Zeta function regularization of the sum of all natural numbers by damped oscillation summation method

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Abstract

This paper converges all natural numbers by damped oscillation summation method of a new zeta function regularization.

The zeta function regularization is a method to assign finite values to divergent integration or sums. One method of the regularization is the analytic continuation. The integral representation of the zeta function converges by the analytic continuation. On the other hand, the zeta function also has a series representation. One of the representation is the sum of all natural numbers. The sum does not converges by the regularization. It diverges. This paper converges the sum by damped oscillation summation method of the new regularization.

It is desirable that both the integral representation and the series representation have same value for mathematical consistency. One method to converge the divergent series is Abel summation method. The method converge the divergent series by multiplying convergence factor. However, the sum of all natural numbers does not converge by the method. Therefore it is an important issue to find a new summation method.

We define new "sum of all natural numbers" by new summation method, damped oscillation summation method. The method converge the divergent series by multiplying convergence factor which is damped and oscillating very slowly. The traditional sum diverges for infinite terms. On the other hand, the new "sum" equals the traditional sum for the finite term. And the "sum" converges on -1/12 for the infinite term.

CONTENTS

1 Introduction

1.1 Issue

The zeta function regularization is a method to assign finite values to divergent integration or sums. One method of the regularization is the analytic continuation. The integral representation of the zeta function converges by the analytic continuation. On the other hand, the zeta function also has a series representation. One of the representation is the sum of all natural numbers. The sum does not converges by the regularization. It diverges. This paper converges the sum by damped oscillation summation method of the new regularization.

1.2 Importance of the issue

It is desirable that both the integral representation and the series representation have same value for mathematical consistency. One method to converge the divergent series is Abel summation method. The method converge the divergent series by multiplying convergence factor. However, the sum of all natural numbers does not converge by the method. Therefore it is an important issue to find a new summation method.

1.3 Research trends so far

Leonhard Euler[2] suggested that the sum of all natural numbers is -1/12 in 1749. Bernhard Riemann[3] show that the integral representation of the zeta function is -1/12 in 1859. Srinivasa Ramanujan[4] proposed that the sum of all natural numbers is -1/12 by Ramanujan summation method.

Niels Abel[5] introduced Abel summation method in order to converge the divergent series in about 1829.

1.4 New derivation method of this paper

We define new "sum of all natural numbers" by new summation method, damped oscillation summation method. The method converge the divergent series by multiplying convergence factor which is damped and oscillating very slowly. The traditional sum diverges for infinite terms. On the other hand, the new "sum" equals the traditional sum for the finite term. And the "sum" converges on -1/12 for the infinite term.

(Damped oscillation summation method for the sum of all natural numbers)

$$\boxed{H(\epsilon) = \sum_{k=1}^{\infty} k \exp(-k\epsilon) \cos(k\epsilon)} \tag{1.1}$$

$$0 < \epsilon \ll 1 \tag{1.2}$$

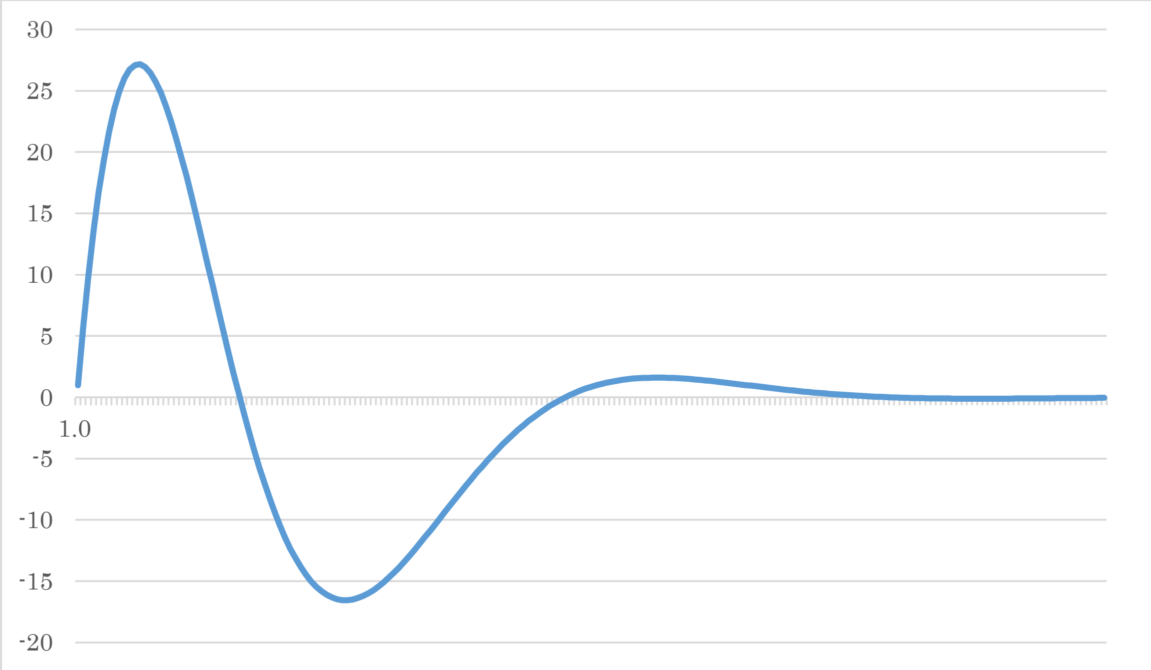

Figure 1.1: Damped oscillation of natural number

2 Confirmations of known results

2.1 Abel summation method

Niels Abel introduced Abel summation method in about 1829.

We consider the following sum of the series.

$$S = \sum_{k=1}^{\infty} a_k \tag{2.1}$$

Then we define the following function.
(Abel summation method)

$$F(\epsilon) = \sum_{k=1}^{\infty} a_k \exp(-k\epsilon) \tag{2.2}$$

$$0 < \epsilon \ll 1 \tag{2.3}$$

We define Abel sum as follows.

$$S_A = \lim_{\epsilon \to 0+} F(\epsilon) \tag{2.4}$$

2.2 Zeta function regularization of the sum of all natural numbers by Abel summation method

We consider the following sum of all natural numbers.

$$S = \sum_{k=1}^{\infty} k \tag{2.5}$$

Then we define the following function.

$$G(\epsilon) = \sum_{k=1}^{\infty} k \exp(-k\epsilon) \tag{2.6}$$

$$0 < \epsilon \ll 1 \tag{2.7}$$

Here, we will use the following formula.
(Formula of the geometric series which has coefficients of natural numbers)

$$\sum_{k=1}^{\infty} kr^k = \frac{r}{(1-r)^2} \tag{2.8}$$

We obtain the following equation by the above formula.

$$G(\epsilon) = \frac{e^{-\epsilon}}{(1-e^{-\epsilon})^2} \tag{2.9}$$

Here, we will use the following formula. In this paper, Bernoulli numbers B_n is Bernoulli polynomial B_n (1).
(Definitional formula of Bernoulli numbers)

$$\frac{ze^z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n \tag{2.10}$$

We will square the both sides of the above formula. And we will divide the both sides by z^2. Then we obtain the following equation.

$$\frac{e^{2z}}{(1-e^z)^2} = \frac{1}{z^2}\left(\sum_{n=0}^{\infty} \frac{B_n}{n!} z^n\right)^2 \tag{2.11}$$

We will multiply the left side of the above formula by e^{-z}. And we will multiply the right side by Maclaurin series of e^{-z}. Then we obtain the following equation.

$$\frac{e^z}{(1-e^z)^2} = \frac{1}{z^2}\left(\sum_{n=0}^{\infty} \frac{B_n}{n!} z^n\right)^2 \sum_{n=0}^{\infty} \frac{1}{n!}(-z)^n \tag{2.12}$$

Therefore we express the function G (ε) as follows.

$$G(\epsilon)=\frac{1}{\epsilon^2}\left(\sum_{n=0}^{\infty}\frac{B_n}{n!}(-\epsilon)^n\right)^2\sum_{n=0}^{\infty}\frac{1}{n!}\epsilon^n \qquad (2.13)$$

We obtain the following equation by calculating the above formula.

$$G(\epsilon)=\frac{1}{\epsilon^2}\left(1-\frac{\epsilon}{2}+\frac{\epsilon^2}{12}+O(\epsilon^3)\right)^2\left(1+\epsilon+\frac{\epsilon^2}{2}+O(\epsilon^3)\right) \qquad (2.14)$$

$$G(\epsilon)=\frac{1}{\epsilon^2}\left(1-\epsilon+\frac{5}{12}\epsilon^2+O(\epsilon^3)\right)\left(1+\epsilon+\frac{\epsilon^2}{2}+O(\epsilon^3)\right) \qquad (2.15)$$

$$G(\epsilon)=\frac{1}{\epsilon^2}\left(1+\epsilon(1-1)+\epsilon^2\left(\frac{1}{2}-1+\frac{5}{12}\right)+O(\epsilon^3)\right) \qquad (2.16)$$

$$G(\epsilon)=\frac{1}{\epsilon^2}\left(1-\frac{1}{12}\epsilon^2+O(\epsilon^3)\right) \qquad (2.17)$$

$$G(\epsilon)=\frac{1}{\epsilon^2}-\frac{1}{12}+O(\epsilon) \qquad (2.18)$$

Here the symbols $O(x)$ are Landau symbos. The symbols mean that the error has the order of the variable x.

The first term diverges. The first term is called singular term.

Therefore the following Abel sum diverges.

$$S_A=\lim_{\epsilon\to 0+}G(\epsilon)=\infty \qquad (2.19)$$

It is purpose of this paper to remove the divergence of this singular term.

3 Zeta function regularization of the sum of all natural numbers by damped oscillation summation method

3.1 Damped oscillation summation method

We consider the following sum of the series.

$$S=\sum_{k=1}^{\infty}a_k \qquad (3.1)$$

Then we define the following new summation method.
(Damped oscillation summation method)

$$\boxed{H(\epsilon) = \sum_{k=1}^{\infty} a_k \exp(-k\phi(\epsilon)) \cos(k\epsilon)} \tag{3.2}$$

$$0 < \epsilon \ll 1 \tag{3.3}$$

$$0 < \phi(\epsilon) \ll 1 \tag{3.4}$$

We determine the function ϕ (ε) by an equation. The equation is shown in the appendix "Singular equation" of this paper.

We define the damped oscillation sum as follows.

$$S_H = \lim_{\epsilon \to 0+} H(\epsilon) \tag{3.5}$$

3.2 Zeta function regularization of the sum of all natural numbers by damped oscillation summation method

We consider the following sum of all natural numbers.

$$S = \sum_{k=1}^{\infty} k \tag{3.6}$$

Then we define the following function.

$$\boxed{H(\epsilon) = \sum_{k=1}^{\infty} k \exp(-k\epsilon) \cos(k\epsilon)} \tag{3.7}$$

$$0 < \epsilon \ll 1 \tag{3.8}$$

Here, we use the following formula.
(Euler's formula)

$$\exp(ix) = \cos(x) + i\ \sin(x) \tag{3.9}$$

We obtain the following formula from the above formula.

$$\cos(x) = \frac{1}{2}(\exp(ix) + \exp(-ix)) \tag{3.10}$$

Hence, we express the function H (ε) as follows.

$$H(\epsilon)=\sum_{k=1}^{\infty}k\exp(-k\epsilon)\frac{1}{2}(\exp(ix)+\exp(-ix)) \tag{3.11}$$

$$H(\epsilon)=\frac{1}{2}\sum_{k=1}^{\infty}k\,exp(-k\epsilon)\,exp(ik\epsilon)+\frac{1}{2}\sum_{k=1}^{\infty}k\,exp(-k\epsilon)\,exp(-ik\epsilon) \tag{3.12}$$

$$H(\epsilon)=\frac{1}{2}\sum_{k=1}^{\infty}k\,exp(-k\epsilon+ik\epsilon)+\frac{1}{2}\sum_{k=1}^{\infty}k\,exp(-k\epsilon-ik\epsilon) \tag{3.13}$$

$$H(\epsilon)=\frac{1}{2}\sum_{k=1}^{\infty}k\,exp(-k(\epsilon-i\epsilon))+\frac{1}{2}\sum_{k=1}^{\infty}k\,exp(-k(\epsilon+i\epsilon)) \tag{3.14}$$

And we define the following function.

$$G(\mathrm{z})=\sum_{k=1}^{\infty}k\exp(-kz) \tag{3.15}$$

$$z\in\mathbb{C} \tag{3.16}$$

We express the function $G\ (z)$.from the formula (2.18).

$$G(z)=\frac{1}{z^2}-\frac{1}{12}+O(z) \tag{3.17}$$

Then, we express the function $H\ (\varepsilon)$.

$$H(\epsilon)=\frac{1}{2}G(\epsilon-i\epsilon)+\frac{1}{2}G(\epsilon+i\epsilon) \tag{3.18}$$

$$H(\epsilon)=\frac{1}{2}\left(\frac{1}{(\epsilon-i\epsilon)^2}-\frac{1}{12}+O(\epsilon)\right)+\frac{1}{2}\left(\frac{1}{(\epsilon+i\epsilon)^2}-\frac{1}{12}+O(\epsilon)\right) \tag{3.19}$$

$$H(\epsilon)=\frac{1}{2(-\epsilon+i\epsilon)^2}+\frac{1}{2(-\epsilon-i\epsilon)^2}-\frac{1}{12}+O(\epsilon) \tag{3.20}$$

The first term is shown below.

$$\frac{1}{2(-\epsilon+i\epsilon)^2}=\frac{1}{2(\epsilon^2-2i\epsilon^2-\epsilon^2)}=\frac{1}{-4i\epsilon^2} \tag{3.21}$$

On the other hand, the second term is shown below.

$$\frac{1}{2(-\epsilon - i\epsilon)^2} = \frac{1}{2(\epsilon^2 + 2i\epsilon^2 - \epsilon^2)} = \frac{1}{4i\epsilon^2} \tag{3.22}$$

Therefore the sum of the first term and the second term is shown below.

$$\frac{1}{2(-\epsilon + i\epsilon)^2} + \frac{1}{2(-\epsilon - i\epsilon)^2} = 0 \tag{3.23}$$

Then we have the following equation.

$$H(\epsilon) = -\frac{1}{12} + O(\epsilon) \tag{3.24}$$

3.3 Interpretation of the sum of all natural numbers

We define the function *S* (*n*) and *H* (*δ*, *n*), and the constant *α* as follows.

$$S(n) = \sum_{k=1}^{n} k \tag{3.25}$$

$$H(\delta, n) = \sum_{k=1}^{n} k \exp(-k\delta) \cos(k\delta) \tag{3.26}$$

$$\alpha = -\frac{1}{12} \tag{3.27}$$

We have the following statement.

$$\lim_{n\to\infty} S(n) = \infty \tag{3.28}$$

We can express the above statement by (*R*, *N*)-definition of the limit as follows.

Given a number $R > 0$, there exists a natural number *N* such that for all *n* satisfying

$$N < n \tag{3.29}$$

we have

$$R < S(n). \tag{3.30}$$

It can be summarized below.

$$\forall R > 0, \exists N \in \mathbb{N} :\ \forall n\ \ (N < n \Rightarrow R < S(n)) \tag{3.31}$$

We have the following statement.

$$\lim_{d \to 0+} H(d, m) = S(m) \tag{3.32}$$

We can express the above statement by (ε, δ)-definition of the limit as follows.

Given a number $\varepsilon > 0$ and a natural number m, there exists a number $\delta > 0$ such that for all d satisfying

$$0 < d < \delta \tag{3.33}$$

we have

$$|H(d, m) - S(m)| < \epsilon. \tag{3.34}$$

It can be summarized below.

$$\forall \epsilon > 0, \forall m \in \mathbb{N}, \exists \delta > 0 : \forall d \ \ (0 < d < \delta \Rightarrow |H(d, m) - S(m)| < \epsilon) \tag{3.35}$$

We have the following statement.

$$\lim_{n \to \infty} \lim_{\delta \to 0+} H(\delta, n) = \alpha \tag{3.36}$$

$$0 \ll \frac{1}{\delta} \ll n \tag{3.37}$$

We can express the above statement by (ε, δ)-definition of the limit as follows.

Given a number $\varepsilon > 0$, there exist a natural number N and a number $\delta > 0$ such that for all n satisfying

$$N < n \tag{3.38}$$

we have

$$|H(\delta, n) - \alpha| < \epsilon. \tag{3.39}$$

It can be summarized below.

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \exists \delta > 0 : \ \forall n \ \ (N < n \Rightarrow |H(\delta, n) - \alpha| < \epsilon\ ) \tag{3.40}$$

Then we define the following symbols.

$$1 + 2 + 3 + \cdots + n \equiv S(n) \tag{3.41}$$

$$1 + 2 + 3 + \cdots \equiv \lim_{n \to \infty} S(n) \tag{3.42}$$

$$"1 + 2 + 3 + \cdots + n" \equiv \lim_{\delta \to 0+} H(\delta, n) \tag{3.43}$$

$$"1 + 2 + 3 + \cdots" \equiv \lim_{n \to \infty} \lim_{\delta \to 0+} H(\delta, n) \tag{3.44}$$

Then we have the following statements.

$$1+2+3+\cdots=\infty \tag{3.45}$$

$$"1+2+3+\cdots+n"=1+2+3+\cdots+n \tag{3.46}$$

$$\boxed{"1+2+3+\cdots"=-\frac{1}{12}} \tag{3.47}$$

The double quotes means the sum of natural numbers defined by damped oscillation summation method.

The traditional sum diverges for infinite terms. On the other hand, the new “sum” equals the traditional sum for the finite term. And the “sum” converges on -1/12 for the infinite term.

4 Conclusion

We obtained the following results in this paper.

- We obtained the zeta function regularization of the sum of all natural numbers by damped oscillation summation method.

5 Future issues

The future issues are shown below.

- To study the relation between the integral representation and series representation defined by damped oscillation summation method.

6 Appendix

6.1 Numerical calculation

We will calculate the following value numerically.

$$H=\sum_{k=1}^{3000} k\exp(-k0.01)\cos(k0.01) \tag{6.1}$$

The result is shown below.

$$H=-0.0833333498\cdots \tag{6.2}$$

This value is very close to the following -1/12.

$$-\frac{1}{12}=-0.0833333333\cdots \tag{6.3}$$

The graph is shown below.

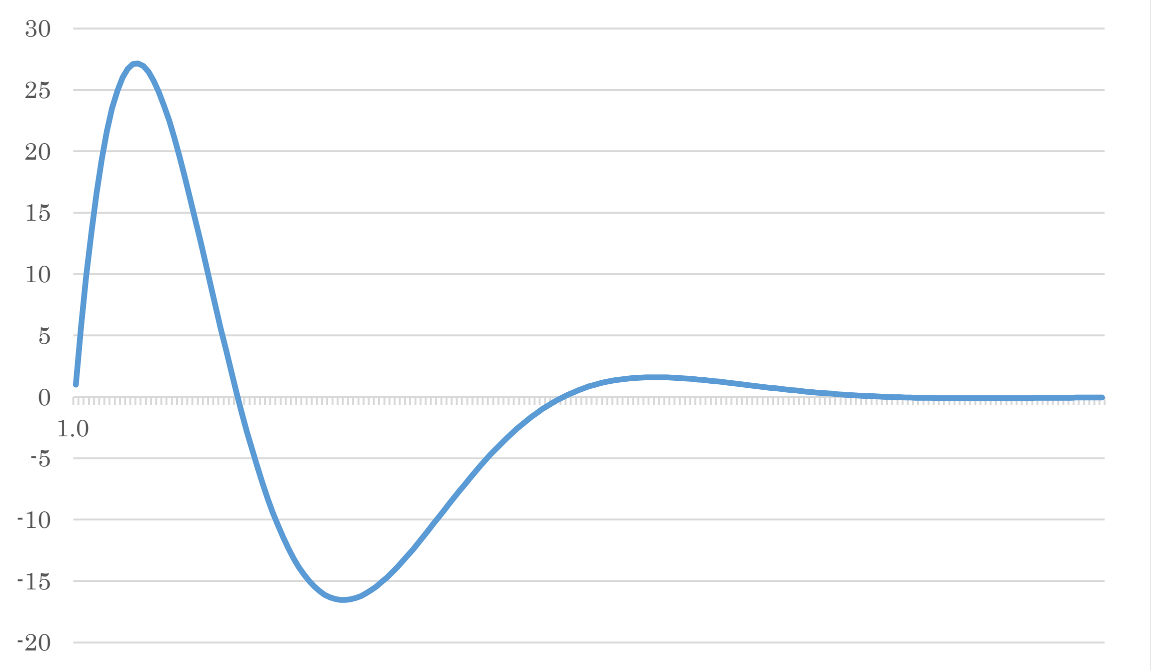

Figure 6.1: Damped oscillation of natural number

Here, we will double the attenuation factor as follows. We don't change the vibration period.

$$H = \sum_{k=1}^{3000} k \exp(-k0.02) \cos(k0.01) \tag{6.4}$$

The result is shown below.

$$H = 1199.9166679 \cdots \tag{6.5}$$

Therefore, a series does not converge if the attenuation factor and the vibration period do not have a special relation. We will consider the relation in the next section.

6.2 Singular equation

We consider the following series.

$$S = \sum_{k=1}^{\infty} a_k \tag{6.6}$$

Then we define the following function.

$$H(\epsilon) = \sum_{k=1}^{\infty} a_k \exp(-k\phi(\epsilon)) \cos(k\epsilon) \tag{6.7}$$

$$0 < \epsilon \ll 1 \tag{6.8}$$

$$0 < \phi(\epsilon) \ll 1 \tag{6.9}$$

And we define the following function.

$$G(\mathrm{z}) = \sum_{k=1}^{\infty} k \exp(-kz) \tag{6.10}$$

$$\mathrm{z} \in \mathbb{C} \tag{6.11}$$

We express the function G (z) by using singular term A (z), and the constant C.

$$G(z) = A(z) + C + O(z) \tag{6.12}$$

Here, we use the following formula.
(Euler's formula)

$$\exp(ix) = \cos(x) + i \sin(x) \tag{6.13}$$

We express the function H (ε) as follows by using the above formula.

$$H(\epsilon) = \frac{1}{2}(G(z) + G(z^*)) \tag{6.14}$$

$$z = \phi(\epsilon) + i\epsilon \tag{6.15}$$

$$z^* = \phi(\epsilon) - i\epsilon \tag{6.16}$$

We obtain the following equation by using the formula (6.12) of the function G (z).

$$H(\epsilon) = \frac{1}{2}\left(A(z) + A(z^*)\right) + C + O(z) \tag{6.17}$$

We determine the function ϕ (ε) by the following singular equation in order to remove the singular term of the above equation.
(Singular equation)

$$\boxed{\frac{1}{2}\left(A(z) + A(z^*)\right) = O(z)} \tag{6.18}$$

The series a_k and the function G (z) and ϕ (ε) are shown below.

Series a_k	Function $G(z)$	Function $\phi(\varepsilon)$	Num.
1	$G(z) = -\frac{1}{z} - \frac{1}{2} + O(z)$	$\phi(\epsilon) = \frac{\epsilon}{\tan(\pi/2)} + O(\epsilon^3)$	(6.19)
k	$G(z) = \frac{1}{z^2} - \frac{1}{12} + O(\mathrm{z})$	$\phi(\epsilon) = \frac{\epsilon}{\tan(\pi/4)} + O(\epsilon^4)$	(6.20)
k^2	$G(z) = -\frac{2}{z^3} + O(z)$	$\phi(\epsilon) = \frac{\epsilon}{\tan(\pi/6)} + O(\epsilon^5)$	(6.21)
k^3	$G(z) = \frac{1}{z^4} + \frac{1}{120} + O(\mathrm{z})$	$\phi(\epsilon) = \frac{\epsilon}{\tan(\pi/8)} + O(\epsilon^6)$	(6.22)

The series and the example of the damped oscillation summation method are shown below.

Series a_k	Example of the damped oscillation summation method	Num.
1	$H(\epsilon) = \sum_{k=1}^{\infty} 1 \exp(-k\epsilon^3) \cos(k\epsilon)$	(6.23)
k	$H(\epsilon) = \sum_{k=1}^{\infty} k \exp(-k\epsilon) \cos(k\epsilon)$	(6.24)
k^2	$H(\epsilon) = \sum_{k=1}^{\infty} k^2 \exp\left(-k\epsilon\sqrt{3}\right) \cos(k\epsilon)$	(6.25)
k^3	$H(\epsilon) = \sum_{k=1}^{\infty} k^3 \exp\left(-k\epsilon\left(1+\sqrt{2}\right)\right) \cos(k\epsilon)$	(6.26)

6.3 General damped oscillation summation method

We consider the following sum of all natural numbers.

$$S = \sum_{k=1}^{\infty} k \tag{6.27}$$

Then we define the following function.

$$H(\epsilon) = \sum_{k=1}^{\infty} k \exp(-k\epsilon) \cos(k\epsilon) \quad (6.28)$$

$$G(z) = \sum_{k=1}^{\infty} k \exp(-kz) \quad (6.29)$$

We express the function H (ε) as follows.

$$H(\epsilon) = \frac{1}{2}(G(\epsilon - i\epsilon) + G(\epsilon + i\epsilon)) \quad (6.30)$$

We interpret the above function as a sum of the function G (z) and G (z*).

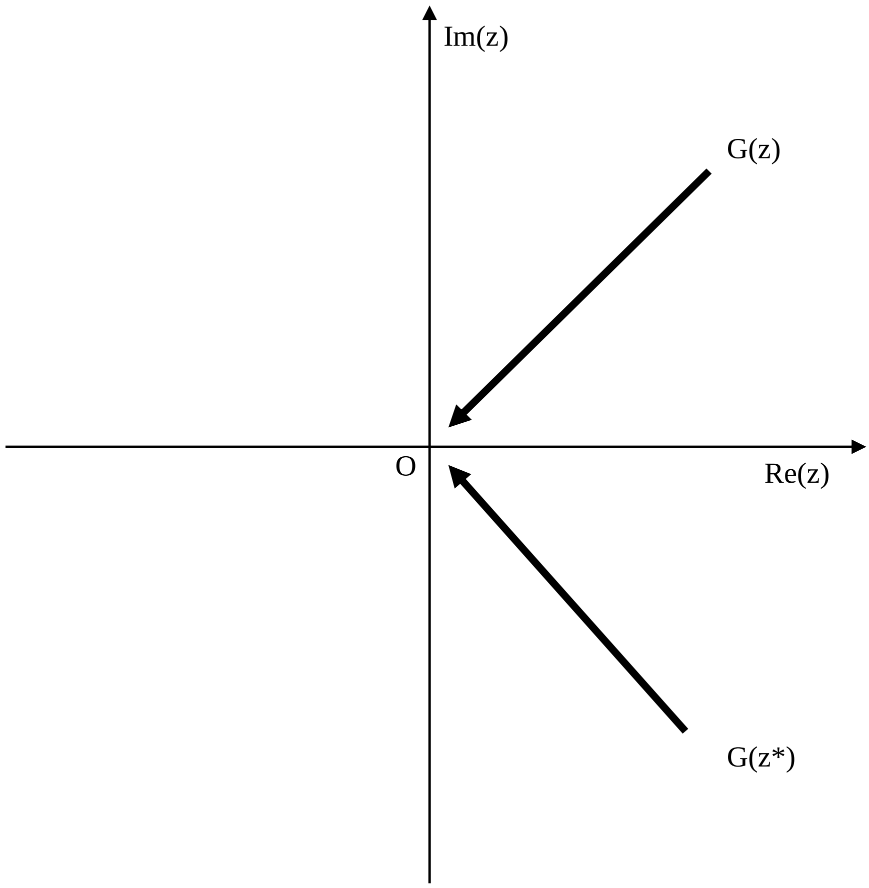

Figure 6.2: Special damped oscillation summation method

In the above figure, the two functions G (z) and G (z*) approach to the origin O at the two angles. This is special condition. Is it possible to make the condition more general?

Then we consider the case that the many functions G (z) approach to the origin at the all angles.

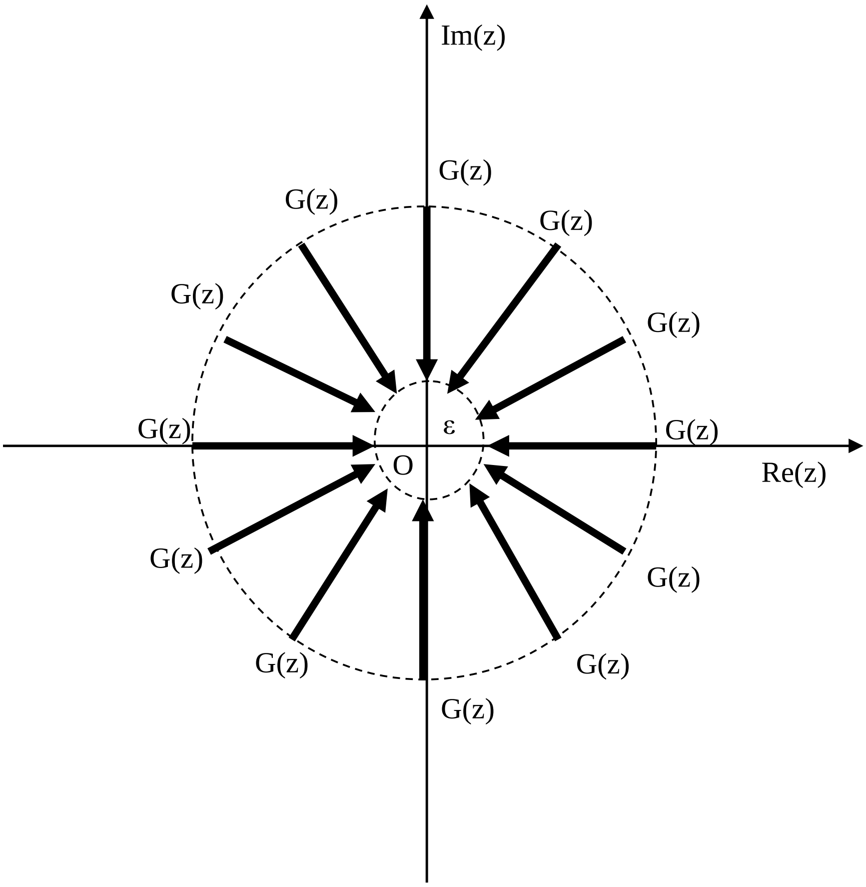

Figure 6.3: General damped oscillation summation method

We express the above consideration in the following equations.

$$H(\epsilon) = \frac{1}{2}(G(z_1) + G(z_2)) \tag{6.31}$$

$$H(\epsilon) = \frac{1}{3}(G(z_1) + G(z_2) + G(z_3)) \tag{6.32}$$

$$H(\epsilon) = \frac{1}{4}(G(z_1) + G(z_2) + G(z_3) + G(z_4)) \tag{6.33}$$

$$H(\epsilon) = \frac{1}{n}(G(z_1) + G(z_2) + G(z_3) + \cdots + G(z_n)) \tag{6.34}$$

We express the above sum as the following integration. Here, $2\pi iz$ is normalization constant.

$$H(\epsilon) = \oint_{|z|=\epsilon} \frac{1}{2\pi iz} G(z)\, dz \tag{6.35}$$

$$H(\epsilon) = \frac{1}{2\pi i} \oint_{|z|=\epsilon} \frac{1}{z} G(z)\, dz \tag{6.36}$$

We obtain the following result by the residue theorem.

$$H(\epsilon) = \frac{1}{2\pi i}\oint_{|z|=\epsilon} \frac{1}{z}\left(\frac{1}{z^2} - \frac{1}{12} + O(z^2)\right) dz = -\frac{1}{12} \tag{6.37}$$

Therefore, we obtain the following new general summation method.
(General damped oscillation summation method)

$$S = \sum_{k=1}^{\infty} a_k \tag{6.38}$$

$$G(z) = \sum_{k=1}^{\infty} a_k \exp(-kz) \tag{6.39}$$

$$H(\epsilon) = \frac{1}{2\pi i}\oint_{|z|=\epsilon} \frac{1}{z} G(z)\, dz \tag{6.40}$$

$$S_H = \lim_{\epsilon \to 0+} H(\epsilon) \tag{6.41}$$

This summation method can sum any series by the residue theorem.

We call the damped oscillation summation method with the residue theorem "**general damped oscillation summation method**."
We call the damped oscillation summation method without the residue theorem "**special damped oscillation summation method**."

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