k} = 1 + \frac{1}{p_j} + \frac{1}{p_j^2} + \cdots = 1 + \frac{1}{p_j} + \frac{1}{p_j^2}\left(1 + \frac{1}{p_j} + \cdots\right)$$

and

$$\frac{1}{p_j^2}\left(1 + \frac{1}{p_j} + \cdots\right) = \frac{1}{p_j^2}\sum_{k=0}^{\infty}\frac{1}{p_j^k} = \frac{1}{p_j^2}\frac{1}{1-\frac{1}{p_j}} = \frac{1}{p_j\left(p_j - 1\right)}$$

and thus

$$\sum_{j=1}^{j=P}\frac{1}{j} < \prod_{j=1}^{j=n}\left(1 + \frac{1}{p_j} + \frac{1}{p_j\left(p_j-1\right)}\right)$$

but

$$1 > \int_{x=1}^{x=2}\frac{\mathrm{d}x}{x}$$

$$\int_{x=1}^{x=2}\frac{\mathrm{d}x}{x} > \frac{1}{2} > \int_{x=2}^{x=3}\frac{\mathrm{d}x}{x}$$

$$\cdots$$

$$\int_{x=j-1}^{x=j}\frac{\mathrm{d}x}{x} > \frac{1}{j} > \int_{x=j}^{x=j+1}\frac{\mathrm{d}x}{x}$$

$$\cdots$$

$$\int_{x=p_n-1}^{x=p_n}\frac{\mathrm{d}x}{x} > \frac{1}{p_n} > \int_{x=p_n}^{x=p_n+1}\frac{\mathrm{d}x}{x}$$

$$\cdots$$

$$\int_{x=P-1}^{x=P}\frac{\mathrm{d}x}{x} > \frac{1}{P} > \int_{x=P}^{x=P+1}\frac{\mathrm{d}x}{x}$$

and thus

$$\left(1 + \int_{x=1}^{x=p_n}\frac{\mathrm{d}x}{x} > \sum_{j=1}^{j=P}\frac{1}{j} > \int_{x=1}^{x=P+1}\frac{\mathrm{d}x}{x}\right)$$

$$\Longleftrightarrow$$

$$\left(1 + [\ln x]_{x=1}^{x=P} > \sum_{j=1}^{j=P}\frac{1}{j} > [\ln x]_{x=1}^{x=P+1}\right)$$

and et

$$\left(1 + \ln\left(P\right) > \sum_{j=1}^{j=P}\frac{1}{j} > \ln\left(p_n + 1\right)\right)$$

$$\Longrightarrow$$

$$\left(1 + \ln\left(P\right) > \sum_{j=1}^{j=P}\frac{1}{j} > \ln\left(P\right)\right)$$

It follows

$$\ln\ln(P) < \ln\prod_{j=1}^{j=n}\left(1+\frac{1}{p_j}+\frac{1}{p_j(p_j-1)}\right)$$

and we have

$$\ln\prod_{j=1}^{j=n}\left(1+\frac{1}{p_j}+\frac{1}{p_j(p_j-1)}\right)=\sum_{j=1}^{j=n}\ln\left(1+\frac{1}{p_j}+\frac{1}{p_j(p_j-1)}\right)$$

Let us now recall that

$$(\forall x\in\mathbb{R})\qquad\left(\left(\exp(x)=\sum_{j=1}^{\infty}\frac{x^j}{j!}\right)\Longrightarrow(\exp(x)\geq 1+x)\right)$$

and thus

$$\exp\left(\frac{1}{p_j}+\frac{1}{p_j(p_j-1)}\right)\geq 1+\frac{1}{p_j}+\frac{1}{p_j(p_j-1)}$$

and therefore

$$\ln\ln(P)\leq\sum_{j=1}^{j=n}\ln\exp\left(\frac{1}{p_j}+\frac{1}{p_j(p_j-1)}\right)$$

or else

$$\ln\ln(P)\leq\sum_{j=1}^{j=n}\left(\frac{1}{p_j}+\frac{1}{p_j(p_j-1)}\right)$$

but

$$\sum_{j=1}^{j=n}\frac{1}{p_j(p_j-1)}<\sum_{j=1}^{j=n}\frac{1}{p_j^2}<\sum_{j=1}^{\infty}\frac{1}{p_j^2}<1$$

and finally

$$\left(\ln\ln(P)\leq 1+\sum_{j=1}^{j=n}\left(\frac{1}{p_j}\right)\right)\Longleftrightarrow\left(\ln\ln(P)-1\leq\sum_{j=1}^{j=n}\left(\frac{1}{p_j}\right)\right)\tag{3.1}$$

3.2 An upper bound of the sum of the inverses of the n first prime numbers

Let us pose, for $1\leq j\leq n$

$$a_j=\frac{1}{\ln p_j}$$

$$b_j=\frac{\ln p_j}{p_j}$$

$$B_j = \sum_{k=1}^{k=j} b_k$$

First of all, let us consider

$$B_j = \sum_{k=1}^{k=j} \frac{\ln p_k}{p_k} = \sum_{k=1}^{k=j} \ln p_k^{\frac{1}{p_k}} = \ln \prod_{k=1}^{k=j} p_k^{\frac{1}{p_k}}$$

We notice that the function

$$y = x^{\frac{1}{x}} = \exp\left(\frac{1}{x}\ln x\right)$$

is differentiable and its derivative is

$$\frac{\mathrm{d}}{dx}y = \frac{\mathrm{d}}{dx}\left(\frac{1}{x}\ln x\right)\exp\left(\frac{1}{x}\ln x\right) = \left(\frac{1}{x^2}\left(1-\ln x\right)\right)x^{\frac{1}{x}}$$

and this derivative is negative when $x > e$. Therefore, for all $k > 2$

$$\frac{\ln p_k}{p_k} < \frac{\ln k}{k}$$

and thus

$$\sum_{k=2}^{k=m} \frac{\ln p_k}{p_k} < \sum_{k=2}^{k=m} \frac{\ln k}{k}$$

but

$$\int_{x=k-1}^{x=k} \frac{\ln x}{x}\mathrm{d}x < \frac{\ln k}{k} < \int_{x=k}^{x=k+1} \frac{\ln x}{x}\mathrm{d}x$$

and hence

$$\sum_{k=2}^{k=m} \frac{\ln k}{k} < \int_{x=2}^{x=m+1} \frac{\ln x}{x}\mathrm{d}x$$

and finally

$$\sum_{k=2}^{k=m} \frac{\ln p_k}{p_k} < \sum_{k=2}^{k=m} \frac{\ln k}{k} < [\frac{1}{2}\left(\ln x\right)^2]_2^{m+1}$$

and

$$\sum_{k=1}^{k=m} \frac{\ln p_k}{p_k} < \frac{\ln 2}{2} + \sum_{k=2}^{j=m} \frac{\ln k}{k} < \frac{1}{2}\left(\left(\ln\left(m+1\right)\right)^2 - \ln 2\left(\ln 2 - 1\right)\right)$$

We numerically check that

$$B_j = \sum_{k=1}^{k=j} \frac{\ln p_k}{p_k} < \ln p_j$$

when $j \leq 10$. Let us assume that this relationship holds for m, then

$$B_{m+1} = \sum_{k=2}^{k=m+1} \frac{\ln p_k}{p_k} = B_m + \frac{\ln p_{m+1}}{p_{m+1}} < \ln p_m + \frac{\ln p_{m+1}}{p_{m+1}}$$

and

$$B_{m+1} = \sum_{k=2}^{k=m+1} \frac{\ln p_k}{p_k} < \ln p_m + \ln p_{m+1}^{\frac{1}{p_{m+1}}}$$

and

$$B_{m+1} = \sum_{k=2}^{k=m+1} \frac{\ln p_k}{p_k} < \ln p_m p_{m+1}^{\frac{1}{p_{m+1}}}$$

Let us also assume

$$\left(p_{m+1}^{\frac{p_m}{p_{m+1}}} < p_m\right) \Longleftrightarrow \left(p_{m+1}^{p_m} < p_m^{p_{m+1}}\right)$$

or stated otherwise

$$p_m \ln p_{m+1} < p_{m+1} \ln p_m$$

yet the **Identity** function increases faster than the ln function. Consequently, there exists a prime number p_n such that

$$\left((\forall p_j > p_n) \quad \left(p_j < p_{j+1}^{\frac{p_j}{p_{j+1}}}\right)\right) \Longrightarrow \left(p_j p_{j+1}^{\frac{1}{p_{j+1}}} < p_{j+1}^{\frac{p_j+1}{p_{j+1}}} < p_{j+1}\right)$$

We check in this instance that $p_n = p_3 = 5$. We thus showed that

$$(\forall j) \quad \left(B_j = \sum_{k=1}^{k=j} \frac{\ln p_k}{p_k} < \ln p_j\right)$$

Let now p_n be the nth prime integer and let us choose the natural integer P such that

$$p_n \leq P < p_{n+1}$$

Let us consider the sequence

$$(a_{j-1} - a_j)\, B_{j-1}$$

and for each of its terms, let us develop. Then

$$\begin{aligned}(a_1 - a_2)\, B_1 &= \left(\frac{1}{\ln p_1} - \frac{1}{\ln p_2}\right) \frac{\ln p_1}{p_1} \\ &= \frac{1}{\ln p_1} \frac{\ln p_1}{p_1} - \frac{1}{\ln p_2} \frac{\ln p_1}{p_1} \\ &= \frac{1}{p_1} - \frac{1}{\ln p_2} \frac{\ln p_1}{p_1}\end{aligned}$$

$$
\begin{aligned}
(a_2 - a_3)\, B_2 &= \left(\frac{1}{\ln p_2} - \frac{1}{\ln p_3}\right)\left(\frac{\ln p_1}{p_1} + \frac{\ln p_2}{p_2}\right) \\
&= \frac{1}{\ln p_2}\left(\frac{\ln p_1}{p_1} + \frac{\ln p_2}{p_2}\right) - \frac{1}{\ln p_3}\left(\frac{\ln p_1}{p_1} + \frac{\ln p_2}{p_2}\right) \\
&= \frac{1}{p_2} + \frac{\ln p_1}{p_1}\frac{1}{\ln p_2} - \left(\frac{\ln p_1}{p_1}\frac{1}{\ln p_3} + \frac{\ln p_2}{p_2}\frac{1}{\ln p_3}\right)
\end{aligned}
$$

$$
\begin{aligned}
(a_3 - a_4)\, B_3 &= \left(\frac{1}{\ln p_3} - \frac{1}{\ln p_4}\right)\left(\frac{\ln p_1}{p_1} + \frac{\ln p_2}{p_2} + \frac{\ln p_3}{p_3}\right) \\
&= \frac{1}{\ln p_3}\left(\frac{\ln p_1}{p_1} + \frac{\ln p_2}{p_2} + + \frac{\ln p_3}{p_3}\right) - \frac{1}{\ln p_4}\left(\frac{\ln p_1}{p_1} + \frac{\ln p_2}{p_2} + \frac{\ln p_3}{p_3}\right) \\
&= \frac{1}{p_3} + \left(\frac{\ln p_1}{p_1}\frac{1}{\ln p_3} + \frac{\ln p_2}{p_2}\frac{1}{\ln p_3}\right) - \left(\frac{\ln p_1}{p_1}\frac{1}{\ln p_4} + \frac{\ln p_2}{p_2}\frac{1}{\ln p_4} + \frac{\ln p_3}{p_3}\frac{1}{\ln p_4}\right)
\end{aligned}
$$

$$\cdots$$

$$
\begin{aligned}
(a_{j-1} - a_j)\, B_{j-1} &= \left(\frac{1}{\ln p_{j-1}} - \frac{1}{\ln p_j}\right)\sum_{k=1}^{k=j-1}\frac{\ln p_k}{p_k} \\
&= \frac{1}{\ln p_{j-1}}\sum_{k=1}^{k=j-1}\frac{\ln p_k}{p_k} - \frac{1}{\ln p_j}\sum_{k=1}^{k=j-1}\frac{\ln p_k}{p_k} \\
&= \frac{1}{p_{j-1}} + \frac{1}{\ln p_{j-1}}\sum_{k=1}^{k=j-2}\frac{\ln p_k}{p_k} - \frac{1}{\ln p_j}\sum_{k=1}^{k=j-1}\frac{\ln p_k}{p_k}
\end{aligned}
$$

$$
\begin{aligned}
(a_j - a_{j+1})\, B_j &= \left(\frac{1}{\ln p_j} - \frac{1}{\ln p_{j+1}}\right)\sum_{k=1}^{k=j}\frac{\ln p_k}{p_k} \\
&= \frac{1}{\ln p_j}\sum_{k=1}^{k=j}\frac{\ln p_k}{p_k} - \frac{1}{\ln p_{j+1}}\sum_{k=1}^{k=j}\frac{\ln p_k}{p_k} \\
&= \frac{1}{p_j} + \frac{1}{\ln p_j}\sum_{k=1}^{k=j-1}\frac{\ln p_k}{p_k} - \frac{1}{\ln p_{j+1}}\sum_{k=1}^{k=j}\frac{\ln p_k}{p_k}
\end{aligned}
$$

$$\cdots$$

$$
\begin{aligned}
(a_{n-1} - a_n)\, B_{n-1} &= \left(\frac{1}{\ln p_{n-1}} - \frac{1}{\ln p_n}\right)\sum_{k=1}^{k=n-1}\frac{\ln p_k}{p_k} \\
&= \frac{1}{\ln p_{n-1}}\sum_{k=1}^{k=n-1}\frac{\ln p_k}{p_k} - \frac{1}{\ln p_n}\sum_{k=1}^{k=n-1}\frac{\ln p_k}{p_k} \\
&= \frac{1}{p_{n-1}} + \frac{1}{\ln p_{n-1}}\sum_{k=1}^{k=n-2}\frac{\ln p_k}{p_k} - \frac{1}{\ln p_n}\sum_{k=1}^{k=n-1}\frac{\ln p_k}{p_k}
\end{aligned}
$$

$$\begin{aligned}\left(a_n - \frac{1}{\ln P}\right) B_n &= \left(\frac{1}{\ln p_n} - \frac{1}{\ln P}\right) \sum_{k=1}^{k=n} \frac{\ln p_k}{p_k} \\ &= \frac{1}{\ln p_n} \sum_{k=1}^{k=n} \frac{\ln p_k}{p_k} - \frac{1}{\ln P} \sum_{k=1}^{k=n} \frac{\ln p_k}{p_k} \\ &= \frac{1}{p_n} + \frac{1}{\ln p_n} \sum_{k=1}^{k=n-1} \frac{\ln p_k}{p_k} - \frac{1}{\ln P} \sum_{k=1}^{k=n} \frac{\ln p_k}{p_k}\end{aligned}$$

Let us make the summation

$$\sum_{j=1}^{j=n-1} (a_j - a_{j+1}) B_j + \left(a_n - \frac{1}{\ln P}\right) \sum_{k=1}^{k=n} \frac{\ln p_k}{p_k} + \frac{1}{\ln P} \sum_{k=1}^{k=n} \frac{\ln p_k}{p_k}$$

with

$$\sum_{k=1}^{k=n} \frac{\ln p_k}{p_k} = B_n$$

and, further to what we already showed

$$(\forall j \in \mathbb{N}^*) \quad (B_j < \ln p_j)$$

we get

$$\left(\sum_{j=1}^{j=n-1} (a_j - a_{j+1}) B_j + \left(a_n - \frac{1}{\ln P}\right) B_n = \sum_{j=1}^{j=n} \frac{1}{p_j} - \frac{1}{\ln P} B_n\right)$$

$$\Longleftrightarrow$$

$$\left(\sum_{j=1}^{j=n} \frac{1}{p_j} = \sum_{j=1}^{j=n-1} (a_j - a_{j+1}) B_j + \left(a_n - \frac{1}{\ln P}\right) B_n + \frac{1}{\ln P} B_n\right)$$

and by writing the terms explicitly

$$\sum_{j=1}^{j=n} \frac{1}{p_j} = \sum_{j=1}^{j=n-1} \left(\frac{1}{\ln p_j} - \frac{1}{\ln p_{j+1}}\right) B_j + \left(\frac{1}{\ln p_n} - \frac{1}{\ln P}\right) B_n + \frac{1}{\ln P} B_n$$

or likewise

$$\sum_{j=1}^{j=n} \frac{1}{p_j} = \sum_{j=1}^{j=n-1} \frac{1}{\ln p_j \ln p_{j+1}} (\ln p_{j+1} - \ln p_j) B_j + \frac{1}{\ln p_n \ln P} (\ln P - \ln p_n) B_n + \frac{1}{\ln P} B_n$$

and thus

$$\sum_{j=1}^{j=n} \frac{1}{p_j} < \sum_{j=1}^{j=n-1} \frac{1}{\ln p_{j+1}} (\ln p_{j+1} - \ln p_j) + \frac{1}{\ln P} (\ln P - \ln p_n) + \frac{1}{\ln P} B_n$$

We have

$$\frac{1}{\ln p_{j+1}}\left(\ln p_{j+1}-\ln p_j\right)<\int_{x=p_j}^{x=p_{j+1}}\frac{1}{\ln x}\mathrm{d}\ln x<\frac{1}{\ln p_j}\left(\ln p_{j+1}-\ln p_j\right)$$

and

$$\frac{1}{\ln p_{j+1}}\sum_{j=1}^{j=n-1}\left(\ln p_{j+1}-\ln p_j\right)<\sum_{j=1}^{j=n-1}\int_{x=p_j}^{x=p_{j+1}}\frac{1}{\ln x}\mathrm{d}\ln x$$

but

$$\sum_{j=1}^{j=n-1}\int_{x=p_j}^{x=p_{j+1}}\frac{1}{\ln x}\mathrm{d}\ln x=\int_{x=p_1}^{x=p_n}\frac{1}{\ln x}\mathrm{d}\ln x=\ln\ln p_n-\ln\ln 2$$

similarly

$$\frac{1}{\ln P}\left(\ln P-\ln p_n\right)<\int_{x=p_n}^{x=P}\frac{1}{\ln x}\mathrm{d}\ln x<\frac{1}{\ln p_n}\left(\ln P-\ln p_n\right)$$

with

$$\int_{x=p_n}^{x=P}\frac{1}{\ln x}\mathrm{d}\ln x=\ln\ln P-\ln\ln p_n$$

We finally obtain the inequality

$$\sum_{j=1}^{j=n}\frac{1}{p_j}<\ln\ln P-\ln\ln 2+\frac{\ln p_n}{\ln P} \tag{3.2}$$

3.3 An approximation of the value of the finite Euler product of rank n

We have in general

$$\left(\forall a\in\mathbb{R}^+\right)\quad\left(\forall b\in\mathbb{R}^+\right)\quad(a<b)\quad\left(\frac{1}{b}<\int_{x=a}^{x=b}\frac{1}{x}\mathrm{d}x<\frac{1}{a}\right)$$

and

$$\int_{x=a}^{x=b}\frac{1}{x}\mathrm{d}x=\ln b-\ln a=\ln\frac{b}{a}$$

Let us pose

$$\frac{b}{a}=\frac{p_j}{p_j-1}=\left(1-\frac{1}{p_j}\right)^{-1}$$

we get

$$\forall p_j\in\mathbb{N}\quad\frac{1}{p_j}<\int_{x=p_j-1}^{x=p_j}\frac{1}{x}\mathrm{d}x<\frac{1}{p_j-1}$$

or likewise

$$\forall p_j \in \mathbb{N} \quad \frac{1}{p_j} < \ln \frac{p_j}{p_j - 1} < \frac{1}{p_j - 1}$$

but

$$\ln \frac{p_j}{p_j - 1} = -\ln \frac{p_j - 1}{p_j} = -\ln \left(1 - \frac{1}{p_j}\right)$$

and thus

$$\forall p_j \in \mathbb{N} \quad \frac{1}{p_j} < -\ln \left(1 - \frac{1}{p_j}\right) < \frac{1}{p_j - 1}$$

Now let us pose

$$-\ln \left(1 - \frac{1}{p_j}\right) = \frac{1}{p_j} + \epsilon_j$$

Clearly

$$0 < \epsilon_j < \frac{1}{p_j - 1} - \frac{1}{p_j} < \frac{1}{(p_j - 1)^2} < \frac{1}{j^2}$$

We have

$$-\sum_{j=1}^{j=n} \ln \left(1 - \frac{1}{p_j}\right) = \sum_{j=1}^{j=n} \frac{1}{p_j} + \sum_{j=1}^{j=n} \epsilon_j$$

but

$$\sum_{j=1}^{j=n} \epsilon_j < \sum_{j=1}^{j=n} \frac{1}{j^2} < 2$$

and hence

$$\sum_{j=1}^{j=n} \frac{1}{p_j} < -\sum_{j=1}^{j=n} \ln \left(1 - \frac{1}{p_j}\right) < \sum_{j=1}^{j=n} \frac{1}{p_j} + 2$$

Yet

$$\sum_{j=1}^{j=n} \ln \left(1 - \frac{1}{p_j}\right) = \ln \prod_{j=1}^{j=n} \left(1 - \frac{1}{p_j}\right)$$

and we can write

$$\sum_{j=1}^{j=n} \frac{1}{p_j} < -\ln \prod_{j=1}^{j=n} \left(1 - \frac{1}{p_j}\right) < \sum_{j=1}^{j=n} \frac{1}{p_j} + 2$$

or likewise, with $p_n \leq P < p_{n+1}$

$$\ln \ln P - 1 < -\ln \prod_{j=1}^{j=n} \left(1 - \frac{1}{p_j}\right) < \ln \ln P - \ln \ln 2 + \frac{\ln p_n}{\ln P} + 2$$

and, by posing $\mathrm{e} = \exp(1)$

$$\ln \left(\frac{\ln P}{\mathrm{e}}\right) < -\ln \prod_{j=1}^{j=n} \left(1 - \frac{1}{p_j}\right) < \ln \left(\frac{\ln P}{\mathrm{e}}\right) - \ln \ln 2 + \frac{\ln p_n}{\ln P} + 3$$

There exists thus a number μ_n such that

$$\left(0 < \ln \mu_n < 3 - \ln \ln 2 + \frac{\ln p_n}{\ln P}\right) \Longleftrightarrow \left(1 < \mu_n < \exp\left(3 - \ln \ln 2 + \frac{\ln p_n}{\ln P}\right)\right)$$

and such that

$$-\ln \prod_{j=1}^{j=n} \left(1 - \frac{1}{p_j}\right) = \ln\left(\frac{\ln P}{\mathrm{e}}\right) + \ln \mu_n = \ln\left(\frac{\mu_n}{\mathrm{e}} \ln P\right)$$

Let us pose

$$\left(\frac{\mu_n}{\mathrm{e}} = m_n\right) \Longleftrightarrow \left(\frac{1}{\mathrm{e}} < m_n < \exp\left(2 - \ln \ln 2 + \frac{\ln p_n}{\ln P}\right)\right)$$

we get

$$\prod_{j=1}^{j=n} \left(1 - \frac{1}{p_j}\right) = \frac{1}{m_n \ln P} = v_n > 0 \tag{3.3}$$

3.4 A possible way to a proof

Let us now revert to the Goldbach's strong conjecture and more specifically in light with what we developped in the previous paragraph. Let us choose the natural non prime integer m, and the two consecutive prime integers p_n and p_{n+1}, such that

$$p_n^2 < 2m < p_{n+1}^2$$

and the fumction G_{m,p_n}

$$\begin{aligned} G_{m,p_n} : \mathbb{R} &\longrightarrow [-1, 1] \\ x &\longmapsto G_{m,p_n}(x) \end{aligned}$$

with

$$G_{m,p_n}(x) = \prod_{j=1}^{j=n} \sin\left(\frac{\pi}{p_j} x\right) \sin\left(\frac{\pi}{p_j}(2m - x)\right)$$

This function is periodic with period

$$TG_{m,p_n} = \prod_{j=1}^{j=n} p_j$$

The divisors of m, which we assumed to be composite, belongs to the set π_{p_n} and thus

$$G_{m,p_n}(m) = 0$$

Furthermore, we know that there exists two consecutive prime integers p_ν and $p_{\nu+1}$, for which the respective periods TG_{m,p_ν} et $TG_{m,p_{\nu+1}}$ of the corresponding functions $G_{m.p_\nu}$ and $G_{m,p_{\nu+1}}$ are such that

$$TG_{m,p_\nu} < 2m < TG_{m,p_{\nu+1}}$$

Let u_k (see the equation 2.10 page 31) et v_k (see the equation 2.11 page 31) be the two sequences we already introduced

$$u_k = \frac{1}{p_k}\prod_{h=1}^{k-1}\left(1-\frac{1}{p_h}\right)$$

$$v_k = \prod_{h=1}^{k}\left(1-\frac{1}{p_h}\right)$$

We have

$$u_k = \frac{1}{p_k}v_{k-1}$$

and thus

$$\sum_{k=1}^{k=n} u_k = \sum_{k=1}^{k=n}\frac{1}{p_k}v_{k-1}$$

Now, in the interval

$$[-\frac{1}{2}TG_{m,p_\nu} + m, \frac{1}{2}TG_{m.p_\nu} + m] \subset [0, 2m]$$

let us consider on the one hand the sets we already defined in the previous chapter

- $\mathbb{A}_{p_k}$ the set of the natural integers the least prime divisor of which is p_k. The cardinal of this set is $|\mathbb{A}_{p_k}|$, and satisfies the inequality (see the equation 2.6 page 29)

$$|\mathbb{A}_{p_k}| \leq \frac{1}{p_k}\prod_{j=1}^{j=k-1}\left(1-\frac{1}{p_j}\right)TG_{m,p_\nu}$$

- $\mathbb{B}_{p_n}$ the set of the natural integers the least prime divisor of which is greater than p_k. The cardinal of this set is $|\mathbb{B}_{p_k}|$ and satisfies the inequality (see the equation 2.7 page 29)

$$|\mathbb{B}_{p_n}| \geq \prod_{j=1}^{j=n}\left(1-\frac{1}{p_j}\right)TG_{m,p_\nu}$$

- $\mathbb{C}_{p_n}$ the set of the natural integers the least prime divisor of which is less than p_n. The cardinal of this set is $|\mathbb{C}_{p_n}|$ (see the equation 2.3 page 30)

$$|\mathbb{C}_{p_n}| = \sum_{k=1}^{k=n} |\mathbb{A}_{p_k}|$$

and satisfies the inequality

$$|\mathbb{C}_{p_n}| \leq TG_{m,p_\nu} \sum_{k=1}^{n} u_k \tag{3.4}$$

and on the other hand, in the interval

$$[-\frac{1}{2}TG_{m,p_\nu} + m, m[$$

let us consider the sets

- $\mathbb{D}_{p_n}$ the set of the natural integers which the function G_{m,p_n} vanishes at. The cardinal of this set is $|\mathbb{D}_{p_n}|$ and satisfies the inequality (see the equation 2.4 page 28)

$$|\mathbb{D}_{p_n}| \leq a_n + b_n$$

- $\mathbb{E}_{p_n}$ the set of the natural integers which the function G_{m,p_n} does not vanish at. The cardinal of this set is $|\mathbb{E}_{p_n}|$ and satisfies the inequality (see the equation 2.5 page 28)

$$|\mathbb{E}_{p_n}| \geq \frac{1}{2}TG_{m,p_\nu} - (a_n + b_n)$$

The Goldbach's strong conjecture would be proved if we could verify

$$\left(|\mathbb{D}_{p_n}| < \frac{1}{2}TG_{m,p_\nu}\right) \Longleftrightarrow (|\mathbb{E}_{p_n}| > 0)$$

3.4.1 Considerations on the set $\mathbb{B}_{p_n}$

Let us consider $\mathbb{B}_{p_n}$ the set of the natural integers belonging to the interval

$$[-\frac{1}{2}TG_{m,p_n} + m, \frac{1}{2}TG_{m,p_n} + m[$$

the least prime divisor of which is greater than p_n We have

$$|\mathbb{B}_{p_n}| \geq \prod_{j=1}^{j=n} \left(1 - \frac{1}{p_j}\right) TG_{m,p_\nu}$$

with

$$TG_{m,p_\nu} = \prod_{j=1}^{j=\nu} p_j$$

Furthermore, we showed that (see the equation 3.3 page 42)

$$\prod_{j=1}^{j=n} \left(1 - \frac{1}{p_j}\right) = \frac{1}{m_n \ln P} = \upsilon_n > 0$$

with

$$\left(\frac{\mu_n}{\mathrm{e}} = m_n\right) \Longleftrightarrow \left(\frac{1}{\mathrm{e}} < m_n < \exp\left(2 - \ln\ln 2 + \frac{\ln p_n}{\ln P}\right)\right)$$
$$\Longleftrightarrow \left(\mathrm{e} > \frac{1}{m_n} > \exp\left(-2 + \ln\ln 2 - \frac{\ln p_n}{\ln P}\right)\right)$$

and

$$p_n \leq P < p_{n+1}$$

and thus

$$\left(|\mathbb{B}_{p_n}| \geq \frac{1}{m_n \ln P} TG_{m,p_\nu}\right) \Longrightarrow \left(|\mathbb{B}_{p_n}| \geq \frac{\exp\left(-2 + \ln\ln 2 - \frac{\ln p_n}{\ln P}\right)}{\ln P} TG_{m,p_\nu}\right)$$

Now, we notice that

$$(TG_{m,p_\nu} \subset [0, 2m]) \Longleftrightarrow ((\exists \lambda \in \mathbb{Q}^*)\,(1 \leq \lambda < p_{\nu+1})\,(\lambda TG_{m,p_\nu} = 2m))$$

with $p_n^2 < 2m < p_{n+1}^2$ and thus

$$\left(p_n^2 < \lambda TG_{m,p_\nu} < p_{n+1}^2 \Longleftrightarrow \frac{p_n^2}{\lambda} < TG_{m,p_\nu} < \frac{p_{n+1}^2}{\lambda}\right)$$
$$\Longrightarrow \left(\frac{p_n^2}{p_{\nu+1}} < TG_{m,p_\nu} < p_{n+1}^2\right)$$

and thus

$$|\mathbb{B}_{p_n}| > \frac{p_n^2}{p_{\nu+1} \ln P} \exp\left(-2 + \ln\ln 2 - \frac{\ln p_n}{\ln P}\right)$$

Yet, P can take any arbitrary value between p_n and p_{n+1}. Let us choose $P = p_n$ and we finally get

$$|\mathbb{B}_{p_n}| > \frac{p_n^2}{p_{\nu+1} \ln p_n} \exp\left(-3 + \ln\ln 2\right)$$

or more explicitly

$$|\mathbb{B}_{p_n}| > \frac{p_n^2}{29 p_{\nu+1} \ln p_n} > \frac{p_n}{29 \ln p_n}$$

One can then see that the cardinal $|\mathbb{B}_{p_n}|$ of the set $\mathbb{B}_{p_n}$ of the natural integers the least prime divisor is greater than p_n numerically satisfies

$$(|\mathbb{B}_{p_n}| > 1) \Longleftrightarrow (p_n \geq p_{35} = 149)$$

which seems to evidence that this set is not empty as soon as $p_n \geq 149$.

3.4.2 Considerations on the set $\mathbb{C}_{p_n}$

Let us consider the set $\mathbb{C}_{p_n}$. Its cardinal satisfies the following relations

$$|\mathbb{C}_{p_n}| = a_n + b_n$$

(see the equation 2.3 page 28) and

$$|\mathbb{C}_{p_n}| \leq TG_{m,p_\nu} \sum_{k=1}^{k=n} u_k$$

(see the equation 3.4 page 44)
Let us focus first on the equation 3.4, we get

$$\sum_{k=1}^{k=n} u_k = \sum_{k=1}^{k=n} \frac{1}{p_k} v_{k-1} = \frac{1}{2} + \sum_{k=2}^{k=n} \frac{1}{p_k} v_{k-1}$$

We can also write (see the equations 2.10 et 3.3, pages 31 and 42)

$$\sum_{k=2}^{k=n} u_k < \sum_{k=1}^{k=n} \frac{1}{m_{k-1} p_k \ln p_{k-1}}$$

or otherwise

$$\sum_{k=2}^{k=n} u_k < \frac{1}{\mathrm{e}} \sum_{k=2}^{k=n} \frac{1}{p_k \ln p_{k-1}} < \frac{1}{\mathrm{e}} \sum_{k=2}^{k=n} \frac{1}{p_k \ln p_k} < \frac{1}{2\mathrm{e}} \sum_{k=2}^{k=n} \frac{p_k - p_{k-1}}{p_k \ln p_k}$$

now

$$\frac{p_k - p_{k-1}}{p_k \ln p_k} < \int_{x=p_{k-1}}^{x=p_k} \frac{\mathrm{d}x}{x \ln x} < \frac{p_k - p_{k-1}}{p_{k-1} \ln p_{k-1}}$$

and

$$\int_{x=p_{k-1}}^{x=p_k} \frac{\mathrm{d}x}{x \ln x} = \int_{x=p_{k-1}}^{x=p_k} \frac{\mathrm{d}\ln x}{\ln x}$$

and hence

$$\sum_{k=2}^{k=n} \frac{p_k - p_{k-1}}{p_k \ln p_k} < \sum_{k=2}^{k=n} \int_{x=p_{k-1}}^{x=p_k} \frac{\mathrm{d}\ln x}{\ln x} < \sum_{k=2}^{k=n} \frac{p_k - p_{k-1}}{p_{k-1} \ln p_{k-1}}$$

or else

$$\sum_{k=2}^{k=n} \frac{p_k - p_{k-1}}{p_k \ln p_k} < \int_{x=p_1}^{x=p_n} \frac{\mathrm{d} \ln x}{\ln x} < \sum_{k=2}^{k=n} \frac{p_k - p_{k-1}}{p_{k-1} \ln p_{k-1}}$$

and finally

$$\sum_{k=2}^{k=n} \frac{p_k - p_{k-1}}{p_k \ln p_k} < [\ln \ln x]_{x=p_1}^{x=p_n} < \sum_{k=2}^{k=n} \frac{p_k - p_{k-1}}{p_{k-1} \ln p_{k-1}}$$

therefore

$$\sum_{k=2}^{k=n} u_k < \frac{1}{2\mathrm{e}} (\ln \ln p_n - \ln \ln 2)$$

In the interval $[-\frac{1}{2}TG_{m,p_\nu} + m, m[$, the number of natural integers which the function G_{m,p_n} vanishes at is less than or equal to $a_n + b_n$. These numbers are either even natural integers, in which case we have

$$(\forall k < m) \left(2k \in [-\frac{1}{2}TG_{m,p_\nu} + m, m[\right) (S_{p_n}(2k) = S_{p_n}(2m - 2k) = 0)$$

or odd natural integers. The cardinal of the set of these odd natural integers in the interval

$$[-\frac{1}{2}TG_{m,p_\nu} + m, \frac{1}{2}TG_{m,p_\nu} + m[$$

is equal to $\frac{1}{2}TG_{m,p_\nu}$ and the following inequalities are satisfied

$$\left(\frac{1}{2}(a_n + b_n) \leq \frac{1}{2}TG_{m,p_\nu} \sum_{k=2}^{k=n} u_k\right) \Longleftrightarrow \left(\frac{1}{2}(a_n + b_n) < \frac{1}{4\mathrm{e}} (\ln \ln p_n - \ln \ln 2)\, TG_{m,p_\nu}\right)$$

Now, the cardinal of the set of the odd natural integers which the function G_{m,p_n} vanishes at in the interval $[-\frac{1}{2}TG_{m,p_\nu} + m, m[$ is also less than or equal to $\frac{1}{2}(a_n + b_n)$. The cardinal of the set of the odd natural integers in the same interval is $\frac{1}{4}TG_{m,p_\nu}$. Let us try and fine the values of p_n for which

$$\left(\frac{1}{4\mathrm{e}} (\ln \ln p_n - \ln \ln 2)\, TG_{m,p_\nu} \leq \frac{1}{4}TG_{m,p_\nu}\right) \Longleftrightarrow ((\ln \ln p_n - \ln \ln 2) \leq \mathrm{e})$$

We get

$$\begin{aligned} ((\ln \ln p_n - \ln \ln 2) \leq \mathrm{e}) &\Longleftrightarrow (\ln \ln p_n \leq \mathrm{e} + \ln \ln 2) \\ &\Longleftrightarrow \left(\ln p_n \leq \mathrm{e}^{\mathrm{e} + \ln \ln 2}\right) \\ &\Longleftrightarrow \left(p_n \leq \mathrm{e}^{\mathrm{e}^{\mathrm{e} + \ln \ln 2}}\right) \end{aligned}$$

and we can numerically verify

$$\mathrm{e}^{\mathrm{e}^{\mathrm{e} + \ln \ln 2}} = 36\,465{,}95$$

Therefore, the cardinal of the set of the odd natural integers which the function G_{m,p_n} vanishes at in the interval

$$[-\frac{1}{2}TG_{m,p_\nu} + m, m[$$

is less than $\frac{1}{4}TG_{m,p_\nu}$ for all prime integer $p_n < 36\,466$. Finally, we notice that

$$\left(\left(\frac{1}{2}p_n^2 < m < \frac{1}{2}p_{n+1}^2\right) \wedge (p_n = 36\,466)\right)$$
$$\Longrightarrow$$
$$\left(\frac{1}{2}1\,329\,765\,293 < m < 2\,(1\,329\,765\,293)\right)$$

3.4.3 A likely conclusion

Based on the previous results, we can now state that on the one hand, the function $G_{m.p_n}$ cannot vanish for all the natural integers belonging to the interval $[-\frac{1}{2}TG_{m,p\nu} + m, \frac{1}{2}TG_{m,p\nu} + m[$ when $p_n < 36\,466$. On the other hand, in the same interval, there exists at least a prime integer greater than p_n as soon as $p_n > p_{35} = 149$. The Goldbach's strong conjecture seems to be partially proved, at least for each natural integer $m \leq \frac{1}{2}1\,329\,765\,293$ and we can fromulate the following theorem

Theorem 4 ***Goldbach's partial*** *For each natural integer* $2 \leq m < \frac{1}{2}1\,329\,765\,293$, *the even natural integer* $2m$ *is the sum of two prime numbers.*

Chapter 4

On an extension of the Joseph Bertrand's conjecture

4.1 Object of the chapter

Joseph Bertand proposed a conjecture later proved by Panufty Tchebychev, which we already mentioned in our introduction

Theorem 5 ***of Bertrand Tchebychev*** *For each $n > 1$, there exists at least one prime integer that belongs to the interval $]n, 2n]$.*

In a similar spirit, and based on numerical results obtained with a computer, we suggest the following conjecture

Conjecture 5 *Let p_n be a prime number, there exists at least one prime number in each and every interval $[kp_n, (k+1)\, p_n[$ for each and every non zero natural integer k such that $(k+1)\, p_n < p_{n+1}^2$.*

We will try over this chapter to prove this conjecture.

4.2 Our tools.

We recall first the definition of the set π_{p_n} that contains each and every prime number p_j less than or equal to a given prime number p_n

$$\pi_{p_n} = \{p_j | \left((c|p_j) \Longleftrightarrow (c \in \{1, p_j\}) \wedge (p_j \leq p_n) \right) \}$$

Let us consider the function

$$S_{p_n} : \mathbb{R} \longrightarrow [-1, 1]$$
$$x \longmapsto S_{p_n}(x)$$

with

$$S_{p_n}(x) = \prod_{j=1}^{j=n} s_{p_j}(x)$$

This function vanishes if and only if x is equal to one element , or the product of several elements, of π_{p_n}. Its period is

$$TS_{p_n} = 2\prod_{j=1}^{j=n} p_j$$

As the function S_{p_n} is the product of sin functions, it is

- odd when n is odd
- even when n is even

In the interval $[0, TS_{p_n}[$, we have

$$S_{p_n}(TS_{p_n}) = S_{p_n}(\frac{TS_{p_n}}{4}) = S_{p_n}(\frac{TS_{p_n}}{2}) = S_{p_n}(\frac{3TS_{p_n}}{4}) = 0$$

We also recall that, for two natural integers x_p and x_q chosen in the interval $[0, TS_{p_n}[$, we have (see the equations 1.1 et 1.2 page 4)

$$\left(x_p + x_q = \frac{1}{4}TS_{p_n}\right) \Longrightarrow \left(S_{p_n}(x_q) = (-1)^{n-1} S_{p_n}(x_p)\right)$$

$$\left(x_p + x_q = \frac{1}{2}TS_{p_n}\right) \Longrightarrow (S_{p_n}(x_q) = (-1)^{n} S_{p_n}(x_p))$$

4.3 Towards an extension of Bertrand Tchebychev's theorem.

4.3.1 The functions S_{p_n} et $S_{p_{n-1}}$ on the interval $[0, \frac{1}{2}TS_{p_n}[$

We notice that

$$[0, \frac{1}{2}TS_{p_n}[= [0, \frac{1}{4}TS_{p_n}[\cup[\frac{1}{4}TS_{p_n}, \frac{1}{2}TS_{p_n}[$$

Let now

$$[\frac{l}{4}TS_{p_{n-1}}, \frac{l+1}{4}TS_{p_{n-1}}] \quad (l \in \mathbb{N})$$

be the sequence of the under-intervals included in the interval $[0, \frac{1}{2}TS_{p_n}]$. There are $2p_n$ of these under-intervals in the interval $[0, \frac{1}{2}TS_{p_n}[$. Let us denote the endpoints of these under-intervals

$$M_0 = O_0 = 0$$

$$
\begin{aligned}
M_1 &= \frac{1}{4} T S_{p_{n-1}} \\
M_2 &= \frac{2}{4} T S_{p_{n-1}} \\
M_3 &= \frac{3}{4} T S_{p_{n-1}} \\
&\dots \\
M_l &= \frac{l}{4} T S_{p_{n-1}} \\
&\dots \\
M_{p_n} &= \frac{p_n}{4} T S_{p_{n-1}} \\
&\dots \\
M_{2p_n} &= \frac{2p_n}{4} T S_{p_{n-1}}
\end{aligned}
$$

All the endpoints M_l are natural integers multiple of p_{n-1}, and we have

$$[M_0, M_{2p_n}[= \bigcup_{l=0}^{l=2p_n-1} [M_l, M_{l+1}[$$

and

$$(\forall l \not\equiv 0 \quad [p_n])\,(M_l \not\equiv 0 \quad [p_n])$$

The figure 4.1 (see page 52) shows the endpoints M_l of each under-intervals in the circular representation of the interval $[0, \frac{1}{2}TS_{pn}[$ in the case where

$$(n = 6) \Longleftrightarrow ((p_n = 13) \wedge (p_{n-1} = 11))$$

Let us consider now the function $S_{p_{n-1}}$ in the interval $[0, \frac{1}{2}TS_{p_n}]$ and let us assume that there exists an under-interval $]A_t, B_t = A_t + p_n[$, in which this function $S_{p_{n-1}}$ vanishes at each and every natural odd integer. A_t is a natural integer assumed to be non zero and is not necessarily a multiple of p_n. This under-interval $]A_t, B_t[$ contains $p_n - 1$ natural integers. The divisors of each of these natural integers belong exclusively to the set $\pi_{p_{n-1}}$. We are then faced with two possibilities

- This under-interval $]A_t, B_t[$ contains a natural integer M_l. Because of the properties of symmetry of the function $S_{p_{n-1}}$, each natural integer M_l in the interval $[M_0, M_{2p_{n-1}}[$ belongs to one of the under-intervals $]A_t, B_t[$. In particular, the natural integer $M_0 = 0$ belongs to one of the under-intervals $]A_t, B_t[$. But we know that $S_{p_{n-1}}(1) \neq 0$. This possibility must therefore be ruled out.
- This under-interval $]A_t, B_t[$ does not contain any of the natural integers M_l. Because of the properties of symmetry of the function $S_{p_{n-1}}$, each under-interval contains an under-interval $]A_t, B_t[$.

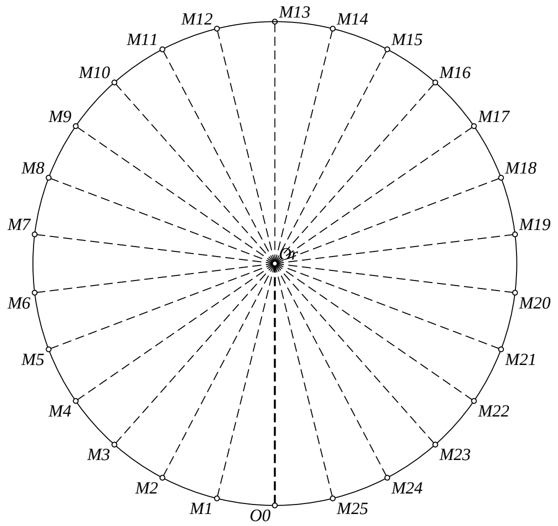

Figure 4.1: The under intervals $[M_l, M_{l+1}[$ on the circular representation of the interval $\left[0, \frac{1}{2}TS_{p_n}\right[$

Because of the properties of symmetry of the function $S_{p_{n-1}}$, each and every of the $2p_n$ under-intervals $[M_l, M_{l+1}[$ included in the interval $[0, \frac{1}{2}TS_{p_n}[$ contains itself an under-interval $]A_t, B_t[$. There are therefore $2p_n$ under-intervals $]A_t, B_t[$ in the interval $[0, \frac{1}{2}TS_{p_n}[$. We denote them

$$]A_0, B_0[$$

$$]A_1, B_1[$$

$$\cdots$$

$$]A_t, B_t[$$

$$]A_{t+1}, B_{t+1}[$$

$$\cdots$$

$$]A_{2p_n-2}, B_{2p_n-2}[$$

$$]A_{2p_n-1}, B_{2p_n-1}[$$

and we have

$$(\forall t \in \{0, 1, 2, \cdots, 2p_n - 2, 2p_n - 1\})\,(A_t \in [M_t, M_{t+1}[\Longleftrightarrow M_t < A_t < M_{t+1})$$

We shall say that the set of the under-intervals $]A_t, B_t[$ is generated by the under-interval $]A_0, B_0[$ and we will define this set as the **indexed family** of the under-intervals $\{]A_t, B_t[\}$. We should note that the under-interval $[M_0, M_1[$ may contain several under-intervals pairwise distinct, which we will denote $]A_0, B_0[_u$, where the index $u \in \mathbb{N}$ can take several different values. Hence, each under-interval $]A_0, B_0[_u$ generates the family $\{]A_t, B_t[_u\}$. In all that follows, we will choose one of these families $\{]A_t, B_t[_u\}$, that we will denote $\{]A_t, B_t[\}$ for the sake of simplicity. For each $t \in \mathbb{N}$ such that $0 \leq t \leq 2p_n - 1$, we have, because of the properties of symmetry of the function $S_{p_{n-1}}$

$$\frac{A_t + A_{t+1}}{2} = M_{t+1} = \frac{t+1}{4} TSp_{n-1}$$

In general, for two natural integers t_1 et t_2, of distinct parity, where

$$0 \leq t_1 < t_2 \leq 2p_n - 1$$

we have

$$\frac{A_{t_1} + A_{t_2}}{2} = M_{\frac{t_1+t_2}{2}+\frac{1}{2}}$$

Hence

$$\frac{A_{t+1} + A_{t+2}}{2} = M_{t+2}$$

and thus

$$\frac{A_{t+2} - A_t}{2} = M_{t+2} - M_{t+1} = \frac{1}{4} TSp_{n-1}$$

and finally

$$A_{t+2} - A_t = \frac{1}{2} TSp_{n-1}$$

and more generally, for $q \in \mathbb{N}$

$$A_{t+2q} - A_t = \frac{q}{2} TSp_{n-1}$$

Similarly, for each t such that $0 \leq t \leq p_n - 1$, we have, because of the properties of symmetry of the function S_{p_n}

$$\left(\frac{1}{2}\left(A_t + A_{2p_n-1-t}\right) = \frac{1}{4} TS_{p_n}\right) \Longleftrightarrow \left(A_{2p_n-1-t} + A_t = \frac{1}{2} TS_{p_n}\right)$$

We can therefore write

$$(\forall p_j \in \pi_{p_n})\,(A_{2p_n-1-t} \equiv -A_t \quad [p_j]) \tag{4.1}$$

In particular, for the natural integer α_t chosen in the set $\mathbb{Z}/p_n\mathbb{Z} = \{0, 1, p_n - 1\}$

$$(A_t \equiv \alpha_t \quad [p_n]) \Longleftrightarrow (A_{2p_n-1-t} \equiv -\alpha_t \quad [p_n])$$

The figure 4.2 shows the position of the under-intervals $]A_t, B_t[$ in the circular representation of the interval $[M_0, M_{2p_n-1}[= [0, \frac{1}{2}TS_{p_n}[$ and in the same manner as in the figure 4.1, where

$$(p_n = 13) \Longleftrightarrow (n = 6)$$

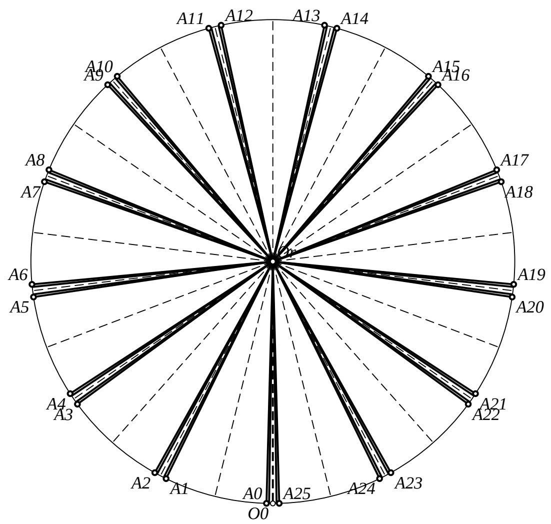

Figure 4.2: The under intervals $]A_t, B_t[$ on the circular representation of the interval $\left[0, \frac{1}{2}TS_{p_n}\right[$

For the sake of clarity, the figure only shows the endpoint A_t of each under-interval $]A_t, B_t[$.
Furthermore, the set of the under-intervals $]A_t, B_t[$ contains itself two subsets the elements of which are respectively the under-intervals $]A_{2\tau}, B_{2\tau}[$ and $]A_{2\tau+1}, B_{2\tau+1}[$, and we have for $q \in \mathbb{N}$ et $0 \leq q \leq \tau \leq p_n - 1$

$$(\forall\tau)\,(\forall q)\left(A_{2\tau+2q} - A_{2\tau} = \frac{q}{2}TS_{p_{n-1}}\right)$$

$$(\forall\tau)\,(\forall q)\left(A_{2\tau+1+2q} - A_{2\tau+1} = \frac{q}{2}TS_{p_{n-1}}\right)$$

These two relations show that for two natural integers t_1 et t_2 with the same parity, where

$$0 \leq t_1 < t_2 \leq p_n - 1$$

$$A_{t_2} \not\equiv A_{t_1} \quad [p_n]$$

Let us then consider the subset of the under-intervals $]A_t, B_t[$ inside the interval $[0, \frac{1}{2}TS_{p_n}[$, where t is chosen even. This set contains p_n under-intervals. The same goes for the other subset of the under-intervals $]A_t, B_t[$, where t is chosen odd. There exists then p_n natural integers A_t with a given parity. Lastly, we note

$$((\forall\tau \in \mathbb{Z}/p_n\mathbb{Z})\,(\forall q \in \mathbb{Z}/p_n\mathbb{Z})\,(q \leq \tau))\left(A_{2\tau+2q} = A_{2\tau} + \frac{q}{2}TS_{p_{n-1}}\right)$$

$$((\forall \tau \in \mathbb{Z}/p_n\mathbb{Z})\,(\forall q \in \mathbb{Z}/p_n\mathbb{Z})\,(q \leq \tau))\left(A_{2p_n-1-2\tau+2q} = A_{2p_n-1-2\tau} + \frac{q}{2}TS_{p_{n-1}}\right)$$

and thus

$$((\forall \tau \in \mathbb{Z}/p_n\mathbb{Z})\,(\forall q \in \mathbb{Z}/p_n\mathbb{Z})\,(q \leq \tau))\left(\forall p_j \in \pi_{p_{n-1}}\right)(A_{2\tau+2q} \equiv -A_{2p_n-1-2\tau+2q} \quad [p_j])$$

We can now enunciate the following lemma

Lemme 1 *Let* $[0, \frac{1}{2}TS_{p_n}[$ *be the interval Let the interval* $[0, \frac{1}{2}TS_{p_n}[$, *where* $p_n \geq 13$ *is the prime number of rank* n *in the set of the prime numbers. Let in this interval the set of the* $2p_n$ *under-intervals* $[\frac{l}{4}TSp_{n-1}, \frac{l+1}{4}TSp_{n-1}[= [M_l, M_{l+1}[$ *and let us assume that there exists at least one under-interval* $]A_t, B_t[$, *where* $B_t = A_t + p_n$, *in which the function* $S_{p_{n-1}}$ *vanishes at all the natural integers it contains, then*

- *this under-interval is entirely included in the under-interval* $[M_t, M_{t+1}[$ *with* $M_t < A_t$
- *there exists one under-interval* $]A_t, B_t[$ *in each of the* $2p_n$ *under-interval* $[\frac{l}{4}TSp_{n-1}, \frac{l+1}{4}TSp_{n-1}[= [M_l, M_{l+1}[$. *We number these under-intervals* $]A_0, B_0[$, $]A_1, B_1[$,..., $]A_t, B_t[$,..., $]A_{2p_{n-2}}, B_{2p_{n-2}}[$, $]A_{2p_{n-1}}, B_{2p_{n-1}}[$, *with*

  $$(\forall t \in \{0, 1, 2, \cdots, 2p_n - 2, 2p_n - 1\})\,(A_t \in [M_t, M_{t+1}[\Longleftrightarrow M_t < A_t < M_{t+1})$$

- *The set of these under-intervals* $]A_t, B_t[$ *contains itself two subsets the elements of which are respectively the under-intervals* $]A_{2k}, B_{2k}[$ *et* $]A_{2k+1}, B_{2k+1}[$, *and we have*

  $$(\forall p_j \in \pi_{p_n})\,(A_t \equiv -A_{2p_n-1-t} \quad [p_j])$$

  In particular, for a given natural integer a_t *chosen in the set*

  $$\mathbb{Z}/p_n\mathbb{Z} = \{0, 1, \cdots, p_n - 1\}$$

  each of these two subsets contains one and only one under-interval $]A_t, B_t[$, *where*

  $$A_t \equiv a_t \quad [p_n]$$

  and

  $$(A_t \equiv a_t \quad [p_n]) \Longleftrightarrow (A_{2p_n-1-t} \equiv -a_t \quad [p_n])$$

Let us pose

$$\frac{1}{2}TSp_{n-1} \equiv \alpha \quad [p_n]$$

$$A_0 \equiv a_0 \quad [p_n]$$

then, for the index τ_1 varying from 1 to p_{n-1}

$$A_2 = A_0 + \frac{1}{2}TSp_{n-1} \equiv a_2 = a_0 + \alpha \quad [p_n]$$

$$A_4 = A_0 + \frac{2}{2}TSp_{n-1} \equiv a_4 = a_0 + 2\alpha \quad [p_n]$$

$$A_6 = A_0 + \frac{3}{2}TSp_{n-1} \equiv a_6 = a_0 + 3\alpha \quad [p_n]$$

$$\cdots$$

$$A_{2\tau_1} = A_0 + \frac{\tau_1}{2}TSp_{n-1} \equiv a_{2\tau_1} = a_0 + \tau_1\alpha \quad [p_n]$$

$$\cdots$$

$$A_{2(p_n-1)} = A_0 + \frac{(p_n-1)}{2}TSp_{n-1} \equiv a_{2(p_n-1)} = a_0 + (p_n-1)\,\alpha \quad [p_n]$$

Similarly, let us pose

$$A_{2p_n-1} \equiv a_{2p_n-1} = -a_0 \quad [p_n]$$

then, for the index τ_2 varying from -1 to $-(p_n-1)$

$$A_{(2p_n-1)-2} = A_{2p_n-1} - \frac{1}{2}TSp_{n-1} \equiv a_{2p_n-3} = a_{2p_n-1} - \alpha \quad [p_n]$$

$$A_{(2p_n-1)-4} = A_{2p_n-1} - \frac{2}{2}TSp_{n-1} \equiv a_{2p_n-5} = a_{2p_n-1} - 2\alpha \quad [p_n]$$

$$A_{(2p_n-1)-6} = A_{2p_n-1} - \frac{3}{2}TSp_{n-1} \equiv a_{2p_n-1-7} = a_{2p_n-1} - 3\alpha \quad [p_n]$$

$$\cdots$$

$$A_{(2p_n-1)-2\tau_2} = A_{2p_n-1} - \frac{\tau_2}{2}TSp_{n-1} \equiv a_{2p_n-1-2\tau_2} = a_{2p_n-1} - \tau_2\alpha \quad [p_n]$$

$$\cdots$$

$$A_{(2p_n-1)-2(p_n-1)} = A_{2p_n-1} - \frac{p_n-1}{2}TSp_{n-1} \equiv a_1 = a_{2p_n-1} - (p_n-1)\,\alpha \quad [p_n]$$

and

$$a_{2p_n-3} \equiv -(a_0+\alpha) \quad [p_n]$$

$$a_{2p_n-5} \equiv -(a_0+2\alpha) \quad [p_n]$$

$$a_{2p_n-1-7} \equiv -(a_0+3\alpha) \quad [p_n]$$

$$\cdots$$

$$a_{2p_n-1-2\tau_2} \equiv -(a_0+\tau_2\alpha) \quad [p_n]$$

$$\cdots$$

$$a_1 = a_{2p_n-1-2(p_n-1)} \equiv -(a_0+(p_n-1)\,\alpha) \quad [p_n]$$

One of the natural integers $a_{2\tau_1}$, which we denote $a_{2\lambda}$, and only one is equal to zero, and

$$a_{2\lambda} = a_0 + \lambda\alpha \equiv 0 \quad [p_n]$$

In the case where $a_0 = 0$, we then notice that

$$A_0 \equiv 0 \quad [p_n]$$

and

$$A_{(2p_n-1)-2\tau_2} \equiv a_{(2p_n-1)-2\tau_2} = -\tau_2\alpha \quad [p_n]$$

Let us pose now $\tau_2 = p_n - \tau_1$

$$A_{(2p_n-1)-2(p_n-\tau_1)} = A_{2\tau_1-1} \equiv a_{2\tau_1-1} = j\alpha \quad [p_n]$$

We can finally write

$$((\forall \tau \in \mathbb{Z}/p_n\mathbb{Z})\,(A_0 \equiv 0 \quad [p_n]) \Longleftrightarrow (A_{2\tau} - A_{2\tau-1} \equiv 0 \quad [p_n])) \tag{4.2}$$

Let us consider again the set of the under-intervals $\{]A_t, B_t[\}$. Let us choose three pair-wise distinct integer indices t_1, t_2 et t_3 such that

$$M_{2t_1} = \frac{2t_1}{4} TSp_{n-1}$$

$$M_{2(p_n-1)-2t_1} = \frac{p_n - 1 - t_1}{4} TSp_{n-1}$$

$$M_{2t_2} = \frac{2t_2}{4} TSp_{n-1}$$

$$M_{2(p_n-1)-2t_2} = \frac{p_n - 1 - t_2}{4} TSp_{n-1}$$

$$M_{2t_3} = \frac{2t_3}{4} TSp_{n-1}$$

$$M_{2(p_n-1)-2t_3} = \frac{p_n - 1 - t_3}{4} TSp_{n-1}$$

then

$$A_{2t_1} = \frac{2t_1}{4} TSp_{n-1} + A_0$$

$$A_{2(p_n-1)-2t_1} = \frac{2\,(p_n - 1 - t_1)}{4} TSp_{n-1} - A_0$$

$$A_{2t_2} = \frac{2t_2}{4} TSp_{n-1} + A_0$$

$$A_{2(p_n-1)-2t_2} = \frac{2\,(p_n - 1 - t_2)}{4} TSp_{n-1} - A_0$$

$$A_{2t_3} = \frac{2t_3}{4} TSp_{n-1} + A_0$$

$$A_{2(p_n-1)-2t_3} = \frac{2\,(p_n - 1 - t_3)}{4} TSp_{n-1} - A_0$$

We get

$$M_{2t_2} - M_{2t_1} = A_{2t_2} - A_{2t_1} = \frac{2\,(t_2 - t_1)}{4} TSp_{n-1}$$

$$M_{2t_3} - M_{2t_2} = A_{2t_3} - A_{2t_2} = \frac{2(t_3 - t_2)}{4} TSp_{n-1}$$
$$M_{2t_1} - M_{2t_3} = A_{2t_1} - A_{2t_3} = \frac{2(t_1 - t_3)}{4} TSp_{n-1}$$

and likewise

$$M_{2(p_n-1)-2t_2} - M_{2(p_n-1)-2t_1} = A_{2(p_n-1)-2t_2} - A_{2(p_n-1)-2t_1} = -\frac{2(t_2 - t_1)}{4} TSp_{n-1}$$
$$M_{2(p_n-1)-2t_3} - M_{2(p_n-1)-2t_2} = A_{2(p_n-1)-2t_3} - A_{2(p_n-1)-2t_2} = -\frac{2(t_3 - t_2)}{4} TSp_{n-1}$$
$$M_{2(p_n-1)-2t_1} - M_{2(p_n-1)-2t_3} = A_{2(p_n-1)-2t_1} - A_{2(p_n-1)-2t_3} = -\frac{2(t_1 - t_3)}{4} TSp_{n-1}$$

Let us now assume

$$A_{2t_1} \equiv 0 \quad [p_n]$$

then

$$A_{2t_2} = \frac{2(t_2 - t_1)}{4} TSp_{n-1}$$
$$A_{2t_3} = \frac{2(t_3 - t_1)}{4} TSp_{n-1}$$

and we have

$$((A_{2t_1} \equiv 0 \quad [p_n]) \wedge (A_{2t_2} + A_{2t_3} \equiv 0 \quad [p_n])) \Longrightarrow (t_2 + t_3 \equiv 2t_1 \quad [p_n]) \tag{4.3}$$

Let us pose now $t_1 = 0$. We already showed that (see the equation 4.2 page 57)

$$(\forall j \in \mathbb{Z}/p_n\mathbb{Z}) \left(A_{2t_1} = A_0 \equiv 0 \quad [p_n] \Longleftrightarrow A_{2t_2} - A_{2t_2-1} \equiv 0 \quad [p_n]\right)$$

and in this case

$$A_{2t_2-1} = A_{2(p_n-1)-2t_3}$$

and thus

$$A_{2t_2} - A_{2t_2-1} = A_{2t_2} - A_{2(p_n-1)-2t_3} = \frac{2t_2}{4} TSp_{n-1} - \frac{2(p_n - 1 - t_3)}{4} TSp_{n-1}$$

and finally

$$A_{2t_2} - A_{2t_2-1} = \frac{2(t_2 - (p_n - 1 - t_3))}{4} TSp_{n-1} = \frac{2(t_2 - p_n + 1 + t_3)}{4} TSp_{n-1}$$

We should therefore have

$$t_2 + t_3 + 1 \equiv 0 \quad [p_n]$$

This leads us to a contradiction as we also showed (see the equation 4.3 page 58)

$$((A_{2t_1} \equiv 0 \quad [p_n]) \wedge (A_{2t_2} + A_{2t_3} \equiv 0 \quad [p_n])) \Longrightarrow (t_2 + t_3 \equiv 2t_1 = 0 \quad [p_n])$$

Consequently

$$(\forall] A_t, B_t [\in \{] A_t, B_t [\}) ((A_t \equiv 0 \quad [p_n]) \Longleftrightarrow (t \neq 0)) \tag{4.4}$$

This result, obtained for a given family $\{]A_t, B_t[_u\}$, is valid for each and every of these families and we can enunciate the following theorem

Theorem 6 *For all prime integer p_n and its associated function S_{p_n}, let the set of the intervals*

$$[kp_n, (k+1)\,p_n[$$

where k is any natural integer, and let the natural integer

$$M_1 = \frac{1}{4} TS_{p_{n-1}}$$

then

$$(\forall k \in \mathbb{N})\,(k < M_1)\,(\exists a \in ([kp_n, (k+1)\,p_n[\cap \mathbb{N}))\,(S_{p_n}(a) \neq 0)$$

Among other consequences, the conjecture that we set out above is verified and we can enunciate what is now a theorem

Theorem 7 *Let p_n be a given prime integer, there exists at least one prime integer in each interval $[kp_n, (k+1)\,p_n[$ for all non-zero natural integer k such that $(k+1)\,p_n < p_{n+1}^2$.*

A formula can be derived from the latter theorem. Let us consider the following sequence of the under-intervals

$$[p_n, 2p_n[$$

$$[2p_n, 3p_n[$$

$$\cdots$$

$$[kp_n, (k+1)\,p_n[$$

$$\cdots$$

$$[(p_n - 1)\,p_n, p_n^2[$$

Each of these under-intervals contains at least one prime integer that we respectively denote $p_{\nu+1}$, $p_{\nu+2}$, $\cdots$, $p_{\nu+k+1}$, $\cdots$, $p_{\nu+p_n}$, and we of course have

$$p_{n+1} \leq p_{\nu+1} \leq 2p_n$$

$$p_{n+2} \leq p_{\nu+2} \leq 3p_n$$

$$\cdots$$

$$p_{n+k+1} \leq p_{\nu+k+1} \leq (k+1)\,p_n$$

$$\cdots$$

$$p_{n+p_n} \leq p_{\nu+p_n} \leq p_n^2$$

and finally

$$\left(\prod_{j=n+1}^{j=n+p_n} p_j \leq p_n! p_n^{p_n - 1} \right) \Longleftrightarrow \left(\prod_{j=n+1}^{j=n+p_n} p_j \leq (p_n - 1)! p_n^{p_n} \right) \tag{4.5}$$

Chapter 5

Some thoughts on two other conjectures.

5.1 A conjecture proposed by Jean Marie legendre.

Jean Marie Legendre proposed the following conjecture.

Conjecture 6 ***of Legendre*** *For all natural integer $n \geqslant 2$, there exists at least a prime integer that belongs to the interval $[n^2, (n+1)^2]$.*

We give an approach that could lead to a rigorous proof of this conjecture. We recall the definition of the function S_{p_n}

$$\begin{aligned} S_{p_n} : \mathbb{R} &\longrightarrow [-1, 1] \\ x &\longmapsto S_{p_n} x \end{aligned}$$

with

$$S_{p_n}(x) = \prod_{j=1}^{j=n} s_{p_j}(x)$$

and

$$s_{p_j}(x) = \sin \frac{\pi}{p_j}(x)$$

We will use the following theorem, which we previously proved (see the theorem 7 page 59).

Theorem 8 *Let p_n be a given prime number, there exists at least one prime integer in each interval $[kp_n, (k+1)\, p_n[$ for all non-zero natural integer k such that $(k+1)\, p_n < p_{n+1}^2$.*

Each and every divisor of both the natural integers k and $k+1$ belongs to π_{p_n}. Neither of these two natural integers is divisible by a prime number greater than p_n. The union of the intervals $\bigcup_{j=1}^{\infty}[p_j, p_{j+1}[$ is the set of the real numbers greater than or equal to 2. We have

$$\bigcup_{j=1}^{\infty}[p_j, p_{j+1}[= \mathbb{R}^+ - \{1\}$$

We check first of all that

$$\begin{aligned} &1^2 < 3 < 2^2 \\ &2^2 < 5 < 7 < 3^2 \\ &3^2 < 11 < 13 < 4^2 \\ &\dots \end{aligned}$$

Let us consider, which is always possible, the natural integer m such that $p_j \leq m < m+1 \leq p_{j+1}$. Then

$$p_j^2 \leq m^2 < (m+1)^2 \leq p_{j+1}^2$$

The interval $[p_j^2, p_{j+1}^2]$ contains a finite set of intervals $[kp_j, (k+1)\,p_j[$, where $k \in N$. There then exists a natural integer K such that

$$Kp_j < p_{j+1}^2 < (K+1)p_j$$

Let us consider m^2 and

$$(m+1)^2 = m^2 + 2m + 1$$

It is clear that

$$(\exists k \in \mathbb{N})\left(m^2 \in [kp_j, (k+1)\,p_j[\right)$$

In order for the Legendre's conjecture to be true, we simply have to show that

$$(\forall k \in \mathbb{N})\left(m^2 \in [kp_j, (k+1)p_j[\right) \Longrightarrow \left((m+1)^2 \geq (k+2)\,p_j\right)$$

and then invoke the here-above mentioned theorem (see the theorem 7 page 59). We just have to show that.

$$2m + 1 > 2p_j$$

One can see that the latter inequality is always true. Indeed

$$2m + 1 > 2p_j \Longleftrightarrow m \geq p_j$$

which is our prerequisite. The conjecture is therefore proved whenever

$$m + 1 < Kp_j$$

Kp_j being the largest natural integer multiple of p_j less than p_{i+1}^2.

We now have to look into the intervals

$$[(K-1)p_j, Kp_j[$$

and

$$[Kp_j, (K+1)p_j[$$

where

$$p_{i+1}^2 \in [Kp_j, (K+1)p_j[$$

We have

$$Kp_j < p_{i+1}^2 < (K+1)p_j$$

and thus the natural integers

$$p_{i+1}^2 - (2m+1)$$

and

$$(m+1)^2 - (2m+1)$$

that is to say m^2, are both strictly less than Kp_j. Indeed

$$\begin{aligned}(m \geq p_j) &\iff \left(p_{i+1}^2 - 2m \leq p_{i+1}^2 - 2p_j\right)\\ &\iff \left(p_{i+1}^2 - (2m+1) < p_{i+1}^2 - 2p_j\right)\end{aligned}$$

and thus

$$(m+1)^2 - (2m+1) < p_{i+1}^2 - (2m+1) < Kp_j$$

This completes the proof of this conjecture and allows to enunciate what is now a theorem

Theorem 9 ***of Legendre*** *For all natural integer $n \geqslant 2$, there exists at least a prime integer that belongs to the interval $[n^2, (n+1)^2]$.*

5.2 A conjecture proposed by Henri Brocard.

For his part, Henri Brocard proposed this other conjecture

Conjecture 7 ***of Brocard*** *For all prime integer $p_n \geqslant 2$, there exists at least four prime integers that belong to the interval $[p_n^2, p_{(n+1)}^2]$.*

We will show that there exists at least four under-intervals $[kp_n, (k+1)p_n[$, with $k \in \mathbb{N}$, that are included in the interval $[p_n^2, p_{n+1}^2[$, for each prime integer p_n. these under-intervals are explicitly of the form

$$[(p_n+k)p_n, (p_n+k+1)p_n[\quad (k \in \mathbb{N}^*)$$

We know that

$$(\forall n \in \mathbb{N}^*)\,(p_{n+1} - p_n \geq 2) \Longleftrightarrow \left(p_{n+1}^2 \geq p_n^2 + 4p_n + 1\right)$$

but $p_n^2 + 4p_n$ is the upper endpoint of the fourth under-interval

$$[(p_n + k)p_n, (p_n + k + 1)p_n[\quad (k = 3)$$

Each of these under-interval contains at least one prime integer, further to the here-above mentioned theorem (see the theorem 7 page 59). The conjecture is therefore proved and we end up with the following theorem

Theorem 10 ***of Brocard*** *For all prime integer $p_n \geqslant 2$, there exists at least four prime integers that belong to the interval $[p_n^2, p_{(n+1)}^2]$.*

Chapter 6

Lemma relating to the function $S^1_{p_n}$.

The functions S_{p_n} et $S^1_{p_n}$ vanish at the same odd natural integers in the interval $[0, TS_{p_n}[$. The study of some properties of the function $S^1_{p_n}$ may thus give us an insight on the behaviour of the function S_{p_n} itself.

6.1 One property of the function $S^1_{p_n}$.

Given a prime number $p_n \geq 13$, let us consider the function $S^1_{p_n}$ in the closed interval $[kp_n, (k+1)\, p_n]$

$$S^1_{p_n}(x) = \prod_{j=2}^{j=n} \sin\left(\frac{\pi}{p_j} x\right)$$

and let us assume that this function vanishes at all the odd natural integers m_h in this interval, with $h \in N^*$. These natural integers are of the form

$$m_h = \left(\prod_{k=2}^{k=n} p_k^{a_k}\right)\left(\prod_{k=n+1}^{k=\nu} p_k^{a_k}\right)$$

Thus, there exists at least one function s_{p_j} that vanishes at each of these natural integers m_h. We have

$$s_{p_j}(m_h) = \sin\frac{\pi}{p_j}(m_h) = 0$$

As m_h is odd, we have for each prime integer p_j that divide p_j

$$\left(s_{p_j}(m_h) = \sin\frac{\pi}{p_j}(m_h) = 0\right) \Longleftrightarrow \left(s_{p_j}\left(\frac{1}{2}m_h\right) = \sin\frac{\pi}{2}\left(\frac{m_h}{p_j}\right) = \pm 1\right)$$

and we can write

$$s_{p_j}\left(\frac{1}{2}m_h\right)=\pm 1 \Longleftrightarrow c_{p_j}\left(\frac{1}{2}m_h\right)=0$$

Let us consider then the function $C^1_{p_n}$ such that

$$C^1_{p_n}(x)=\prod_{j=2}^{j=n}\cos\left(\frac{\pi}{p_j}x\right)$$

This function vanishes at each number $\frac{1}{2}m_h$ in the closed interval

$$[\frac{1}{2}kp_n,\frac{1}{2}(k+1)p_n]$$

All these numbers are strictly rational and we have

$$(\forall h)\left((m_{h+1}-m_h=2)\Longleftrightarrow\left(\frac{1}{2}m_{h+1}-\frac{1}{2}m_h=1\right)\right)$$

and

$$(\forall h)\left(\left(m_h\pm\frac{1}{2}\right)\in N\right)$$

Furthermore, we note that

$$C^1_{p_n}(x)=C^1_{p_n}\left(x+\frac{1}{2}-\frac{1}{2}\right)=\prod_{j=2}^{j=n}\cos\left(\frac{\pi}{p_j}\left(x+\frac{1}{2}\right)-\frac{1}{2}\right)$$

Let us now consider

$$\cos\left(\frac{\pi}{p_j}\left(x-\frac{1}{2}\right)\right)=\cos\left(\frac{\pi}{p_j}\left(x-\frac{1}{2}\right)+(2l_j+1)\frac{\pi}{2}-(2l_j+1)\frac{\pi}{2}\right)$$

with $l_j\in\mathbb{N}$. Then

$$\begin{aligned}\cos\left(\frac{\pi}{p_j}\left(x-\frac{1}{2}\right)\right)&=\cos\left(\frac{\pi}{p_j}\left(x-\frac{1}{2}\right)+(2l_j+1)\frac{\pi}{p_j}\frac{p_j}{2}-(2l_j+1)\frac{\pi}{2}\right)\\&=\cos\left(\frac{\pi}{p_j}\left(\left(x-\frac{1}{2}\right)+(2l_j+1)\frac{p_j}{2}\right)-(2l_j+1)\frac{\pi}{2}\right)\end{aligned}$$

and

$$\cos\left(\frac{\pi}{p_j}\left(x-\frac{1}{2}\right)\right)=\pm\sin\left(\frac{\pi}{p_j}\left(x+\frac{1}{2}\left((2l_j+1)p_j-1\right)\right)\right)$$

Therefore

$$\prod_{j=2}^{j=n}\cos\left(\frac{\pi}{p_j}\left(x-\frac{1}{2}\right)\right)=\pm\prod_{j=2}^{j=n}\sin\left(\frac{\pi}{p_j}\left(x+\frac{1}{2}\left((2l_j+1)p_j-1\right)\right)\right)$$

Let us pose

$$\alpha = \frac{1}{2}\left(\left(2l_j + 1\right)p_j - 1\right)$$

and let us chose α such that α is independent from the index j, then α can be equal to

$$\alpha = \frac{1}{2}\left(\prod_{j=2}^{j=n} p_j - 1\right)$$

and we write

$$\prod_{j=2}^{j=n} \cos\left(\frac{\pi}{p_j}\left(x - \frac{1}{2}\right)\right) = \pm \prod_{j=2}^{j=n} \sin\left(\frac{\pi}{p_j}\left(x + \alpha\right)\right)$$

In particular, whenever the function $S^1_{p_n}$ vanishes at each of the odd natural integers m_h in the interval $[kp_n, (k+1)p_n]$ then the function $C^1_{p_n}$ vanishes at each of the rational numbers $\frac{1}{2}m_h$ in the interval $[\frac{1}{2}k_{p_n}, \frac{1}{2}(k+1)p_n]$ and, in this same interval, we have

$$\prod_{j=2}^{j=n} \cos\left(\frac{\pi}{p_j}\left(\frac{1}{2}m_h\right)\right) = \pm \prod_{j=2}^{j=n} \sin\left(\frac{\pi}{p_j}\left(\frac{1}{2}\left(m_h + 1\right) + \alpha\right)\right) = 0$$

This means that the function $S^1_{p_n}$ vanishes at each of the integers in the interval $[\frac{1}{2}(kp_n + 1) + \alpha, \frac{1}{2}((k+1)p_n + 1) + \alpha]$. Hence the following lemma

Lemme 2 *Let a prime number $p_n \geq 13$ and the function $S^1_{p_n}$, if this function vanishes at all the odd natural numbers of the interval $[kp_n, (k+1)p_n]$, then there exists a number $\alpha = \frac{1}{2}\left(\prod_{j=2}^{j=n} p_j - 1\right)$ such that the function $S^1_{p_n}$ vanishes at all the natural integers of the interval $[\frac{1}{2}(kp_n + 1) + \alpha, \frac{1}{2}((k+1)p_n + 1) + \alpha]$.*

Let us pose

$$\frac{1}{2}\left(kp_n + 1\right) + \alpha = a$$

$$\frac{1}{2}\left(\left(k+1\right)p_n + 1\right) + \alpha = b$$

It is clear that one and only on of the two numbers a and b is a natural integer depending on the parity of the natural integer k. Let now m_{h_1} and m_{h_2} be two distinct natural integers chosen in the interval $[kp_n, (k+1)p_n]$ such that $m_{h_1} < m_{h_2}$, then their respective images in the interval $[a, b]$ are $\alpha + \frac{1}{2}(m_{h_1} + 1)$ and $\alpha + \frac{1}{2}(m_{h_2} + 1)$. These images are distinct and we have

$$\alpha + \frac{1}{2}\left(m_{h_2} + 1\right) - \alpha + \frac{1}{2}\left(m_{h_1} + 1\right) = \frac{1}{2}m_{h_2} - \frac{1}{2}m_{h_1} > 0$$

Thus, the function S^1_{13} vanishes at all the odd natural integers in the interval $[2184, 2197[$, where $k = 168$, and all the natural integers of the interval $[8599.5, 8606]$ (see figure 6.1 page 68). Similarly, the same function vanishes at all the odd natural integers in the interval $[9113, 9126[$, where $k = 701$, and all the natural integers of the interval $[12064, 12070.5]$ (see figure 6.2 page 68).

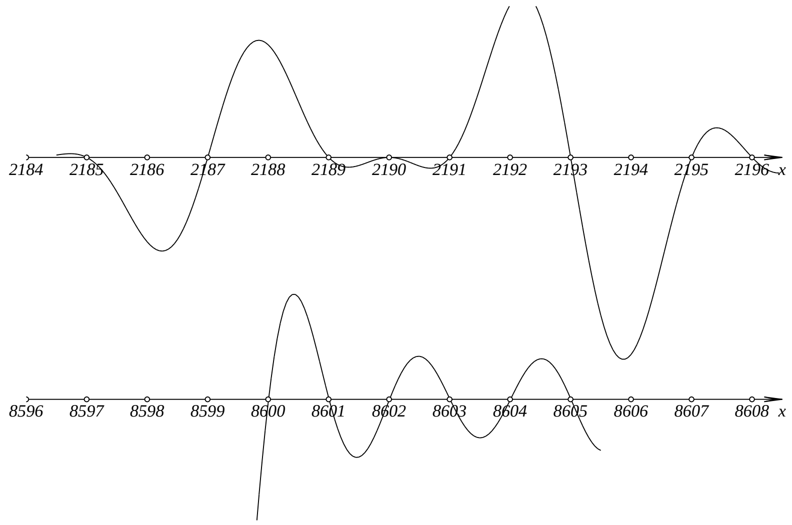

Figure 6.1: The function S^1_{13} on the intervals $[2184, 2197[$ et $[8599.5, 8606[$

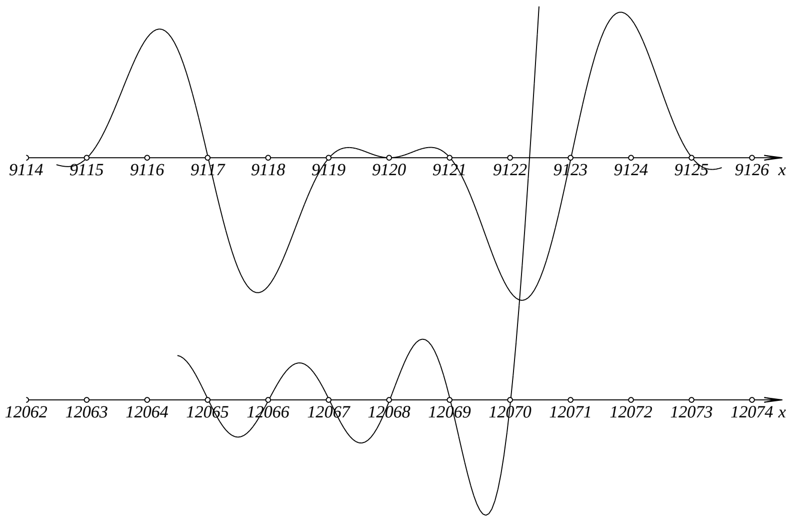

Figure 6.2: The function S^1_{13} on the intervals $[9113, 9126[$ et $[12064, 12070.5[$

Acknowledgements

6.2 Acknowledgements

I would like to thank here all the people who, thanks to their support and encouragements, made me love and admire this world and try and understand it. This memoir is also a testimony that their efforts have born fruit.

6.3 Softwares used

This work would have never been possible without the existence of the following softwares

- LaTeX. This remarkable piece of software was simply indispensable. The ability to work on documents that were always clear and easily modifiable gave me the opportunity to develop my ideas without paper drafts. My warmest thanks to the LaTeX community.

- WinGCLC. This geometry software proved to be easy to use. It allowed me to make all the illustrations contained in this work. I would like to thank its author, Mister Pedrag Janicic of the University of Belgrade, as well as the numerous co-authors.

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