A PROPERTY FOR A COUNTEREXAMPLE TO CARMICHAËL'S CONJECTURE

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Carmichaël has conjectured that:
$(\forall)\ n \in \mathbb{N},\ \ (\exists)\ m \in \mathbb{N}$, with $m \neq n$, for which $\varphi(n) = \varphi(m)$, where φ is Euler's totient function.

There are many papers on this subject, but the author cites the papers which have influenced him, especially Klee's papers.

Let n be a counterexample to Carmichaël's conjecture.

Grosswald has proved that n is a multiple of 32, Donnelly has pushed the result to a multiple of 2^{14}, and Klee to a multiple of $2^{42} \cdot 3^{47}$, Smarandache has shown that n is a multiple of $2^2 \cdot 3^2 \cdot 7^2 \cdot 43^2$. Masai & Valette have bounded $n > 10^{10000}$.

In this note we will extend these results to: n is a multiple of a product of a very large number of primes.

We construct a recurrent set M such that:

a) the elements $2, 3 \in M$;

b) if the distinct elements $2, 3, q_1, ..., q_r \in M$ and $p = 1 + 2^a \cdot 3^b \cdot q_1 \cdots q_r$ is a prime, where $a \in \{0, 1, 2, ..., 41\}$ and $b \in \{0, 1, 2, ..., 46\}$, then $p \in M$; $r \geq 0$;

c) any element belonging to M is obtained only by the utilization (a finite number of times) of the rules a) or b).

Of course, all elements from M are primes.

Let n be a multiple of $2^{42} \cdot 3^{47}$;
if $5 \nmid n$ then there exists $m = 5n/4 \neq n$ such that $\varphi(n) = \varphi(m)$; hence
$5 \mid n$; whence $5 \in M$;
if $5^2 \nmid n$ then there exists $m = 4n/5 \neq n$ with our property; hence $5^2 \mid n$;
analogously, if $7 \nmid n$ we can take $m = 7n/6 \neq n$, hence $7 \mid n$; if $7^2 \nmid n$ we can take $m = 6n/7 \neq n$; whence $7 \in M$ and $7^2 \mid n$; etc.

The method continues until it isn't possible to add any other prime to M, by its construction.

For example, from the 168 primes smaller than 1000, only 17 of them do not belong to M (namely: 101, 151, 197, 251, 401, 491, 503, 601, 607, 677, 701, 727, 751, 809, 883, 907, 983); all other 151 primes belong to M.

Note $M = \{2, 3, p_1, p_2, ..., p_s, ...\}$, then n is a multiple of $2^{42} \cdot 3^{47} \cdot p_1^2 \cdot p_2^2 \cdots p_s^2 \cdots$
From our example, it results that M contains at least 151 elements, hence $s \geq 149$.

If M is infinite then there is no counterexample n, whence Carmichaël's conjecture is solved.

(The author conjectures M is infinite.)

Using a computer it is possible to find a very large number of primes, which divide n, using the construction method of M, and trying to find a new prime p if $p-1$ is a product of primes only from M.

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