Odd perfect numbers conjecture

Idan Raman

Abstract

When defining O(N) as the sum of all divisors of N including himself, it is to be proved that there is no odd number which satisfy the equation:

$$O(N)=2N$$

And from that proof, it follows that there is no odd prime by definition.

The work showed here, is based on the work Euler did on the subject.

Contents

Introduction

Let N be an odd natural number. O(N) is a function of N defined as the sum of all the positive divisors of N, including himself.

If we take this sum and decrease N from it, we will get the sum of N's proper positive divisors – which is defined in this way. As also defined, a number is called perfect if its sum of proper positive divisors is equal to the number N itself.

$$O(N) - N = N$$

Or

$$O(N) = 2N$$

In this paper, I will show beyond doubt that for no N this equality is being satisfied.

This problem has been around for the last 3 centuries, and has been investigated by a lot of known mathematicians. Some of them even showed some interesting and groundbreaking progress, in what conditions we can relate to N to satisfy the equation.
In this paper there is a use of some of this progress, mainly made be Euler. The following condition will be proved in part 3.

N is of the form $k^w {p_1}^{a_1} {p_2}^{a_2} \cdots {p_n}^{a_n}$ *where* $a_i\ [1 \le i \le n]$ *is even,* $k, p_1, \dots, p_n$ *are distinct primes, and* w *is odd.*

Notations

p,k – a prime number.

N- a natural number.

L(q) – the group of the positive divisors of q.

O(q) – the sum of the positive divisors of q.

E(q) – the group of the unique primed divisors of q.

D(q) – the sum of the unique primed divisors of q.

|q| - the amount of terms in q

Lemmas

I The formula

$$|\mathbf{L(N)}| = (\boldsymbol{a_1} + \mathbf{1})(\boldsymbol{a_2} + \mathbf{1}) \cdots (\boldsymbol{a_n} + \mathbf{1})$$

first of all, we know that every natural number N we can divide to prime divisors

$$\mathrm{N} = {p_1}^{a_1} \cdot {p_2}^{a_2} \cdots {p_n}^{a_n}$$

when $(a_1 \ldots a_n) > 0$ and we also know, that since N is odd, none of its prime divisors is equal to 2.

The amount of positive divisors of N is defined as all of the ways to choose a set of $a_{i1}, a_{i2}, a_{i3} \ldots\ a_{in}$ when $0 \le a_{i1} \le \mathrm{a_i}$.

The amount of options for each a_{it} , is $a_t + 1$ because the range is all the numbers below a_t , adding 0.

That gives us the formula:

$$|\mathrm{L(N)}| = (a_1 + 1)(a_2 + 1) \cdots (a_n + 1)$$

II If $2^{\mathrm{j}}|\mathrm{A} \rightarrow 2^{\mathrm{j}}|2^{\mathrm{A}}$

First we will show that A>j.

$$2^j \cdot b = A \ \rightarrow 2^j \le A$$

And we can also see that $2^j > j$ for any value of j, because if we will take the derivative of

$$m(j) = 2^j - j \ \rightarrow m'(j) = ln(2) \cdot 2^j - 1 = 0 \ \rightarrow \frac{1}{ln(2)} = 2^j \ \rightarrow j = log_2\left(\frac{1}{ln(2)}\right) = 0.5287$$

$$m(0.5287) = 0.913 > 0$$

$$m''(j) = ln^2(2) \cdot 2^j \ \rightarrow\ m''(0.5287) = 0.693 > 0 \ \rightarrow (0.5287, 0.913)min.$$

so $2^j > j$, and so $j < 2^j \le A \ \rightarrow j < A$.

now we can see that $2^{\mathrm{j}} < 2^{\mathrm{A}}$, and $2^{\mathrm{j}}|2^{\mathrm{A}}$.

III Only one of $\boldsymbol{a_1, a_2, a_3 \dots\ a_n}$ is odd.

As we know N is odd, and 2N divides by 2 only once.
Every term in L(n) is odd, and so we can relate the terms as $2f_i$+1, Where $1 \leq i \leq |L(n)|$ the sum of these terms will be:

$$O(N) = \sum_{i=1}^{|L(n)|} 2f_i + 1 = |L(n)| + \sum_{i=1}^{|L(n)|} 2f_i = |\mathrm{L(n)}| + 2^{|L(n)|} \sum_{i=1}^{|L(n)|} f_i$$

By lemma II we know that the power of 2 in $2^{|L(n)|}$ will be bigger than the power of 2 in |L(N)|, and the total minimum power of 2 in O(N) is L(N). Now as we know, if each of $a_{i1}, a_{i2}, a_{i3} \dots\ a_{in}$ is odd, then the power of 2 in |L(N)| will get bigger by 1 so we have 2 options to keep it divisible by 2 only once:

1. all $\boldsymbol{a_1, a_2, a_3 \dots\ a_n}$ are even

We will revoke this option negatively.
We know that all the terms in L(N) are odd, since $\mathrm{p_1}, \dots, \mathrm{p_n}$ are odd too.
If all the terms are odd, and the amount of them is odd too, then the sum O(N) is odd which contradicts the fact that O(N) needs to be divisible by 2 once.

2. only one of $\boldsymbol{a_1, a_2, a_3 \dots\ a_n}$ is odd.

This is the only possibility, as showed.

IV For N to be able to satisfy the equation, N needs to be of the following form:

$\mathbf{k^w p_1{}^{a_1} p_2{}^{a_2} \cdots p_n{}^{a_n}}$ **where** $\mathbf{a_i}$ $[\mathbf{1 \leq i \leq n}]$ **is even and k,** $\mathbf{p_1, \dots, p_n}$ **are distinct primes.**

This one is easy to prove by lemma III.
we have already showed that only one of $a_1, a_2, a_3 \dots\ a_n$ is odd, and we will mark it as w.

V The formula:

$$\boldsymbol{O(N)} = \left[\frac{\boldsymbol{k^{w+1}-1}}{\boldsymbol{k-1}}\right] \cdot \left[\frac{\boldsymbol{{p_1}^{a_1+1}-1}}{\boldsymbol{p_1-1}}\right] \cdots \left[\frac{\boldsymbol{{p_n}^{a_n+1}-1}}{\boldsymbol{p_n-1}}\right]$$

We will prove by induction, on the amount of unique prime divisors (n).

For n=1:

$N = {p_1}^{a_1}$, we know we have $a_1 + 1$ options and they are generated by the formula ${p_1}^{a_i}$ when $1 \leq a_i \leq a_n$. This condition answers to the requires of a geometric sum, when $b_1 = 1$, $q = p_1$, $n = a_n$ and by the formula:

$$O(N) = S_{a_n} = \left[\frac{{p_1}^{a_1+1}-1}{p_1-1}\right]$$

Because k, $p_1, \ldots, p_n$ are independent primes, the total O(N) is achieved by the following multiply:

$$O(N) = \left[\frac{k^{w+1}-1}{k-1}\right] \cdot \left[\frac{{p_1}^{a_1+1}-1}{p_1-1}\right] \cdots \left[\frac{{p_n}^{a_n+1}-1}{p_n-1}\right]$$

Summation

In this part, we will do use of the proved lemmas to show the solution to the conjecture. We will also do use of the functions E(q) and D(q).

Lets return to the first equation, which is equivalent to the conjecture itself:

$$O(N) = 2N$$

We will put both sides of the equation into the function E(q) first to determinate is it possible for the two numbers to be equal:

$$2N = 2k^{w}p_1^{a_1}p_2^{a_2} \cdots p_n^{a_n} \rightarrow E(N) = n + 2$$

Because by definition 2,k, and $p_1, \dots, p_n$ are distinct primes.

Now we will see the value of E(O(N)):

$$O(N) = \left[\frac{k^{w+1} - 1}{k - 1}\right] \cdot \left[\frac{{p_1}^{a_1+1} - 1}{p_1 - 1}\right] \cdots \left[\frac{{p_n}^{a_n+1} - 1}{p_n - 1}\right] \rightarrow E\big(O(N)\big) \geq n + 2$$

Because $\left[\frac{{p_1}^{a_1+1}-1}{p_1-1}\right] \cdots \left[\frac{{p_n}^{a_n+1}-1}{p_n-1}\right]$ are distinct primes and $\left[\frac{k^{w+1}-1}{k-1}\right]$ is divisible by 2, the minimum amount of divisors is n+2 and **we get it only when** $\left[\frac{k^{w+1}-1}{k-1}\right], \left[\frac{{p_1}^{a_1+1}-1}{p_1-1}\right] \ldots \left[\frac{{p_n}^{a_1+1}-1}{p_n-1}\right]$ **are primes.**

This fact is useable for us- since there are exactly equal amount of prime divisors to N, O(N) we know that there should be an one to one correspondence between the factors – and if the factor k appears in the prime divisors of N, one of the divisors of O(N) must be equal to it.

Now we will use the function D(q) to determinate if the conjecture to be true, using also the previews discovery.

$$2N = 2\mathrm{k}^{\mathrm{w}}\mathrm{p}_1{}^{\mathrm{a}_1}\mathrm{p}_2{}^{\mathrm{a}_2}\cdots \mathrm{p}_{\mathrm{n}}{}^{\mathrm{a}_{\mathrm{n}}} \rightarrow D(2N) = \sum_{i=1}^{n} p_i + k + 2$$

$$O(N) = 2\left[\frac{k^{w+1}-1}{2k-2}\right]\cdot\left[\frac{p_1{}^{a_1+1}-1}{p_1-1}\right]\cdots\left[\frac{p_n{}^{a_n+1}-1}{p_n-1}\right] \rightarrow D\big(O(N)\big) = \sum_{I=1}^{n}\left[\frac{p_i{}^{a_i+1}-1}{p_i-1}\right] + \left[\frac{k^{w+1}-1}{2k-2}\right] + 2$$

Now we will look at the difference D(O(N))-D(N):

$$D(O(N)) - D(2N) = \sum_{I=1}^{n}\left[\frac{p_i{}^{a_i+1}-1}{p_i-1} - p_i\right] + \left[\frac{k^{w+1}-1}{2k-2} - k\right]$$

We will look at the minimum of first term of the sum:

$$\sum_{I=1}^{n}\left[\frac{p_i{}^{a_i+1}-1}{p_i-1} - p_i\right] \; for\ a\ minimal\ value, a_i\ needs\ to\ be\ minimal\ (2) =$$

$$\sum_{I=1}^{n}\left[\frac{p_i{}^{3}-1}{p_i-1} - p_i\right] = \sum_{I=1}^{n}[p_i{}^2 + p_i + 1]\; for\ a\ minimal\ value, p_i\ needs\ to\ be\ minimal\ (3) =$$

$$\sum_{I=1}^{n}[3^2 + 3 + 1] = 13n > 0$$

So the first term is always positive. Now we'll check whether the second term is always positive or not:

$$\frac{k^{w+1}-1}{2k-2} - k\; for\ a\ minimal\ value, w\ needs\ to\ be\ minimal\ (1) =$$

$$\frac{k^2-1}{2k-2} - k = \frac{k+1}{2} - k = \frac{1-k}{2} < 0$$

So the conjecture is possible when w=1. For w=3 or bigger:

$$\frac{k^3-1}{2k-2} - k = \frac{k^2+k+1}{2} - k = \frac{k^2-k+1}{2} > 0$$

So we can conclude, that w=1.

$$2N = 2\mathrm{k}\cdot\mathrm{p}_1{}^{\mathrm{a}_1}\mathrm{p}_2{}^{\mathrm{a}_2}\cdots \mathrm{p}_{\mathrm{n}}{}^{\mathrm{a}_{\mathrm{n}}}$$

We also know, that the following inequality must be true:

$$\sum_{I=1}^{n}\left[\frac{{p_i}^{a_i+1}-1}{p_i-1}-p_i\right] < \frac{k-1}{2}$$

For that to happen, k needs to be max.

The biggest value that k can get, is the biggest of the terms $\left[\frac{{p_1}^{a_1+1}-1}{p_1-1}\right]\cdots\left[\frac{{p_n}^{a_n+1}-1}{p_n-1}\right]$,which we'll mark as:

$$k=\left[\frac{{p_t}^{a_t+1}-1}{p_t-1}\right]$$

So now we will put the value of k in the formula above:

$$\sum_{I=1}^{n}\left[\frac{{p_i}^{a_i+1}-1}{p_i-1}-p_i\right] < \frac{\left[\frac{{p_t}^{a_t+1}-1}{p_t-1}\right]-1}{2} \rightarrow 2\sum_{I=1}^{n}\left[\frac{{p_i}^{a_i+1}-1}{p_i-1}-p_i\right]+1 < \left[\frac{{p_t}^{a_t+1}-1}{p_t-1}\right] \rightarrow$$

$$\textit{Because t is in } [1,n] \textit{ we know that}: 2\sum_{I=1}^{n}\left[\frac{{p_i}^{a_i+1}-1}{p_i-1}-p_i\right]+1 \geq 2\left[\frac{{p_t}^{a_t+1}-1}{p_t-1}\right]+1$$

$$\textit{and by transitive law}: 2\left[\frac{{p_t}^{a_t+1}-1}{p_t-1}\right]+1 < \left[\frac{{p_t}^{a_t+1}-1}{p_t-1}\right] \rightarrow \left[\frac{{p_t}^{a_t+1}-1}{p_t-1}\right] < -1$$

$$\boldsymbol{\textit{which is impossible, because as known}\left[\frac{{p_t}^{a_t+1}-1}{p_t-1}\right]\textit{is positive}.}$$

Completion

Now we will go back through the steps, to make the proof more clear:

- We showed that no k is possible, in a way that $D\big(O(N)\big) - D(2N) \leq 0$, So we know that $D\big(O(N)\big) \neq D(2N)$.

- If O(N) and N were equal, we would have gotten $D\big(O(N)\big) = D(2N)$ So we also know that $\mathrm{O(N)} \neq 2N$

- We showed that N of the form presented in lemma IV, and so there is no other option for N values which will might get the equality $D\big(O(N)\big) = D(2N)$, to be prefect numbers.

- And finally, we showed that the Odd Perfect Conjecture is equivalent to the equality $\mathrm{O(N)} = 2N$, and by showing it never holds we have solved the conjecture.