Derivation of two-valuedness and angular momentum of spin-1/2 from rotation of 3-sphere

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We derive the two-valuedness and the angular momentum of a spin-1/2 from a rotation of 3-dimensional surface of a sphere existing in extra 4-dimensional space other than normal 3-dimensional space, in this paper.

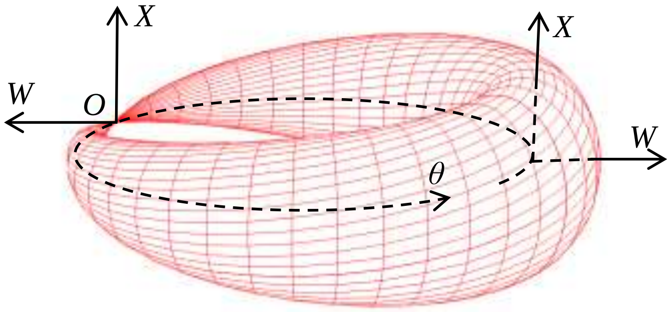

Figure 3-8: Wave function of spin-1/2 particle

We will derive the two-valuedness of the spin as follows.
We introduce 3-dimensional surface of a sphere S^3 existing in extra 4-dimensional space (W, X, Y, Z) other than normal 3-dimensional space (x, y, z). We interpret the angle of rotation of the 3-sphere S^3 as the phase of a wave function. We interpret the 3-sphere S^3 as the absolute value of a wave function.

We can express 3-sphere as the manifold with a constant sum of squares of the radius of two circles. When one circle's radius becomes the maximum, the other circle's radius becomes zero. Therefore, we can turn the circle inside out naturally. If we combine the circle turned inside out with the original circle, the manifold becomes a torus with a node. If we rotate the node of the torus by 360 degrees, we can turn the torus inside out. If we rotate the node of the torus 720 degrees, we can return the torus to the original state. This property is consistent with the property of the spin.

We derive the angular momentum of the spin as follows.
We make 3-dimensional solid sphere by removing one point from 3-sphere S^3. On the other hand, we can make boundary like a 3-sphere S^3 by removing one point from normal 3-dimensional space (x, y, z). We combine the boundaries of them. By repeating this, we can construct 3-dimensional helical space.

The angle of rotation of the 3-sphere S^3 is the angle of rotation of 3-dimensional helical space. On the other hand, we can interpret the angle of the rotation in the helical space as the coordinates (x, y,

z) of the normal 3-dimensional space. Therefore, we can interpret the angular momentum of the 3-sphere S^3 as the angular momentum of normal space.

CONTENTS

1 Introduction

1.1 Subject

In order to unify gravity and electromagnetic force, we introduced 1-sphere in extra dimensional space. In order to derive two-valuedness and angular momentum of spin-1/2, we introduce 3-sphere in extra dimensional space in this paper.

1.2 Importance of Subject

Although many researchers have tried quantization of gravity, they have not resulted in the success. Quantization of the gravity has been an important issue of physics.

One method for quantizing gravity was to interpret a point particle as a string that is 1-dimensional manifold. Therefore, we can deduce that it is effective to interpret a wave function of a particle as a manifold.

1.3 History of research

1.3.1 History of research of spin

George Eugene Uhlenbeck and Samuel Abraham Goudsmit discovered the spin of the electron in 1925. Wolfgang Pauli formulated the spin by the Pauli matrices in 1927. Paul Adrien Maurice Dirac derived the spin by the Dirac equation in 1928.

1.3.2 History of research of manifold

Albert Einstein constructed the general theory of relativity by the 4-dimensional Riemann manifold in 1916. Theodor Kaluza[2] and Oskar Klein[3] constructed proposed in the Kaluza-Klein theory by the 1-dimensional circle in 1926.

1.4 New construction method of this paper

We derive the angular momentum of the spin as follows.
We make 3-dimensional solid sphere by removing one point from 3-sphere. On the other hand, we can make boundary like a 3-sphere by removing one point from normal 3-dimensional space. We combine the boundaries of them. By repeating this, we can construct 3-dimensional helical space.

The angle of rotation of the 3-sphere is the angle of rotation of 3-dimensional helical space. On the other hand, we can interpret the angle of the rotation in the helical space as the coordinates of the normal 3-dimensional space. Therefore, we can interpret the angular momentum of the 3-sphere as the angular momentum of normal space.

2 Confirmation of the traditional research of the spin

2.1 Pauli matrices and quaternion

In order to express the spin, Pauli defined the following Pauli matrices in 1927.

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.1}$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.2}$$

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{2.3}$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.4}$$

The products are shown below.

$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = -i\sigma_1\sigma_2\sigma_3 = \sigma_0 \tag{2.5}$$

Here, we define the following matrices.

$$E = \sigma_0 \tag{2.6}$$

$$I = i\sigma_3 \tag{2.7}$$

$$J = i\sigma_2 \tag{2.8}$$

$$K = i\sigma_1 \tag{2.9}$$

The products are shown below.

$$I^2 = J^2 = K^2 = IJK = -E \tag{2.10}$$

These are the matrix representations of the following quaternions.

$$i^2 = j^2 = k^2 = ijk = -1 \tag{2.11}$$

William Rowan Hamilton discovered the quaternions in 1843.

2.2 Confirmation of the experiment of two-valuedness and angular momentum of spin

2.2.1 Verification of two-valuedness of the spin by experiment

H. Rauch[4] and S. A. Werner[5] verified the two-valuedness of the spin by the Neutron Interference Experiment in 1975. In this section, we confirm the two-valuedness of the spin.

We can express the wave function of a particle rotating about the *z*-axis by the Pauli matrices as follows.

$$\psi(\theta) = \exp\left(-\frac{i\sigma_3\theta}{2}\right) \tag{2.12}$$

The rotating the angle of the rotation θ by 360 degrees does not bring it back to the same state, but to the state with the opposite phase. The rotating the angle of the rotation θ by 720 degrees brings it back to the original state.

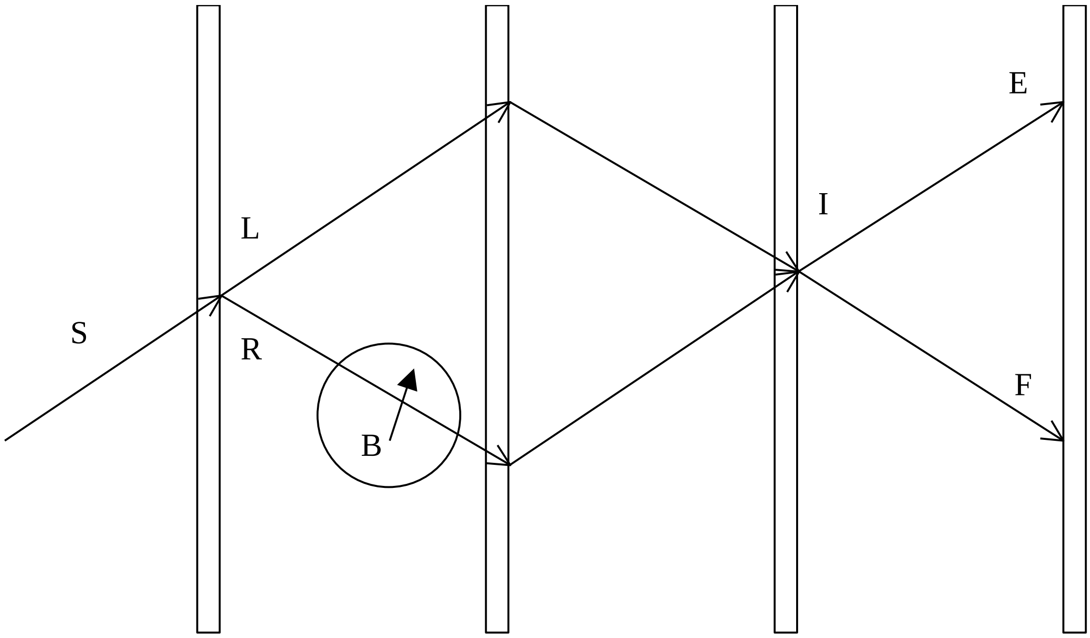

Figure 2-1: Experiment for verification of two-valuedness of spin

We divide Neutron to path L and path R. Neutron of the path L goes through a domain without magnetic field. Neutron of the path R goes through a domain with magnetic field. As a result, the magnetic field changes the phase of the neutron of path R. Quantity of the change of the phase $\Delta\phi$ is as follows.

$$\Delta\phi = \exp\left(\frac{i\omega T}{2}\right) \tag{2.13}$$

$$\omega = g_n \frac{e}{2m} B \tag{2.14}$$

Here, the variable ω is the angular frequency of precession of the spin of the neutron. The variable T is the time neutron passes through the magnetic field. The variable g_n is a g-factor. Constant e is the elementary charge. Variable B is the strength of the magnetic field. Variable m is the mass of the neutron.

The neutron that passed along the path L and path R joins at the position I. We can observe it at position E or position F.

Since superposition of a wave function occurs when it joins at position E or position F, we can observe the phase shift. The phase shift was observed as the result of the experiment actually.

It has been clarified that a spin has two-valuedness by this experiment.

2.2.2 Verification of angular momentum of the spin by an experiment

Albert Einstein and Wander Johannes de Haas[6] verified the angular momentum of the spin by the following experiment in 1915.

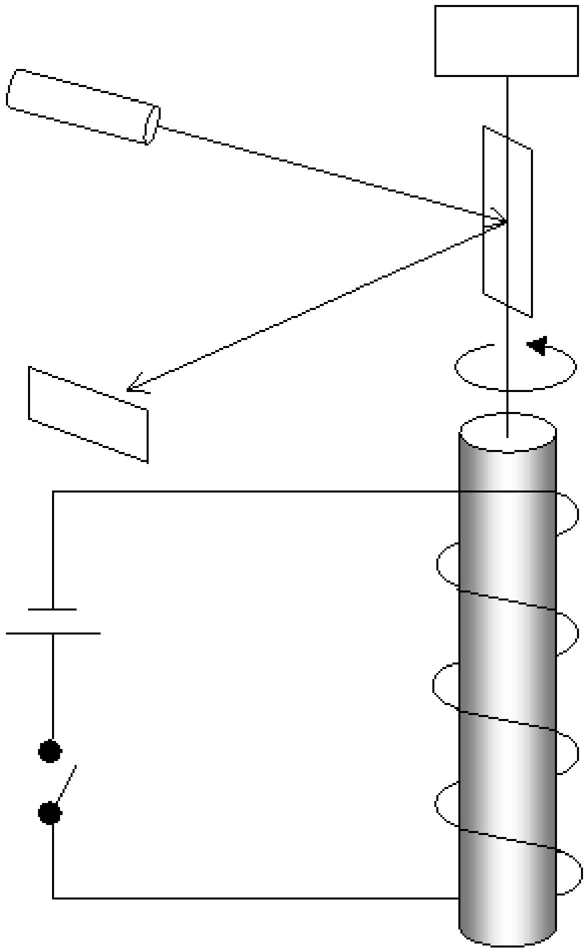

Figure 2-2: The experiment to verify the angular momentum of the spin

The experiment was performed as follows.

We apply the magnetic field to the disk of the magnetic material. Then we make the disk stationary state. After that, we stop the magnetic field. Then disk begins to turn around. This effect is called "Einstein-de Haas effect." It has been clarified that a spin has angular momentum by this experiment.

3 Derivation of the two-valuedness and angular momentum of spin

3.1 Derivation of the two-valuedness of spin

The point particle cannot rotate, because the point particle has the radius of rotation zero. We need infinite momentum to get finite angular momentum by the radius of rotation zero.

We can express angular momentum L by using the radius r and momentum p as follows. The operator × is outer product.

$$L = r \times p \tag{3.1}$$

If L is finite and radius r is zero, momentum p becomes infinite.

On the other hand, we cannot derive the two-valuedness of the spin by a rotation of 2-dimensional surface of a sphere (2-sphere). Therefore, we consider the rotation of 3-dimensional surface of a sphere (3-sphere).

3.1.1 Consideration of 3-sphere

We can express 3-sphere S^3 by combining two 3-dimensional solid sphere $B^3{}_1$ and $B^3{}_2$ in the following figure.

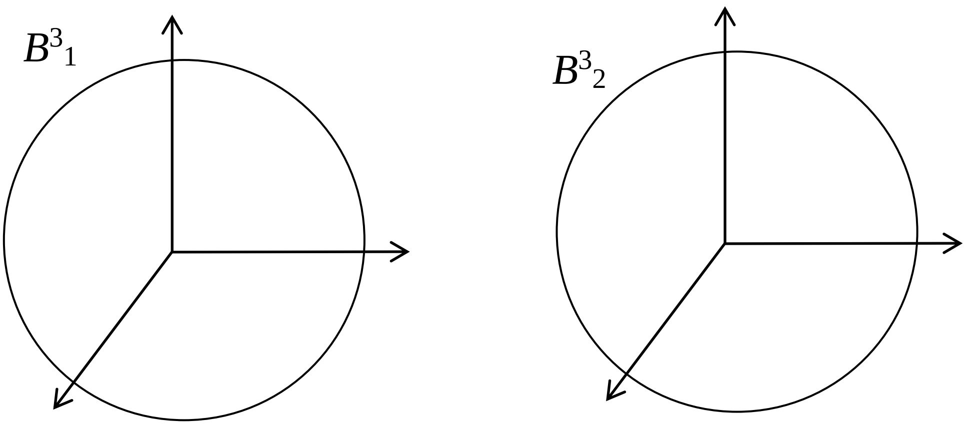

Figure 3-1: 3-sphere

3-sphere has 6 kinds of spin R_1, R_2, R_3, R_4, R_5, and R_6 like the following figure.

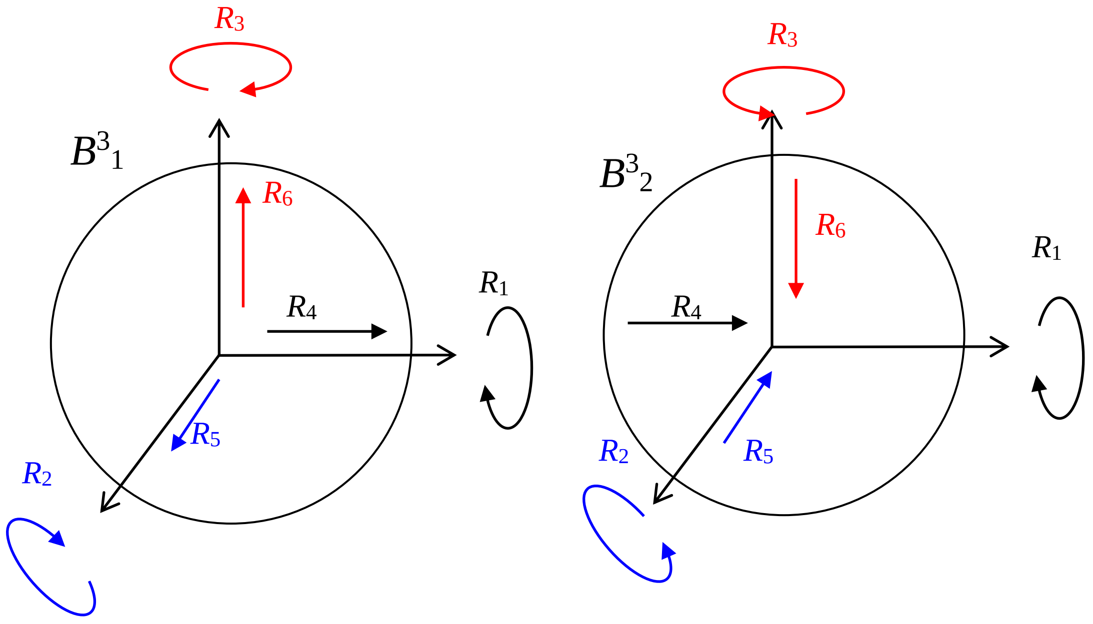

Figure 3-2: Rotation of 3-sphere

It is not difficult to consider the spin R_1, R_2, R_3. However, it is difficult to consider spin R_4, R_5, R_6.

It is difficult to consider 3-sphere because the 3-sphere exists in the 4-dimensional space. Then we try to consider the 3-sphere by taking a view of two sections of 3-sphere simultaneously. We call the method to take a view of the two sections simultaneously like this the **simultaneous sections method**.

First, we try to apply the simultaneous sections method to 2-sphere because it is easier to consider 2-sphere than 3-sphere.

3.1.2 Taking a view of 2-sphere by the simultaneous sections method

We suppose that 2-sphere S^2 in extra 3-dimensional space specified by the coordinates (X, Y, Z). If the radius of the 2-sphere is 1, 2-sphere satisfies the following equation.

$$X^2 + Y^2 + Z^2 = 1 \tag{3.2}$$

We can express this sphere by a sectional view of X-Y plane and the position on the Z-axis.

$$X^2 + Y^2 = \sin^2\theta \tag{3.3}$$

$$Z^2 = \cos^2\theta \tag{3.4}$$

Here, angle θ satisfies the following equation.

$$\cos^2\theta + \sin^2\theta = 1 \tag{3.5}$$

We show the 2-sphere that is applied the simultaneous sections method to in the following figure.

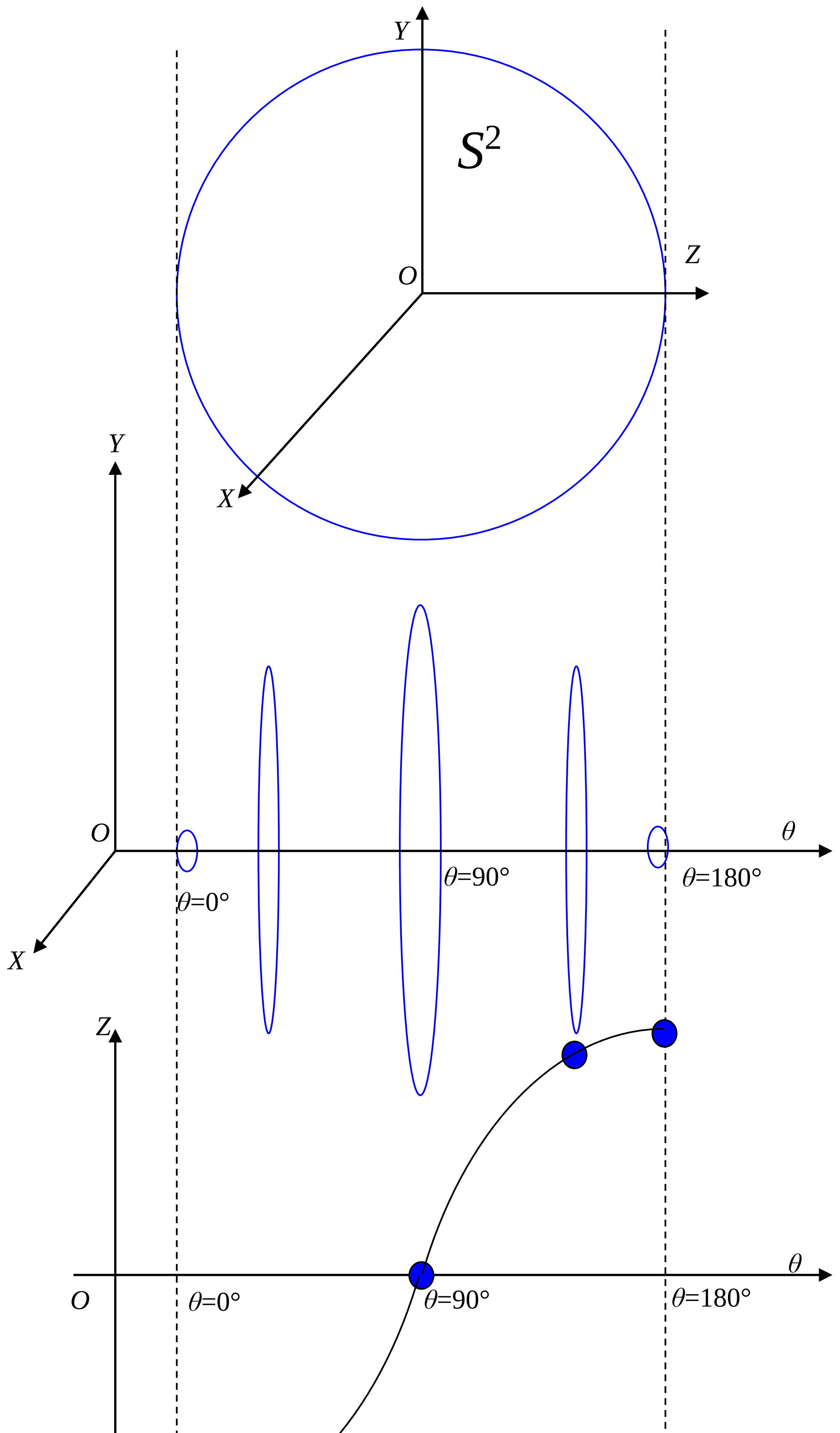

Figure 3-3: The simultaneous sections of 2-sphere

We express the radius of the circle in the X-Y plane and position Z at the angle θ.

Table 3-1: The radius of the circle in the X-Y plane and position Z at the angle θ

Angle θ	Radius of the circle in the X-Y plane	Position Z
0°	0	-1
90°	1	0
180°	0	1

We can consider the structure of the 2-sphere by taking a view of the radius of the circle in the X-Y plane and position Z simultaneously, like this.

Then, we apply the simultaneous sections method to 3-sphere.

3.1.3 Taking a view of 3-sphere by the simultaneous sections method

We suppose that 3-sphere S^3 in 4-dimensional space specified by the coordinates (W,X,Y,Z). If the radius of the 3-sphere is 1, 3-sphere satisfies the following equation.

$$W^2 + X^2 + Y^2 + Z^2 = 1 \tag{3.6}$$

We can express this sphere by a sectional view of X-Y-Z space and position on the W-axis.

$$X^2 + Y^2 + Z^2 = \sin^2\theta \tag{3.7}$$

$$W^2 = \cos^2\theta \tag{3.8}$$

We show the 3-sphere that is applied the simultaneous sections method to in the following figure.

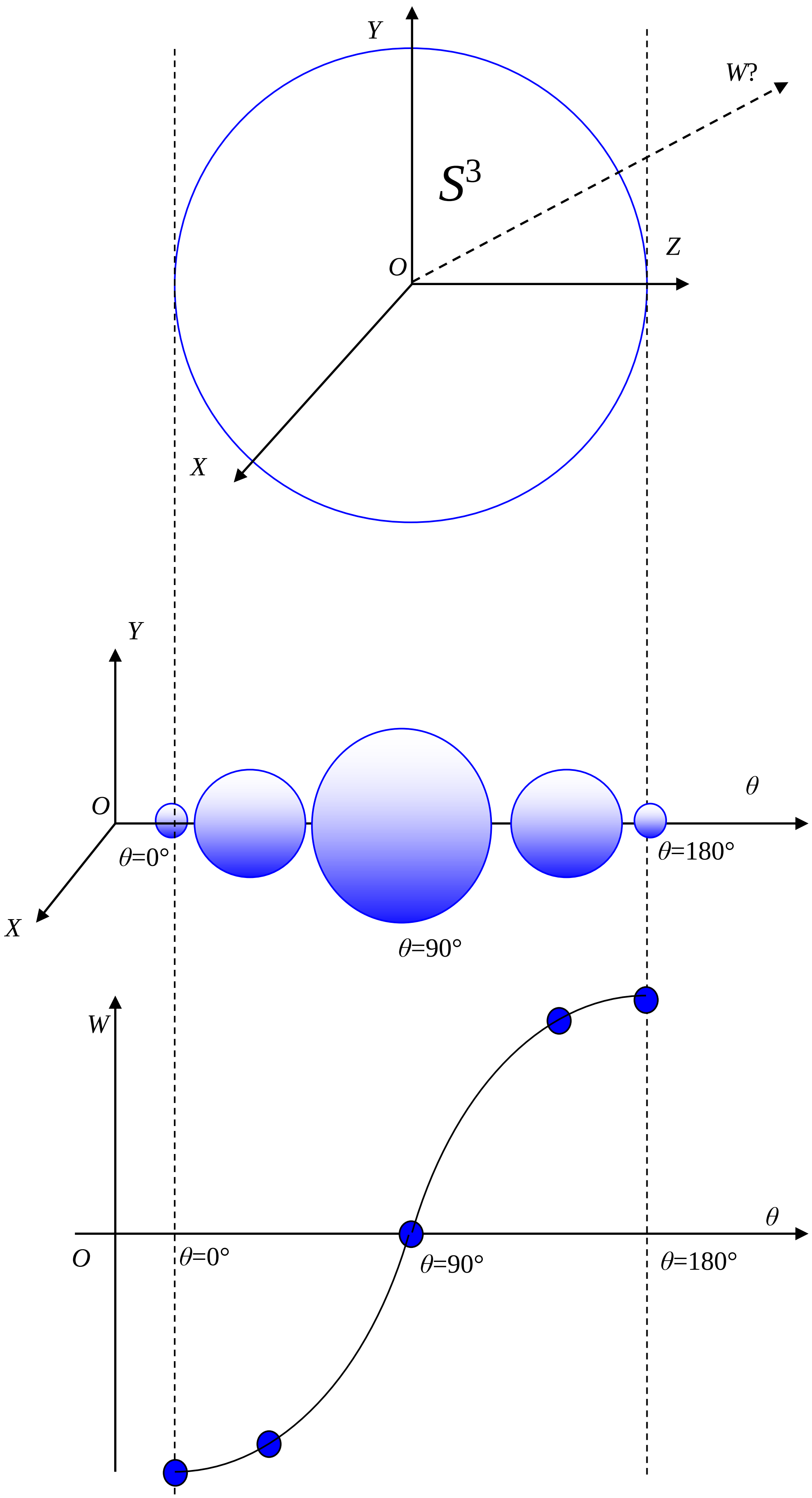

Figure 3-4: The simultaneous sections of 3-sphere

We express the radius of the sphere in *X*-*Y*-*Z* space plane and position *Z* at the angle θ.

Table 3-2: The radius of the sphere in *X*-*Y*-*Z* space and position *Z* at the angle θ

Angle θ	Radius of the sphere in *X*-*Y*-*Z* space	Position *Z*
0°	0	-1
90°	1	0
180°	0	1

In this section, we divided 3-sphere to 2-sphere and position on an axis. However, we can divide 3-sphere by the other way, too. We consider the way in the next section.

3.1.4 Taking a view of 3-sphere by the simultaneous sections method (The other way)

We suppose that 3-sphere S^3 in extra 4-dimensional space specified by the coordinates (*W*,*X*,*Y*,*Z*). If the radius of the 3-sphere is 1, 3-sphere satisfies the following equation.

$$W^2 + X^2 + Y^2 + Z^2 = 1 \qquad (3.9)$$

We can express this sphere by a sectional view of *W*-*X* plane and *Y*-*Z* plane.

$$W^2 + X^2 = \sin^2\theta \qquad (3.10)$$

$$Y^2 + Z^2 = \cos^2\theta \qquad (3.11)$$

This is the **Hopf fibration** which Heinz Hopf found in 1931.

We show the 3-sphere that is applied the simultaneous sections method to in the following figure.

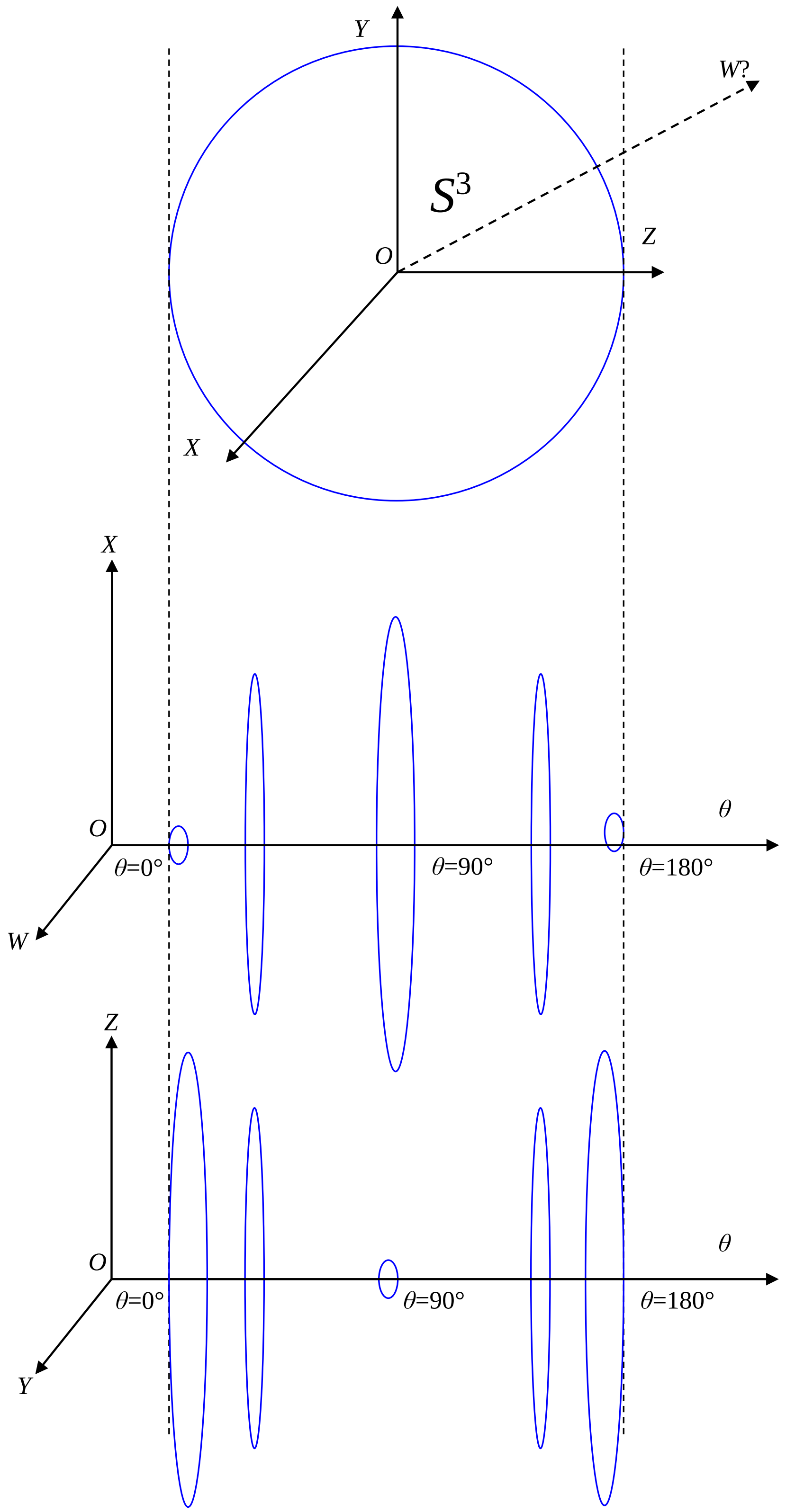

Figure 3-5: The simultaneous sections of 3-sphere (The other way)

We express the radius of the circle in *W*-*X* plane and circle in the *Y*-*Z* plane at the angle *θ*.

Table 3-3: The radius of the circle in *W*-*X* plane and circle in the *Y*-*Z* plane at the angle θ

Angle θ	Radius of the circle in *W*-*X* plane	Radius of the circle in the *Y*-*Z* plane
0°	0	1
90°	1	0
180°	0	1

Here we can connect the circle in the *W*-*X* plane at the angle θ = 0° and the circle in the *W*-*X* plane at the angle θ = 180° because they have the same radius 0. In addition, we can also connect the circle in the *Y*-*Z* plane at the angle θ = 0° and the circle in the *Y*-*Z* plane at the angle θ = 180° because they have the same radius 1.

Therefore, we can interpret the angle θ as an angle of rotation of the manifold.

This rotation turns the circle inside out. For example, the circle in the *Y*-*Z* plane is turned inside out at the angle of rotation θ = 180°. Therefore, this rotation is strange spin that is different from the normal spin.

We call the strange spin "**toric spin**." In addition, we call normal spin "**spheric spin**."

3.1.5 Even torus and odd torus

Here we express 3-sphere as follows.

$$W^2 + X^2 = \sin^2 \theta \tag{3.12}$$

$$Y^2 + Z^2 = \cos^2 \theta \tag{3.13}$$

We express the 3-sphere by the simultaneous sections method in the following figure.

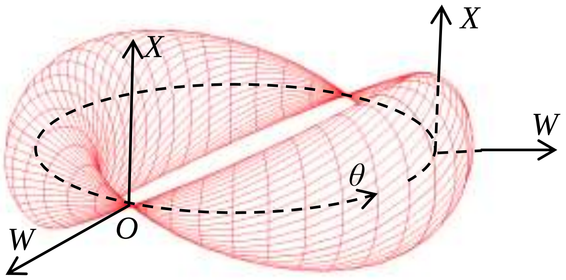

Figure 3-6: Wave function of spin-1 particle(*W*-*X*-*θ*)

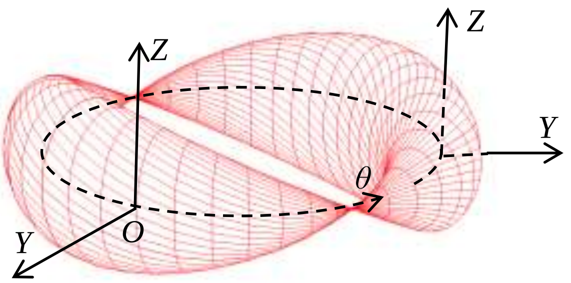

Figure 3-7: Wave function of spin-1 particle(*Y*-*Z*-*θ*)

We can interpret the above torus as a wave function of spin-1 particle. We can express it by the complex function as follows.

$$\psi(\theta) = \exp(i\theta) \tag{3.14}$$

Next, we express 3-sphere as follows.

$$W^2 + X^2 = \sin^2\left(\frac{\theta}{2}\right) \tag{3.15}$$

$$Y^2 + Z^2 = \cos^2\left(\frac{\theta}{2}\right) \tag{3.16}$$

We can express the 3-sphere by the simultaneous sections method in the following figure.

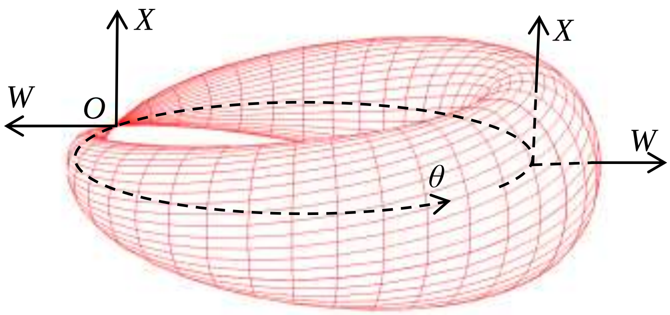

Figure 3-8: Wave function of spin-1/2 particle (*W*-*X*-*θ*)

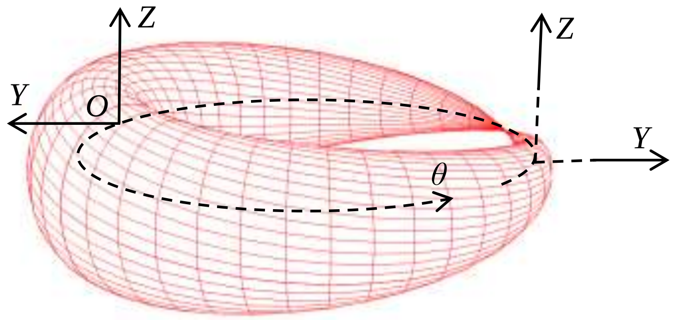

Figure 3-9: Wave function of spin-1/2 particle (*Y*-*Z*-*θ*)

We can interpret the above torus as a wave function of spin-1/2 particle. We can express it by the complex function as follows.

$$\psi(\theta) = \exp\left(\frac{i\theta}{2}\right) \tag{3.17}$$

Here we express 3-sphere as follows.

$$W^2 + X^2 = \sin^2\left(\frac{n\theta}{2}\right) \tag{3.18}$$

$$Y^2 + Z^2 = \cos^2\left(\frac{n\theta}{2}\right) \tag{3.19}$$

Variable n is an integer. We call the torus that has even n **even torus**. We call the torus that has odd n **odd torus**.

3.2 Derivation of angular momentum of spin

In this paper, we interpret spin as a rotation of 3-sphere. Why does the rotation of 3-sphere have the same angular momentum as the angular momentum in the 3-dimensional normal space.

In this section, we consider the possibility that the 3-sphere connects to the 3-dimensional normal space.

3.2.1 Construction of 1-dimensional helical space

We can construct **1-dimensional helical space** as follows.

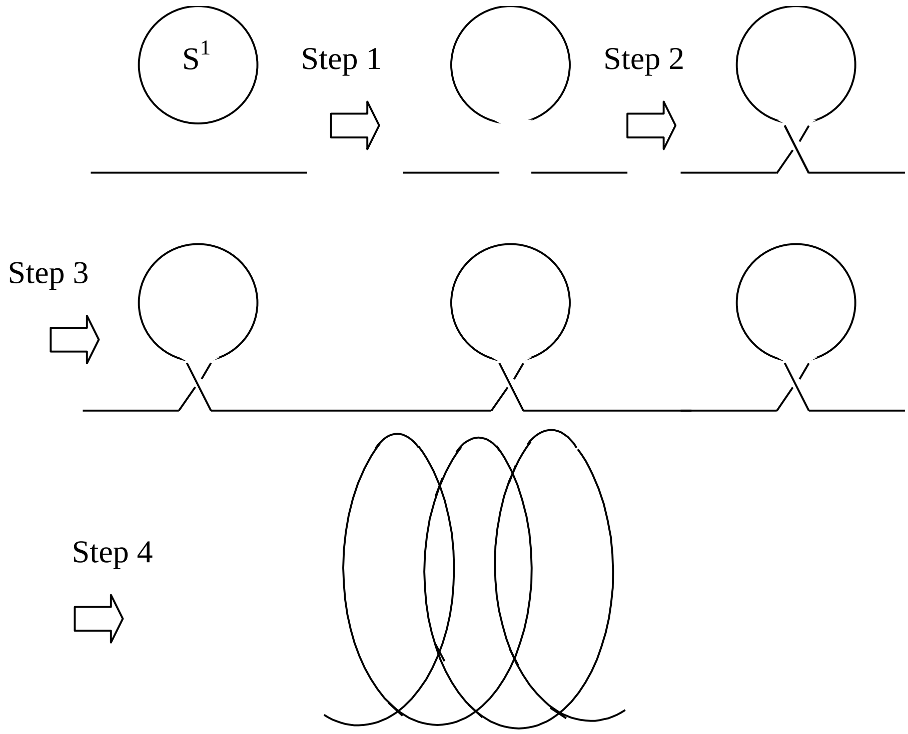

Figure 3-10: Construction of 1-dimensional helical space

We explain the transformation of each step in the following table.

Table 3-4: Construction of 1-dimensional helical space

Step	Method of construction
1	If we remove one point from a circle, we can get an arc. On the other hand, if we remove one point from a segment of a line, we can get two boundaries.
2	We connect their boundaries.
3	If we repeat this process, we can connect many circles.
4	If we change the orientation of the circle, we can construct 1-dimensional helical space.

We can express 1-dimensional helical space of the matrix representations of complex numbers as follows.

$$ET + IX = R\exp(I\Theta) \tag{3.20}$$

The variables T and X are the coordinates of an extra space. The variable R is a radius of the extra space. The variable Θ is the angle in the extra space.

The symbols $\{E, I\}$ are matrix representations of complex numbers.

$$I^2 = -E \tag{3.21}$$

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.22}$$

$$I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{3.23}$$

We express the coordinate x of a normal space by a wavelength λ as follows.

$$x = \frac{\lambda}{2\pi}\Theta \tag{3.24}$$

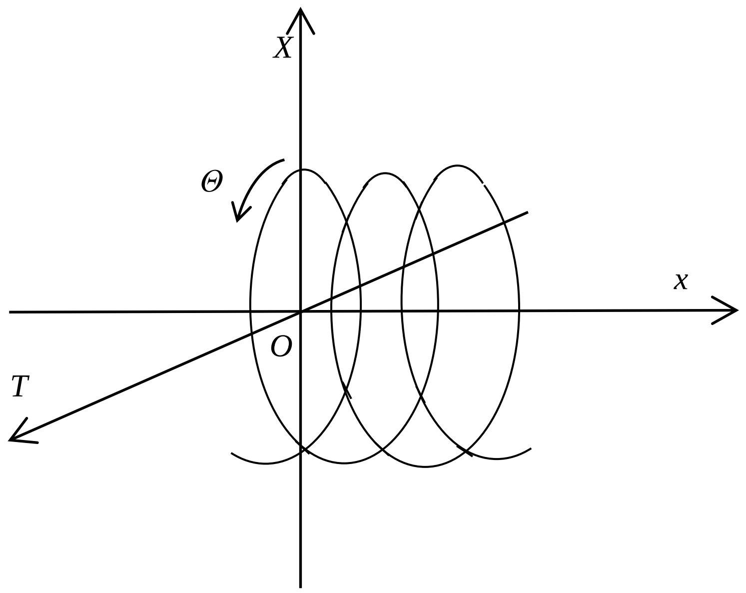

Figure 3-11: 1-dimensional helical space

If we combine the both ends, we can get **1-dimensional helical circle**.

$$EW + IX = R\exp(In\Theta) \tag{3.25}$$

$$x = \frac{\lambda}{2\pi}\Theta \tag{3.26}$$

Here, n is an integer. The variable Θ is the angle of rotation of the major radius of helical circle. The variable r is the major radius of helical circle. The variable R is the minor radius of helical circle. The variables $\{W, X\}$ are coordinates of an extra space. The variables x are coordinates of a normal space.

The matrix representations of complex numbers $\{E, I\}$ and the complex numbers $\{1, i\}$ commute.

$$Ei = iE, \qquad Ii = iI \tag{3.27}$$

We express the 1-dimensional helical circle in the following figure.

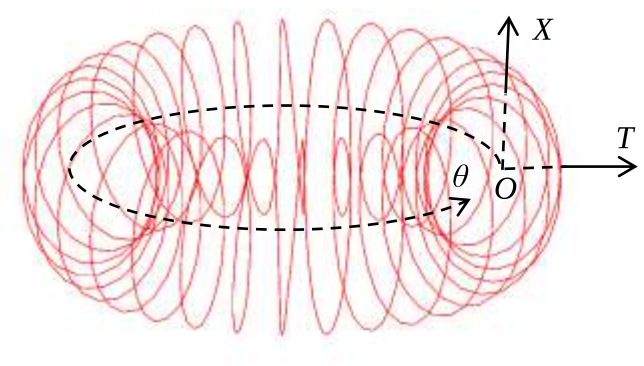

Figure 3-12: 1-dimensional helical circle

Is it possible to do the same thing in 2-dimensional space? We consider it in the next section.

3.2.2 Construction of 2-dimensional helical space

We can construct 2-dimensional helical space as follows.

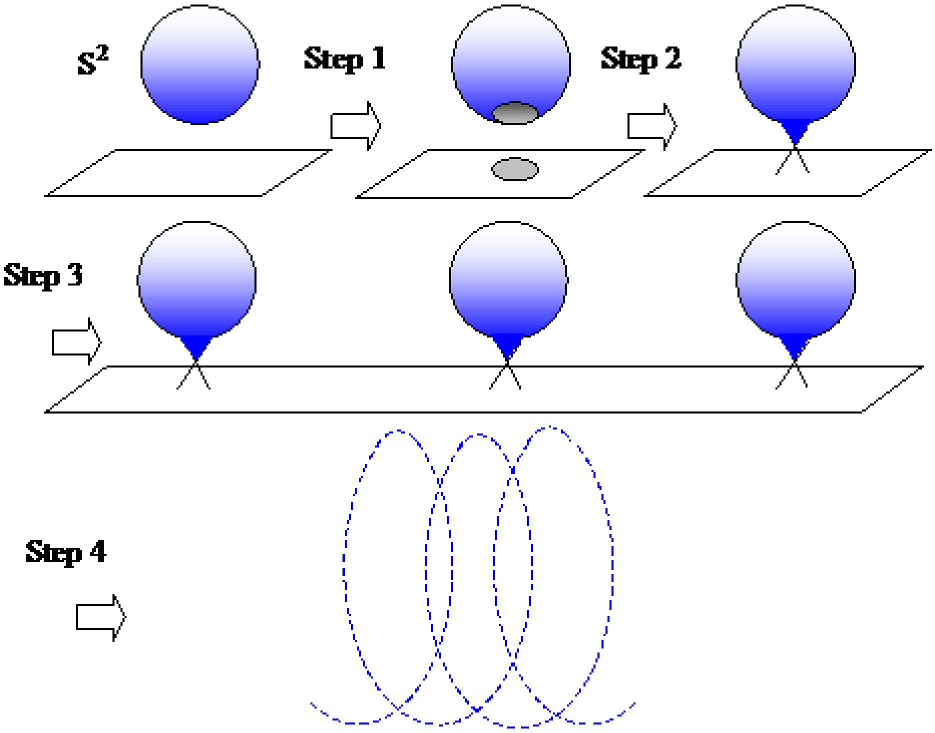

Figure 3-13: Construction of 2-dimensional helical space

We explain the transformation of each step in the following table.

Table 3-5: Construction of 2-dimensional helical space

Step	Method of construction
1	If we remove one point from a 2-sphere, we can get 2-dimensional disk. On the other hand, if we remove one point from a plane, we can get a boundary like a circle.
2	We connect their boundaries.
3	If we repeat this process, we can connect many 2-sphere.
4	If we change the orientation of the 2-sphere, we can construct 2-dimensional helical space.

We cannot express 2-dimensional helical space by the trigonometric functions. We cannot express 2-dimensional helical space by the complex function, too. Therefore, I do not guess 2-dimensional helical space exists. However, 3- dimensional helical space might exist. We consider it in the next section.

3.2.3 Construction of 3-dimensional helical space

We can construct **3-dimensional helical space** as follows.

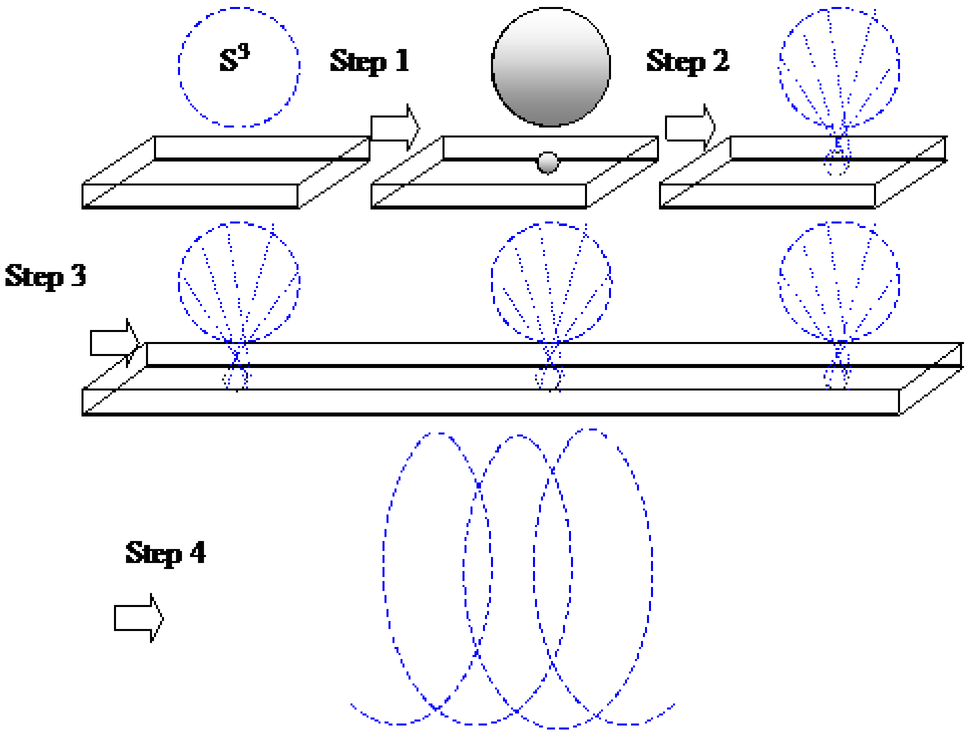

Figure 3-14: Construction of 3-dimensional helical space

We explain the transformation of each step in the following table.

Table 3-6: Construction of 3-dimensional helical space

Step	Method of construction
1	If we remove one point from a 3-sphere, we can get 3-dimensional solid sphere. On the other hand, if we remove one point from 3-dimensional space, we can get a boundary like 2-sphere.
2	We connect their boundaries.
3	If we repeat this process, we can connect many 3-sphere.
4	If we change the orientation of the 3-sphere, we can construct 3-dimensional helical space.

We can express 3-dimensional helical space of the matrix representations $\{E, I, J, K\}$ of quaternions as follows.

$$EW + IX + JY + KZ = R\exp(I\Theta_1 + J\Theta_2 + K\Theta_3) \qquad (3.28)$$

The variables $\{W, X, Y, Z\}$ are coordinates of an extra space. $\{\Theta_1, \Theta_2, \Theta_3\}$ are the angle in the extra space. R is a radius of extra space.

The matrix representations $\{E, I, J, K\}$ of quaternions are shown below.

$$I^2 = J^2 = K^2 = IJK = -E \tag{3.29}$$

$$E = \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix} \tag{3.30}$$

$$I = \begin{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & -\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \end{pmatrix} \tag{3.31}$$

$$J = \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \tag{3.32}$$

$$K = \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \tag{3.33}$$

We express the coordinate (x, y, z) of the normal space by the wavelength $\{\lambda_1, \lambda_2, \lambda_3\}$ as follows.

$$x = \frac{\lambda_1}{2\pi}\Theta_1 \tag{3.34}$$

$$y = \frac{\lambda_2}{2\pi}\Theta_2 \tag{3.35}$$

$$z = \frac{\lambda_3}{2\pi}\Theta_3 \tag{3.36}$$

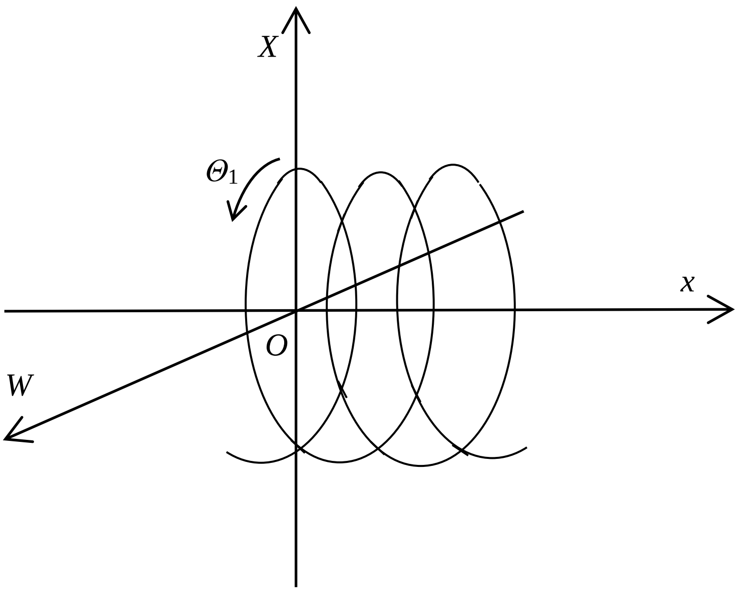

Figure 3-15: 3-dimensional helical space

If we combine the both ends, we can obtain **3-dimensional helical sphere**.

$$EW + IX + JY + KZ = R\exp(In_1\Theta_1 + Jn_2\Theta_2 + Kn_3\Theta_3) \tag{3.37}$$

$$x = \frac{\lambda_1}{2\pi}\Theta_1 \tag{3.38}$$

$$y = \frac{\lambda_2}{2\pi}\Theta_2 \tag{3.39}$$

$$z = \frac{\lambda_3}{2\pi}\Theta_3 \tag{3.40}$$

Here, $\{n_1, n_2, n_3\}$ are integers. $\{\Theta_1, \Theta_2, \Theta_3\}$ are the angles of rotation of the major radius of helical circle. The variable r is the major radius of helical circle. The variable R is the minor radius of helical circle. The variables $\{x, y, z\}$ are the coordinates of normal space.

The matrix representations of quaternions $\{E, I, J, K\}$ and the quaternions $\{1, i, j, k\}$ commute.

$$Ei = iE, \qquad Ii = iI, \qquad Ji = iJ, \qquad Ki = iK \tag{3.41}$$

$$Ej = jE, \qquad Ij = jI, \qquad Jj = jJ, \qquad Kj = jK \tag{3.42}$$

$$Ek = kE, \qquad Ik = kI, \qquad Jk = kJ, \qquad Kk = kK \tag{3.43}$$

We can express the 3-dimensional helical sphere symbolically in the following figure.

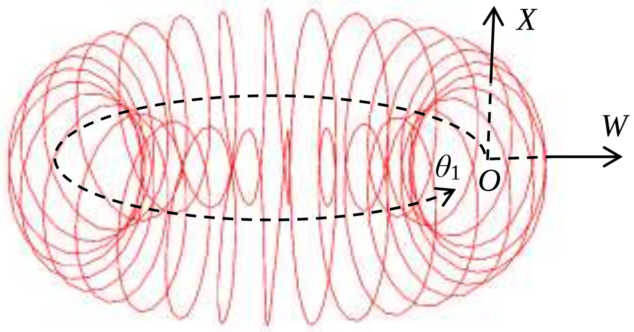

Figure 3-16: 3-dimensional helical sphere

3.2.4 Consideration of 3-dimensional helical space

1-dimensional helical space corresponded to complex. On the other hand, 3-dimensional helical space corresponded to quaternions. I guess 2-dimensional helical space does not exist because triples of numbers do not exist.

We can interpret a position in 3-dimensional helical sphere as the position in normal 3-dimensional space. Therefore, we can interpret an angular momentum in 3-dimensional helical sphere as an angular momentum in normal 3-dimensional space. In other words, we can interpret the spin of the quantum mechanics as the rotation of a particle.

4 Conclusion

In this paper, we derived the following property of the spin.
(1) Two-valuedness of a spin
(2) Angular momentum of a spin

5 Future Issues

Future issues are shown as follows.
- Derivation of Dirac equation

6 Supplement

6.1 Spin of 3-sphere

We introduced the 3-sphere as the wave function in 3-space in this paper.
We express the 3-sphere by the quaternionic functions as follows.

$$f(\theta,\phi,\psi) = W + iX + jY + kZ \in \mathbb{H} \qquad (6.1)$$

$$f(\theta,\phi,\psi) = \sin\theta \exp(i\phi) + \cos\theta \exp(i\psi)\, j \qquad (6.2)$$

$$W = \sin\theta \cos\phi \qquad (6.3)$$

$$X = \sin\theta \sin\phi \qquad (6.4)$$

$$Y = \cos\theta \cos\psi \qquad (6.5)$$

$$Z = \cos\theta \sin\psi \qquad (6.6)$$

This is the **Hopf fibration**.
We express the coordinate (W, X) for the rotational angle θ in the following figure.

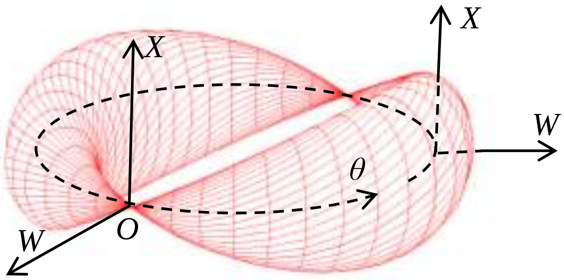

Figure 6-1: Wave function of the particle of spin 1

The value of the quaternionic function f of the rotational angle 180 degrees becomes the (-1) times of the value of the quaternionic function f of the rotational angle 0 degrees.

$$f(\theta,\phi,\psi) = -f(\theta+\pi,\phi,\psi) \qquad (6.7)$$

The value of the quaternionic function f of the rotational angle 360 degrees becomes the same value of the quaternionic function f of the rotational angle 0 degrees.

$$f(\theta,\phi,\psi) = f(\theta+2\pi,\phi,\psi) \qquad (6.8)$$

We interpret the manifold as the wave function of a particle of spin 1. We interpret the rotational angle of the manifold as the phase of the wave function. We interpret the surface area of the manifold as the absolute value of the wave function.

Here we change the angle θ to the half angle.

$$\theta \to \frac{\theta}{2} \qquad (6.9)$$

Then we express the quaternionic function as follows.

$$f(\theta,\phi,\psi) = \sin\left(\frac{\theta}{2}\right)\exp(i\phi) + \cos\left(\frac{\theta}{2}\right)\exp(i\psi)\,j \tag{6.10}$$

$$W = \sin\left(\frac{\theta}{2}\right)\cos(\phi) \tag{6.11}$$

$$X = \sin\left(\frac{\theta}{2}\right)\sin(\phi) \tag{6.12}$$

$$Y = \cos\left(\frac{\theta}{2}\right)\cos(\psi) \tag{6.13}$$

$$Z = \cos\left(\frac{\theta}{2}\right)\sin(\psi) \tag{6.14}$$

We express the coordinate (*W*, *W)* for the rotational angle θ in the following figure.

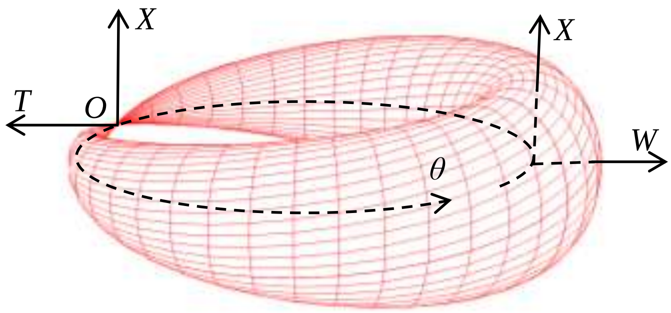

Figure 6-2: Wave function of a particle of spin 1/2

The value of the quaternionic function f of the rotational angle 360 degrees becomes the (-1) times of the value of the quaternionic function f of the rotational angle 0 degrees.

$$f(\theta,\phi,\psi) = -f(\theta + 2\pi,\phi,\psi) \tag{6.15}$$

The value of the quaternionic function f of the rotational angle 720 degrees becomes the same value of the quaternionic function f of the rotational angle 0 degrees.

$$f(\theta,\phi,\psi) = f(\theta + 4\pi,\phi,\psi) \tag{6.16}$$

We interpret the manifold as the wave function of a particle of spin 1/2. We interpret the rotational angle of the manifold as the phase of the wave function. We interpret the surface area of the manifold as the absolute value of the wave function.

7 Appendix

7.1 Jacobian

Carl Gustav Jacob Jacobi[7] introduced the Jacobian in 1841.

We suppose that the map f transform the coordinates (x,y) to the coordinates (u,v).

$$f: (x, y) \to (u, v) \tag{7.1}$$

Then, we can interpret the variable u as a function $u\ (x, y)$ of the coordinates (x, y).

$$u(x, y) \tag{7.2}$$

In the same way, we can interpret the variable v as a function $v\ (x, y)$ of the coordinates (x, y).

$$v(x, y) \tag{7.3}$$

We use abbreviations u, v for the functions $u\ (x, y)$ and $v\ (x, y)$.

$$u \equiv u(x, y) \tag{7.4}$$

$$v \equiv v(x, y) \tag{7.5}$$

We use the following abbreviations for the partial differentiation of functions.

$$u_x \equiv \frac{\partial u}{\partial x}, u_y \equiv \frac{\partial u}{\partial y} \tag{7.6}$$

$$v_x \equiv \frac{\partial v}{\partial x}, v_y \equiv \frac{\partial v}{\partial y} \tag{7.7}$$

Then, we describe the Jacobi-matrix as follows.
(Jacobi-matrix)

$$J = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \tag{7.8}$$

We use the symbol of derived function as the abbreviation for the Jacobi-matrix because the Jacobi-matrix is a generalization of derived function.

$$\frac{\partial(u, v)}{\partial(x, y)} \equiv \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \tag{7.9}$$

We call the determinant of the Jacobi-matrix Jacobian.
(Jacobian)

$$|J| = \left|\frac{\partial(u, v)}{\partial(x, y)}\right| \tag{7.10}$$

We express a surface area of a manifold by integration of a solid angle in this paper.
We transform the polar coordinates, complex numbers, and quaternions to the solid angle by Jacobian.

7.1.1 One-dimensional sphere (The circular polar coordinates)

We express the position (x, y) on the surface of the one-dimensional sphere S by the following circular polar coordinates.

$$x = r\cos\theta \tag{7.11}$$

$$y = r\sin\theta \tag{7.12}$$

Jacobian of the circular polar coordinates is shown below.

$$|J| = \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = r \tag{7.13}$$

We express the surface area *A* of the one-dimensional sphere *S* as follows.

$$A = \int_S |dS| \tag{7.14}$$

$$A = \int_0^{2\pi} d\theta\,|J| \tag{7.15}$$

$$A = \int_0^{2\pi} d\theta\, r = 2\pi r \tag{7.16}$$

Here we introduce the solid angle *ω*.

$$d\omega = d\theta \tag{7.17}$$

We express the surface area *A* by the solid angle *ω* as follows.

$$A = \int_S r d\omega \tag{7.18}$$

If we suppose that the radius *r* is the function of the solid angle *ω,* we have the following formula.

$$A = \int_S r(\omega) d\omega \tag{7.19}$$

Here we introduce the following new spherical harmonics.

$$h(\omega) = r(\omega) \tag{7.20}$$

We express the surface area *A* by the spherical harmonics as follows.

$$A = \int_S h(\omega) d\omega \tag{7.21}$$

7.1.2 One-dimensional sphere (Complex number)

We express the position (*x*, *y*) on the surface of the one-dimensional sphere *S* by the following complex number.

$$S = x + iy \tag{7.22}$$

On the other hand, we have the following formula.
(Euler's formula)

$$\exp(i\theta) = \cos\theta + i\sin\theta \tag{7.23}$$

Therefore, we have the following equations.

$$S = r\exp(i\theta) \tag{7.24}$$

$$x = r\cos\theta \tag{7.25}$$

$$y = r\sin\theta \tag{7.26}$$

Jacobian of the complex number is shown below.

$$|J| = \left|\frac{\partial(x, y)}{\partial(r, \theta)}\right| = r \tag{7.27}$$

We express the surface area *A* of the one-dimensional sphere *S* as follows.

$$A = \int_S |dS| \tag{7.28}$$

$$A = \int_0^{2\pi} d\theta\, |J| \tag{7.29}$$

$$A = \int_0^{2\pi} d\theta\, r = 2\pi r \tag{7.30}$$

Here we introduce the solid angle ω.

$$d\omega = d\theta \tag{7.31}$$

We express the surface area *A* by the solid angle ω as follows.

$$A = \int_S r d\omega \tag{7.32}$$

If we suppose that the radius *r* is the function of the solid angle ω, we have the following formula.

$$A = \int_S r(\omega) d\omega \tag{7.33}$$

Here we introduce the following new spherical harmonics.

$$h(\omega) = r(\omega) \tag{7.34}$$

We express the surface area *A* by the spherical harmonics as follows.

$$A = \int_S h(\omega) d\omega \tag{7.35}$$

Though the solid angle is scalar, we change it to the complex number as follows.

$$d\omega = i d\theta \exp(i\theta) \tag{7.36}$$

We call the solid angle **complex solid angle** in this paper.

Therefore, we express the surface area *A* as follows.

$$A = \int_S h(\omega)|d\omega| \tag{7.37}$$

7.1.3 Two-dimensional sphere (The spherical polar coordinates)

We express the position (x, y, z) on the surface of the two-dimensional sphere S by the following spherical polar coordinates.

$$x = r\sin\theta\cos\phi \tag{7.38}$$

$$y = r\sin\theta\sin\phi \tag{7.39}$$

$$z = r\cos\theta \tag{7.40}$$

Jacobian of the spherical polar coordinates is shown below.

$$|J| = \left|\frac{\partial(x, y, z)}{\partial(r, \phi, \theta)}\right| = r^2|\sin\theta| \tag{7.41}$$

We express the surface area A of the two-dimensional sphere S as follows.

$$A = \int_S |dS| \tag{7.42}$$

$$A = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, |J| \tag{7.43}$$

$$A = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, r^2|\sin\theta| = 4\pi r^2 \tag{7.44}$$

Here we introduce the solid angle ω.

$$d\omega = d\theta d\phi \sin\theta \tag{7.45}$$

We express the surface area A by the solid angle ω as follows.

$$A = \int_S r^2|d\omega| \tag{7.46}$$

If we suppose that the radius r is the function of the solid angle ω, we have the following formula.

$$A = \int_S \big(r(\omega)\big)^2|d\omega| \tag{7.47}$$

Here we introduce the following new spherical harmonics.

$$h(\omega) = \big(r(\omega)\big)^2 \tag{7.48}$$

We express the surface area A by the spherical harmonics as follows.

$$A = \int_S h(\omega)|d\omega| \tag{7.49}$$

7.1.4 Three-dimensional sphere (The spherical polar coordinates)

We express the position (τ, x, y, z) on the surface of the three-dimensional sphere S by the following spherical polar coordinates.

$$\tau = r \sin\psi \ \sin\theta \cos\phi \tag{7.50}$$

$$x = r \sin\psi \sin\theta \sin\phi \tag{7.51}$$

$$y = r \sin\psi \cos\theta \tag{7.52}$$

$$\mathrm{z} = r \cos\psi \tag{7.53}$$

Jacobian of the spherical polar coordinates is shown below.

$$|J| = \left| \frac{\partial(\tau, x, y, z)}{\partial(r, \phi, \theta, \psi)} \right| = r^3 |\sin\theta \sin^2\psi| \tag{7.54}$$

We express the surface area A of the three-dimensional sphere S as follows.

$$A = \int_S |dS| \tag{7.55}$$

$$A = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \int_0^{\pi} d\psi \, |J| \tag{7.56}$$

$$A = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \int_0^{\pi} d\psi \, r^3 |\sin\theta \sin^2\psi| = 2\pi^2 r^3 \tag{7.57}$$

Here we introduce the solid angle ω.

$$d\omega = d\phi d\theta d\psi \sin\theta \sin^2\psi \tag{7.58}$$

We express the surface area A by the solid angle ω as follows.

$$A = \int_S r^3 |d\omega| \tag{7.59}$$

If we suppose that the radius r is the function of the solid angle ω, we have the following formula.

$$A = \int_S \left(r(\omega)\right)^3 |d\omega| \tag{7.60}$$

Here we introduce the following new spherical harmonics.

$$h(\omega) = \left(r(\omega)\right)^3 \tag{7.61}$$

We express the surface area A by the spherical harmonics as follows.

$$A = \int_S h(\omega) |d\omega| \tag{7.62}$$

7.1.5 Three-dimensional sphere (the Hopf fibration)

We express the position (τ, x, y, z) on the surface of the three-dimensional sphere S by the following Hopf fibration.

$$S = \tau + ix + jy + kz \tag{7.63}$$

$$S = r\sin\phi\exp(i\theta) + r\cos\phi\exp(i\psi)\,j \tag{7.64}$$

$$\tau = r\sin\phi\ \cos\theta \tag{7.65}$$

$$x = r\sin\phi\ \sin\theta \tag{7.66}$$

$$y = r\cos\phi\ \cos\psi \tag{7.67}$$

$$\mathrm{z} = r\cos\phi\ \sin\psi \tag{7.68}$$

This is **the Hopf fibration** that Heinz Hopf found in 1931.

Jacobian of the Hopf fibration is shown below.

$$|J| = \left|\frac{\partial(\tau, x, y, z)}{\partial(r, \phi, \theta, \psi)}\right| = r^3|\cos\phi\sin\phi| \tag{7.69}$$

We express the surface area *A* of the three-dimensional sphere *S* as follows.

$$A = \int_S |dS| \tag{7.70}$$

$$A = \int_0^{\pi/2} d\phi \int_0^{2\pi} d\theta \int_0^{2\pi} d\psi\,|J| \tag{7.71}$$

$$A = \int_0^{\pi/2} d\phi \int_0^{2\pi} d\theta \int_0^{2\pi} d\psi\, r^3|\cos\phi\sin\phi| = 2\pi^2 r^3 \tag{7.72}$$

Here we introduce the solid angle ω.

$$d\omega = d\phi d\theta d\psi \cos\phi\sin\phi \tag{7.73}$$

We express the surface area *A* by the solid angle ω as follows.

$$A = \int_S r^3|d\omega| \tag{7.74}$$

If we suppose that the radius *r* is the function of the solid angle ω, we have the following formula.

$$A = \int_S \big(r(\omega)\big)^3|d\omega| \tag{7.75}$$

Here we introduce the following new spherical harmonics.

$$h(\omega) = \big(r(\omega)\big)^3 \tag{7.76}$$

We express the surface area *A* by the spherical harmonics as follows.

$$A = \int_S h(\omega)|d\omega| \tag{7.77}$$

7.1.6 Three-dimensional sphere (Quaternions)

We express the position (τ, x, y, z) on the surface of the three-dimensional sphere S by the following quaternions.

$$S = \tau + ix + jy + kz \tag{7.78}$$

$$S = r\exp(i\phi + j\theta + k\psi) \tag{7.79}$$

$$\tau = r(\cos\phi\cos\theta\cos\psi - \sin\phi\sin\theta\sin\psi) \tag{7.80}$$

$$x = r(\sin\phi\cos\theta\cos\psi + \cos\phi\sin\theta\sin\psi) \tag{7.81}$$

$$y = r(\cos\phi\sin\theta\cos\psi - \sin\phi\cos\theta\sin\psi) \tag{7.82}$$

$$z = r(\cos\phi\cos\theta\sin\psi + \sin\phi\sin\theta\cos\psi) \tag{7.83}$$

Jacobian of the quaternions is shown below.

$$|J| = \left|\frac{\partial(\tau, x, y, z)}{\partial(r, \phi, \theta, \psi)}\right| = r^3|\cos(2\theta)| \tag{7.84}$$

We express the surface area A of the three-dimensional sphere S as follows.

$$A = \int_S |dS| \tag{7.85}$$

$$A = \int_0^{2\pi} d\phi \int_0^{\pi/4} d\theta \int_0^{2\pi} d\psi\, |J| \tag{7.86}$$

$$A = \int_0^{2\pi} d\phi \int_0^{\pi/4} d\theta \int_0^{2\pi} d\psi\, r^3|\cos(2\theta)| = 2\pi^2 r^3 \tag{7.87}$$

Here we introduce the solid angle ω.

$$d\omega = d\phi d\theta d\psi \cos(2\theta) \tag{7.88}$$

We express the surface area A by the solid angle ω as follows.

$$A = \int_S r^3|d\omega| \tag{7.89}$$

If we suppose that the radius r is the function of the solid angle ω, we have the following formula.

$$A = \int_S \big(r(\omega)\big)^3|d\omega| \tag{7.90}$$

Here we introduce the following new spherical harmonics.

$$h(\omega) = \big(r(\omega)\big)^3 \tag{7.91}$$

We express the surface area A by the spherical harmonics as follows.

$$A = \int_S h(\omega)|d\omega| \tag{7.92}$$

Though the solid angle is scalar, we change it to the quaternions as follows.

$$d\omega = -d\phi d\theta d\psi \exp^3(i\phi + j\theta + k\psi)\cos(2\theta) \tag{7.93}$$

We call the solid angle **quaternionic solid angle** in this paper.
Therefore, we express the surface area *A* as follows.

$$A = \int_S h(\omega)|d\omega| \tag{7.94}$$

7.2 Arrangement of Terms

Table 7-1: Spin and so on

Term	Explanation
Spin	Rotation of the object that contains the axis of rotation.
Spheric spin	Rotation that does not include the inside out circle.
Toric spin	Rotation, including the inside out circle.

Table 7-2: Helical space and so on

Category	Term
Space	Helical space
Circle	Helical circle
Sphere	Helical sphere

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9 References

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