ON CARMICHAËL'S CONJECTURE

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Carmichaël's conjecture is the following: "the equation $\varphi(x) = n$ cannot have a unique solution, $(\forall)n \in \mathbb{N}$, where φ is the Euler's function". R. K. Guy presented in [1] some results on this conjecture; Carmichaël himself proved that, if n_0 does not verify his conjecture, then $n_0 > 10^{37}$; V. L. Klee [2] improved to $n_0 > 10^{400}$, and P. Masai & A. Valette increased to $n_0 > 10^{10000}$. C. Pomerance [4] wrote on this subject too.

In this article we prove that the equation $\varphi(x) = n$ admits a finite number of solutions, we find the general form of these solutions, also we prove that, if x_0 is the unique solution of this equation (for a $n \in \mathbb{N}$), then x_0 is a multiple of $2^2 \cdot 3^2 \cdot 7^2 \cdot 43^2$ (and $x_0 > 10^{10000}$ from [3]).

§1. Let x_0 be a solution of the equation $\varphi(x) = n$. We consider n fixed. We'll try to construct another solution $y_0 \neq x_0$.

The first method:

We decompose $x_0 = a \cdot b$ with a, b integers such that $(a, b) = 1$.

we look for an $a' \neq a$ such that $\varphi(a') = \varphi(a)$ and $(a', b) = 1$; it results that $y_0 = a' \cdot b$.

The second method:

Let's consider $x_0 = q_1^{\beta_1} ... q_r^{\beta_r}$, where all $\beta_i \in \mathbb{N}^*$, and $q_1, ..., q_r$ are distinct primes two by two; we look for an integer q such that $(q, x_0) = 1$ and $\varphi(q)$ divides $x_0 / (q_1, ..., q_r)$; then $y_0 = x_0 q / \varphi(q)$.

We immediately see that we can consider q as prime.

The author conjectures that for any integer $x_0 \geq 2$ it is possible to find, by means of one of these methods, a $y_0 \neq x_0$ such that $\varphi(y_0) = \varphi(x_0)$.

Lemma 1. The equation $\varphi(x) = n$ admits a finite number of solutions, $(\forall)n \in \mathbb{N}$.

Proof. The cases $n = 0,1$ are trivial.

Let's consider n to be fixed, $n \geq 2$. Let $p_1 < p_2 < ... < p_s \leq n+1$ be the sequence of prime numbers. If x_0 is a solution of our equation (1) then x_0 has the form $x_0 = p_1^{\alpha_1} ... p_s^{\alpha_s}$, with all $\alpha_i \in \mathbb{N}$. Each α_i is limited, because:

$(\forall)i \in \{1, 2, ..., s\}$, $(\exists)a_i \in \mathbb{N}$: $p_i^{\alpha_i} \geq n$.

Whence $0 \le \alpha_i \le a_i + 1$, for all i. Thus, we find a wide limitation for the number of solutions: $\prod_{i=1}^{s} (a_i + 2)$

Lemma 2. Any solution of this equation has the form (1) and (2):

$$x_0 = n \cdot \left(\frac{p_1}{p_1 - 1} \right)^{\varepsilon_1} \ldots \left(\frac{p_s}{p_s - 1} \right)^{\varepsilon_s} \in \mathbb{Z} ,$$

where, for $1 \le i \le s$, we have $\varepsilon_i = 0$ if $\alpha_i = 0$, or $\varepsilon_i = 1$ if $\alpha_i \ne 0$.

Of course**,** $n = \varphi(x_0) = x_0 \left(\frac{p_1}{p_1 - 1} \right)^{\varepsilon_1} \ldots \left(\frac{p_s}{p_s - 1} \right)^{\varepsilon_s}$,

whence it results the second form of x_0.

From (2) we find another limitation for the number of the solutions: $2^s - 1$ because each ε_i has only two values, and at least one is not equal to zero.

§2. We suppose that x_0 is the unique solution of this equation.

Lemma 3. x_0 is a multiple of $2^2 \cdot 3^2 \cdot 7^2 \cdot 43^2$.

Proof. We apply our second method.

Because $\varphi(0) = \varphi(3)$ and $\varphi(1) = \varphi(2)$ we take $x_0 \ge 4$.

If $2 \nmid x_0$ then there is $y_0 = 2x_0 \ne x_0$ such that $\varphi(y_0) = \varphi(x_0)$, hence $2 \mid x_0$; if $4 \nmid x_0$, then we can take $y_0 = x_0 / 2$.

If $3 \nmid x_0$ then $y_0 = 3x_0 / 2$, hence $3 \mid x_0$; if $9 \nmid x_0$ then $y_0 = 2x_0 / 3$, hence $9 \mid x_0$; whence $4 \cdot 9 \mid x_0$.

If $7 \nmid x_0$ then $y_0 = 7x_0 / 6$, hence $7 \mid x_0$; if $49 \nmid x_0$ then $y_0 = 6x_0 / 7$ hence $49 \mid x_0$; whence $4 \cdot 9 \cdot 49 \mid x_0$.

If $43 \nmid x_0$ then $y_0 = 43x_0 / 42$, hence $43 \mid x_0$; if $43^2 \nmid x_0$ then $y_0 = 42x_0 / 43$, hence $43^2 \mid x_0$; whence $2^2 \cdot 3^2 \cdot 7^2 \cdot 43^2 \mid x_0$.

Thus $x_0 = 2^{\gamma_1} \cdot 3^{\gamma_2} \cdot 7^{\gamma_3} \cdot 43^{\gamma_4} \cdot t$, with all $\gamma_i \ge 2$ and (t, $2 \cdot 3 \cdot 7 \cdot 43$) = 1 and $x_0 > 10^{10000}$ because $n_0 > 10^{10000}$.

§3. Let's consider $\gamma_l \geq 3$. If $5 \nmid x_0$ then $5x_0 / 4 = y_0$, hence $5 \mid x_0$; if $25 \nmid x_0$ then $y_0 = 4x_0 / 5$, whence $25 \mid x_0$.

We construct the recurrent set M of prime numbers:

a) the elements $2, 3, 5 \in M$;
b) if the distinct odd elements $e_1, \ldots, e_n \in M$ and $b_m = 1 + 2^m \cdot e_1, \ldots, e_n$ is prime, with $m = 1$ or $m = 2$, then $b_m \in M$;
c) any element belonging to M is obtained by the utilization (a finite number of times) of the rules a) or b) only.

The author conjectures that M is infinite, which solves this case, because it results that there is an infinite number of primes which divide x_0. This is absurd.

For example 2, 3, 5, 7, 11, 13, 23, 29, 31, 43, 47, 53, 61, … belong to M.

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The method from §3 could be continued as a tree (for $\gamma_2 \geq 3$ afterwards $\gamma_3 \geq 3$, etc.) but its ramifications are very complicated…

REFERENCES

[1] R. K. Guy, Monthly unsolved problems - 1969-1983. Amer. Math. Monthly, Vol. 90, No. 10/1983, p. 684.

[2] V. L. Klee, Amer. Math. Monthly 76, (969), p. 288.

[3] P. Masai & A. Valette, A lower bound for a counter-example to Carmichaël's conjecture, Boll. Unione Mat. Ital, (6) A_1 (1982), pp. 313-316.

[4] C. Pomerance, Math. Reviews: 49:4917.

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