Geodesic equation in spherical surface

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Abstract. The notion of a geodesic line (also: geodesic) is a geometric concept which is a generalization of the concept of a straight line (or a segment of a straight line) in Euclidean geometry to spaces of a more general type. In this paper I have derived the geodesic equation for a 2 dimensional spherical surface following a local local minkowski frame with the minkowski metric localized .

PACS numbers: 02.40.Ky, 02.40.Pc, 02.30.Hq, 03.30.+p, 03.50.-z, 04.20.-q

1. Introduction

One of the central problems in general relativity is the determination of geodesic curves. These are the curves that a free particle (that is, a particle upon which no force acts, where force in this case excludes gravity, since the effects of gravity are felt entirely through the curvature of space-time) will follow in a curved space-time.

The shortest path between two points in a curved space can be found by writing the equation for the length of a curve (a function f from an open interval of R to the manifold), and then minimizing this length using the calculus of variations. This has some minor technical problems, because there is an infinite dimensional space of different ways to parameterize the shortest path. It is simpler to demand not only that the curve locally minimize length but also that it is parameterized "with constant velocity", meaning that the distance from f(s) to f(t) along the geodesic is proportional to $|s-t|$. Equivalently, a different quantity may be defined, termed the energy of the curve; minimizing the energy leads to the same equations for a geodesic (here "constant velocity" is a consequence of minimisation). Intuitively, one can understand this second formulation by noting that an elastic band stretched between two points will contract its length, and in so doing will minimize its energy. The resulting shape of the band is a geodesic.

In Riemannian geometry geodesics are not the same as "shortest curves" between two points, though the two concepts are closely related. The difference is that geodesics are only locally the shortest distance between points, and are parameterized with "constant velocity".

Going the "long way round" on a great circle between two points on a sphere is a geodesic but not the shortest path between the points. The map t → t2 from the unit interval to itself gives the shortest path between 0 and 1, but is not a geodesic because the velocity of the corresponding motion of a point is not constant.

Geodesics are commonly seen in the study of Riemannian geometry and more generally metric geometry. In general relativity, geodesics describe the motion of point particles under the influence of gravity alone. In particular, the path taken by a falling rock, an orbiting satellite, or the shape of a planetary orbit are all geodesics in curved space-time. More generally, the topic of sub-Riemannian geometry deals with the paths that objects may take when they are not free, and their movement is constrained in various ways.

This article presents the mathematical formalism involved in defining, finding, and proving the existence of geodesics, in the case of a spherical 2 dimensional surface.

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2. Geodesics in Spherical Surface

Before deriving the geodesic in a spherical surface, it is inevitable that one must be able to derive the general geodesic equation which can then be applied to any metric field that spans over a space. hence the first section contains the general over view of the geodesic equation

2.1. Geodesic Equation

geodesic curves are found by maximizing the proper time between two events, where the proper time interval is given by

$$\Delta\tau = \int_0^1 \sqrt{-g_{ij}\frac{dx^i}{d\sigma}\frac{dx^j}{d\sigma}}d\sigma \qquad (1)$$

We can do this by solving the set of Lagrangian differential equations given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_a}\right) - \frac{\partial L}{\partial q_a} = 0 \qquad (2)$$

where the Lagrangian function is given by

$$L = \sqrt{-g_{ij}\left(x^k\left(\sigma\right)\right)\frac{dx^i}{d\sigma}\frac{dx^j}{d\sigma}}$$

Here, the objects world path is given in parametric form by $x^i\left(\sigma\right)$. The metric can depend on the position, so it too is ultimately a function of the parameter σ, which plays the role of t in 2. For this Lagrangian, the coordinates are $q_a = x^a$ and the generalized velocities are $\dot{q}_a = \frac{dx^a}{d\sigma} \equiv \dot{x^a}$. We therefore have

$$\frac{\partial L}{\partial \dot{x}^a} = -\frac{1}{2L} g_{ij} \left(\delta^i{}_a \frac{dx^j}{d\sigma} - \frac{dx^i}{d\sigma} \delta^j{}_a \right) = -\frac{g_{aj}}{L} \frac{dx^j}{d\sigma}$$

This follows because g_{ij} does not depend on $\dot{x}^a$(it depends only on the coordinates, not the velocities), and also because $g_{aj} = g_{ja}$.

The other derivative comes out to

$$\frac{\partial L}{\partial x^a} = -\frac{1}{2L} \frac{\partial g_{ij}}{\partial x^a} \frac{dx^i}{d\sigma} \frac{dx^j}{d\sigma}$$

We can get rid of L in these two results by noticing from 1 that

$$\frac{d\tau}{d\sigma} = \sqrt{-g_{ij} \frac{dx^i}{d\sigma} \frac{dx^j}{d\sigma}} = L$$

We therefore get

$$\frac{\partial L}{\partial \dot{x}^a} = -\frac{g_{aj}}{L} \frac{dx^j}{d\sigma} = -g_{aj} \frac{d\sigma}{d\tau} \frac{dx^j}{d\sigma} = -g_{aj} \frac{dx^j}{d\tau}$$

$$\frac{\partial L}{\partial x^a} = -\frac{1}{2L} \frac{\partial g_{ij}}{\partial x^a} \frac{dx^i}{d\sigma} \frac{dx^j}{d\sigma} = -\frac{1}{2} \frac{d\sigma}{d\tau} \frac{\partial g_{ij}}{\partial x^a} \frac{dx^i}{d\sigma} \frac{dx^j}{d\sigma} = -\frac{1}{2} \frac{\partial g_{ij}}{\partial x^a} \frac{dx^i}{d\tau} \frac{dx^j}{d\sigma}$$

Putting it all together using 2, we get

$$-\frac{d}{d\sigma} \left(g_{aj} \frac{dx^j}{d\tau} \right) + \frac{1}{2} \frac{\partial g_{ij}}{\partial x^a} \frac{dx^i}{d\tau} \frac{dx^j}{d\sigma} = 0$$

We can now eliminate the parameterσ by multiplying through by $-\frac{d\sigma}{d\tau}$:

$$\frac{d}{d\tau} \left(g_{aj} \frac{dx^j}{d\tau} \right) - \frac{1}{2} \frac{\partial g_{ij}}{\partial x^a} \frac{dx^i}{d\tau} \frac{dx^j}{d\tau} = 0 \qquad (3)$$

This is one form of the geodesic equation, which is a second order ordinary differential equation. (Its an ordinary differential equation despite the appearance of the partial derivative$\frac{\partial g_{ij}}{\partial x^a}$ because we will know the metric as a function of the coordinates, so this derivative will be known when we set out to solve the ODE.)

2.2. Curved 2-d surface of a sphere

For a curved 2-d surface of a sphere of radius R. The metric for this space is, using the usual spherical coordinates θ and ϕ

$$g_{ij} = \begin{bmatrix} R^2 & 0 \\ 0 & (R\sin\theta)^2 \end{bmatrix}$$

The required derivatives of g_{ij} are

$$\partial_\theta g_{ij} = \begin{bmatrix} 0 & 0 \\ 0 & 2R^2 \sin\theta\cos\theta \end{bmatrix}$$

$$\partial_\phi g_{ij} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Then 3 becomes (using s as the parameter in place of τ) for $a = \theta$:

$$R^2\frac{d^2\theta}{ds^2} - R^2\sin\theta\cos\theta\left(\frac{d\phi}{ds}\right)^2 = 0$$

$$\frac{d^2\theta}{ds^2} - \sin\theta\cos\theta\left(\frac{d\phi}{ds}\right)^2 = 0$$

And for $a = \phi$:

$$\frac{d}{ds}\left((R\sin\theta)^2\frac{d\phi}{ds}\right) = 0$$

$$\frac{d}{ds}\left(\sin^2\theta\frac{d\phi}{ds}\right) = 0$$

The last equation can be integrated once to get

$$\sin^2\theta\frac{d\phi}{ds} = \frac{k}{R} \qquad (4)$$

where k is a constant with dimensions of length, so that k/R is dimensionless. Substituting this into the other ODE gives

$$\frac{d^2\theta}{ds^2} - \left(\frac{k}{R\sin^2\theta}\right)^2\sin\theta\cos\theta = 0$$

$$\frac{d^2\theta}{ds^2} = \frac{k^2\cos\theta}{R^2\sin^3\theta}$$

We have decoupled the equations, although this latest ODE isnt exactly easy to solve. We can make a bit of progress by observing that in 2-d space, the infinitesimal interval is given by $ds^2 = g_{ij}dx^idx^j$, so

$$g_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds} = \left(\frac{ds}{ds}\right)^2 = +1$$

Here, this gives us

$$R^2\left(\frac{d\theta}{ds}\right)^2 + R^2\sin^2\theta\left(\frac{d\phi}{ds}\right)^2 = 1$$

$$R^2\left(\frac{d\theta}{ds}\right)^2 + R^2\sin^2\theta\left(\frac{k}{R\sin^2\theta}\right)^2 = 1$$

$$\frac{d\theta}{ds} = \pm\frac{1}{R}\sqrt{1-\frac{k^2}{\sin^2\theta}} = \pm\frac{1}{R\sin\theta}\sqrt{\sin^2\theta - k^2}$$

Although we could integrate this directly (using software), the answer isnt terribly illuminating. We can take a different approach by rearranging the equation to get

$$ds = \pm\frac{R\sin\theta d\theta}{\sqrt{\sin^2\theta - k^2}} = \pm\frac{R\sin\theta d\theta}{\sqrt{1-\cos^2\theta - k^2}} = \pm\frac{R\sin\theta d\theta}{\sqrt{a^2-\cos^2\theta}} \text{ where } a^2 = 1-k^2$$

Now we can use the substitution $u = \cos\theta$ with $du = -\sin\theta d\theta$
and we get

$$\int ds = \pm R\int\frac{\sin\theta d\theta}{\sqrt{a^2-\cos^2\theta}} = \mp R\int\frac{du}{\sqrt{a^2-u^2}}$$

$$s = \mp R\arctan\left(\frac{u}{\sqrt{a^2-u^2}}\right) + u_0$$

If we want the path length to be zero at $\theta = \pi/2$, this corresponds to $u = 0$, so we take $u_0 = 0$. This gives

$$\tan\frac{s}{R} = \mp\frac{u}{\sqrt{a^2-u^2}}$$

Defining $\psi \equiv s/R$ and squaring both sides, we get

$$\tan^2\psi = \frac{u^2}{a^2-u^2}$$

$$a^2\tan^2\psi = u^2\left(1+\tan^2\psi\right)$$

$$u = \cos\theta = \pm a\sin\psi$$

Returning to 4, we can now eliminate θ,
since $\sin^2\theta = 1-\cos^2\theta = 1-a^2\sin^2\psi$

$$\left(1-a^2\sin^2\psi\right)\frac{d\phi}{ds} = \frac{k}{R}$$

$$\frac{d\phi}{ds} = \frac{k}{R\left(1-a^2\sin^2\frac{s}{R}\right)}$$

Integrating this using software gives the rather cryptic result

$$\phi = k\text{arctanh}\left(\sqrt{-1+a^2}\tan\left(\frac{s}{R}\right)\right)\frac{1}{\sqrt{-1+a^2}} + \phi_0$$

We can convert this into something more meaningful if we remember that $k = \sqrt{1-a^2}, so\sqrt{-1+a^2} = ik$. Also, since

$$\tanh ix = \frac{e^{ix}-e^{-ix}}{e^{ix}+e^{-ix}} = i\frac{\sin x}{\cos x} = i\tan x$$

the inverse functions are related by

$$\operatorname{arctanh} ix = i \arctan x$$

Putting this together, we get

$$\phi - \phi_0 = \frac{1}{i} i \tan\left(k \tan\left(\frac{s}{R}\right)\right)$$
$$\tan(\phi - \phi_0) = k \tan\left(\frac{s}{R}\right)$$

We now have equations giving θ and ϕ in terms of s:

$$\cos\theta = \pm a \sin\frac{s}{R}$$
$$\tan(\phi - \phi_0) = k \tan\left(\frac{s}{R}\right)$$

We know that the geodesics on a sphere should be arcs from great circles, that is, arcs from circles formed by the intersection of a plane containing the center of the sphere with the sphere itself. Note that were looking for great circles that connect any two points on the sphere, so these circles need not go through the poles. We can define these circles by considering a plane with equation $z = my$ where m is a constant, and its intersection with the sphere $x^2 + y^2 + z^2 = R^2$. All these planes will contain the x axis, but we are free to define the x axis pointing in any direction from the center of the sphere so we havent really restricted the solution in any way.

The intersection of the plane and sphere is given by converting to spherical coordinates:

$$x^2 + y^2 + (my)^2 = R^2$$

$$R^2 \sin^2\theta \cos^2\phi + \left(1 + m^2\right) R^2 \sin^2\theta \sin^2\phi = R^2$$
$$\sin^2\theta \left(\cos^2\phi + \sin^2\phi + m^2 \sin^2\phi\right) = 1$$
$$\pm m \sin\phi = \sqrt{\frac{1}{\sin^2\theta} - 1} = \cot\theta$$

Thus the equation $\pm m \sin\phi = \cot\theta$ is the equation of a great circle that includes the intersection of the x axis with the sphere. which agrees with the geodesic equation as shown earlier

Conventional spherical coordinates requires $\phi = 0$ along the x axis and since were passing all our planes through that axis, we need to choose the constant ϕ_0 above to match this. Weve taken $s = 0$ on the equator, which also intersects the x axis, so we need to take $\phi_0 = 0$ as well. Therefore the geodesics are

$$\cos\theta = \pm a \sin\frac{s}{R}$$
$$\tan\phi = k \tan\left(\frac{s}{R}\right)$$

We need to use a bit of trigonometric wizardry to convert these equations. First,

$\sin^2 x = 1 - \cos^2 x = 1 - 1/\left(1 + \tan^2 x\right) = \tan^2 x/\left(1 + \tan^2 x\right)$, so

$$\cos^2\theta = a^2 \frac{\tan^2 \frac{s}{R}}{1 + \tan^2 \frac{s}{R}}$$

$$\frac{1}{1 + \tan^2\theta} = a^2 \frac{\tan^2\phi}{k^2 + \tan^2\phi}$$

$$\frac{\cot^2\theta}{1 + \cot^2\theta} = a^2 \frac{\sin^2\phi}{k^2\left(1 - \sin^2\phi\right) + \sin^2\phi}$$

$$= \frac{\left(1 - k^2\right)\sin^2\phi}{k^2 + \left(1 - k^2\right)\sin^2\phi} = \frac{\frac{1-k^2}{k^2}\sin^2\phi}{1 + \frac{1-k^2}{k^2}\sin^2\phi}$$

The LHS and RHS are equal if

$$\cot^2\theta = \frac{1 - k^2}{k^2}\sin^2\phi$$

$$\cot\theta = \pm\sqrt{\frac{1 - k^2}{k^2}}\sin\phi$$

That is, these are great circles if we identify

$$m = \sqrt{\frac{1 - k^2}{k^2}}$$

References

[1] Bernard Schutz, Article name, *First Course in General Relativity*
[2] Prof. Susskind, *General Relativity Lecture Series OCW*
[3] Wikipedia
[4] Prof. Carl bender, *Lecture Notes on Mathematical Physics*