Solid Angle of the Off-Axis Circle Sector

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The solid angle of a circular sector specified by circle radius, angle of the sector, and distance of the circle plane to the observer is calculated in terms of various trigonometric and cyclometric functions. This generalizes previous results for the full circle that have appeared in the literature.

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I. GEOMETRY

The solid angle covered by a circle at general position relative to the observer has been discussed in the literature [1–8]. The principal generic application is the calibration of particle detectors of circular shape that receive isotropic radiation from sources which are far reaching—photonic or closer than the Bragg peak to a particle detector. The following calculation generalizes the case of the full circle to the case of a circular sector, characterized by the interior angle α, radius R and planar area $\alpha R^2/2$ of the section.

The solid angle of a circular segment might be calculated by subtraction of the solid angles of the associated circular sector and of the common triangle of both regions. This solid angle of the triangle is published [9–11]; so this work also assists to the calculation of the circular sections.

The main variables of the geometry are characterized in Figure 1:

- The shortest (perpendicular) distance $h > 0$ of the observer to the plane of the circle.
- The radius $R > 0$ of the circle.
- The interior angle α of the sector, in the range $0 \le \alpha \le \pi$.
- The distance $D \ge 0$ between the location of the observer perpendicularly projected onto the plane of the circle and the circle center. We fasten the origin of the plane of the circle to that foot point of the observer's position.

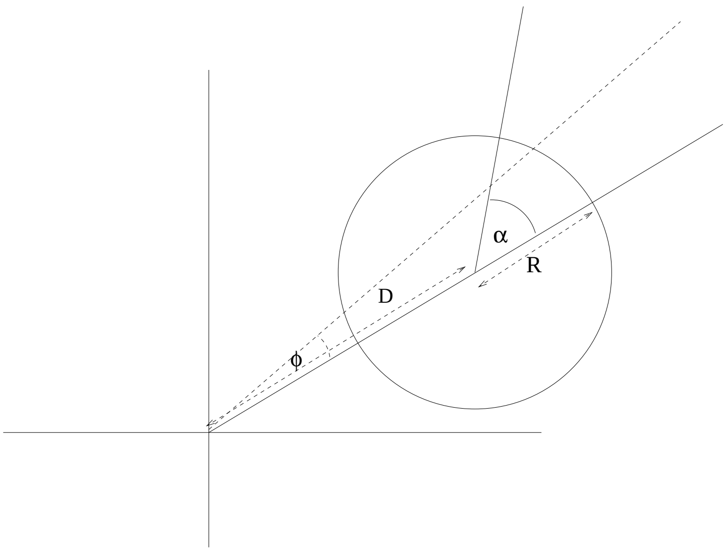

FIG. 1: The circle of radius R at distance D, its sector of angle α, and the azimuthal angle ϕ of the spherical coordinates seen by the observer looking into the direction $+z$.

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We calculate only the case where the direction from the (projected) observer's position to the circle center is aligned with one of the straight edges of the sector. In general, the sector will be rotated with respect to this aligned geometry. The solid angle is then obtained by subtraction of the two solid angles of two circular sectors that both have a common edge aligned with the viewing direction and a difference or sum of their interior angles equaling the actual interior angle of the desired sector. In the language of analysis this means we calculate a first integral over the angle α with lower limit at zero, defined by the line of intersection between (i) the plane of the circle and (ii) the orthogonal plane which contains the observer and the center of the circle. The other integrals are then differences between these first integrals.

II. TRANSFORMATION TO OFF-CENTER CIRCLE COORDINATES

In a spherical coordinate system centered at the observer, with polar angle θ ($0 \le \theta \le \pi$) and azimuth angle ϕ ($0 \le \phi \le 2\pi$), the solid angle of the object is

$$\Omega = \int \sin\theta \, d\theta \, d\phi, \tag{1}$$

where the two angular coordinates scan the surface of the object. We transform Cartesian coordinates to spherical coordinates and perform the double integral, using the generic

$$x = r\sin\theta\cos\phi, \tag{2}$$
$$y = r\sin\theta\sin\phi, \tag{3}$$
$$z = r\cos\theta, \tag{4}$$

and the inverse transformation

$$\phi = \arctan\frac{y}{x}, \quad \theta = \arccos\frac{z}{\sqrt{x^2+y^2+z^2}}. \tag{5}$$

If the coordinate system is oriented such that the azimuth angle ϕ starts at zero in the direction of the circle center seen from the projected observer's position (Figure 1), the equation of the circle is

$$(x-D)^2 + y^2 = R^2; \quad z = h. \tag{6}$$

Because z is fixed, so $r = h/\cos\theta$, the relevant 2D coordinates for the computation of the solid angle in (2) and (3) are

$$x = h\tan\theta\cos\phi; \quad y = h\tan\theta\sin\phi. \tag{7}$$

The same point (x, y) is characterized in a a circular coordinate system of radial ρ and azimuthal γ centered at the circle center as

$$x - D = \rho\cos\gamma; \quad y = \rho\sin\gamma. \tag{8}$$

Eliminating x and y from the previous equations, the transformation between the (θ,ϕ) and the (ρ,γ) system is

$$h\tan\theta\cos\phi - D = \rho\cos\gamma; \tag{9}$$
$$h\tan\theta\sin\phi = \rho\sin\gamma. \tag{10}$$

Adding the squares or building the ratio of these two equations yields

$$\rho = \sqrt{(h\tan\theta\cos\phi - D)^2 + (h\tan\theta\sin\phi)^2}; \quad \tan\gamma = \frac{h\tan\theta\sin\phi}{h\tan\theta\cos\phi - D} \tag{11}$$

and

$$h^2\tan^2\theta = (D+\rho\cos\gamma)^2 + (\rho\sin\gamma)^2 = D^2 + 2D\rho\cos\gamma + \rho^2 > 0, \tag{12}$$

and

$$\theta = \arctan\frac{\sqrt{D^2+2D\rho\cos\gamma+\rho^2}}{h}; \quad \phi = \arctan\frac{\rho\sin\gamma}{D+\rho\cos\gamma}. \tag{13}$$

The Jacobi determinant is derived from the four partial derivatives of these representations of θ and ϕ:

$$\left|\begin{array}{cc}\partial\theta/\partial\rho & \partial\theta/\partial\gamma\\ \partial\phi/\partial\rho & \partial\phi/\partial\gamma\end{array}\right| = h\rho\frac{1}{\sqrt{D^2+2D\rho\cos\gamma+\rho^2}(h^2+D^2+2D\rho\cos\gamma+\rho^2)}. \tag{14}$$

All factors in this equation are positive. We transform (1) to (ρ,γ) coordinates,

$$\Omega = \int\sin\theta d\theta\int d\phi = \int_0^R d\rho\int_0^\alpha d\gamma\left|\begin{array}{cc}\partial\theta/\partial\rho & \partial\theta/\partial\gamma\\ \partial\phi/\partial\rho & \partial\phi/\partial\gamma\end{array}\right|\sin\theta \tag{15}$$

and substitute (14) as well as

$$\sin\theta = \frac{\tan\theta}{\sqrt{1+\tan^2\theta}} = \frac{\frac{\sqrt{D^2+2D\rho\cos\gamma+\rho^2}}{h}}{1+\frac{D^2+2D\rho\cos\gamma+\rho^2}{h^2}}; = h\frac{\sqrt{D^2+2D\rho\cos\gamma+\rho^2}}{h^2+D^2+2D\rho\cos\gamma+\rho^2}, \tag{16}$$

such that the square roots cancel:

$$\Omega = \int_0^R d\rho\int_0^\alpha d\gamma\frac{h^2\rho}{(h^2+D^2+2D\rho\cos\gamma+\rho^2)^2}. \tag{17}$$

We may define the two unitless variables

$$\delta\equiv D/h\geq 0,\quad \beta\equiv R/h>0. \tag{18}$$

As expected from a homogeneous scaling of all lengths, the solid angle depends only on these length ratios:

$$\Omega(\beta,\delta,\alpha) = \beta^2\int_0^1 d\xi\int_0^\alpha d\gamma\frac{\xi}{(1+\delta^2+2\delta\beta\xi\cos\gamma+\beta^2\xi^2)^2}. \tag{19}$$

Up to here, the solid angle has merely been rephrased in an off-center circular coordinate system of the planar target. The following calculation works out the double integral supposing that the target shape is a circular sector.

III. CIRCULAR SECTOR

We integrate (19) over the radial coordinate ξ with the aid of [12, 2.175.2], where the discriminant $\Delta = 4(1+\delta^2)\beta^2 - 4\delta^2\beta^2\cos^2\gamma = 4\beta^2(1+\delta^2\sin^2\gamma) > 0$ has positive sign:

$$\begin{aligned}\Omega &= \beta^2\int_0^\alpha d\gamma[-\frac{2(1+\delta^2)+(2\delta\beta\cos\gamma)\xi}{4\beta^2(1+\delta^2\sin^2\gamma)(1+\delta^2+2\delta\beta\xi\cos\gamma+\beta^2\xi^2)}\,|_{\xi=0}^1\\ &\quad -\frac{2\delta\beta\cos\gamma}{4\beta^2(1+\delta^2\sin^2\gamma)}\int_0^1 d\xi\frac{1}{1+\delta^2+2\delta\beta\xi\cos\gamma+\beta^2\xi^2}]\end{aligned} \tag{20}$$

$$= \beta^2\int_0^\alpha d\gamma[\frac{1}{2\beta}\frac{\delta\cos\gamma+\beta}{(1+\delta^2\sin^2\gamma)(1+\delta^2+2\delta\beta\cos\gamma+\beta^2)} - \frac{\delta\cos\gamma}{2\beta(1+\delta^2\sin^2\gamma)}\int_0^1 d\xi\frac{1}{1+\delta^2+2\delta\beta\xi\cos\gamma+\beta^2\xi^2}] \tag{21}$$

and with [12, 2.172]

$$\Omega = \beta^2\int_0^\alpha d\gamma[\frac{1}{2\beta}\frac{\delta\cos\gamma+\beta}{(1+\delta^2\sin^2\gamma)(1+\delta^2+2\delta\beta\cos\gamma+\beta^2)} - \frac{\delta\cos\gamma}{2\beta^2(1+\delta^2\sin^2\gamma)^{3/2}}\arctan\frac{2\delta\beta\cos\gamma+2\beta^2\xi}{2\beta\sqrt{1+\delta^2\sin^2\gamma}}\,|_{\xi=0}^1] \tag{22}$$

$$\begin{aligned}&= \int_0^\alpha d\gamma\quad \Big\{\frac{\beta}{2}\frac{\delta\cos\gamma+\beta}{(1+\delta^2\sin^2\gamma)(1+\delta^2+2\delta\beta\cos\gamma+\beta^2)}\\ &\qquad +\frac{\delta\cos\gamma}{2(1+\delta^2\sin^2\gamma)^{3/2}}[\arctan\frac{\delta\cos\gamma}{\sqrt{1+\delta^2\sin^2\gamma}} - \arctan\frac{\delta\cos\gamma+\beta}{\sqrt{1+\delta^2\sin^2\gamma}}]\Big\}\end{aligned} \tag{23}$$

$$\equiv \frac{\beta}{2}[I_{\beta,\delta}(\alpha) - I_{\beta,\delta}(0)] + \frac{\delta}{2}[J_{\beta,\delta}(\alpha) - J_{\beta,\delta}(0)]. \tag{24}$$

The main task is to calculate the indefinite integrals I and J for variable upper limits:

$$I_{\beta,\delta}(\gamma) \equiv \int d\gamma \frac{\delta\cos\gamma + \beta}{(1+\delta^2\sin^2\gamma)(1+\delta^2+2\delta\beta\cos\gamma+\beta^2)}; \tag{25}$$

$$J_{\beta,\delta}(\gamma) \equiv \int d\gamma \frac{\cos\gamma}{(1+\delta^2\sin^2\gamma)^{3/2}}[\arctan\frac{\delta\cos\gamma}{\sqrt{1+\delta^2\sin^2\gamma}} - \arctan\frac{\delta\cos\gamma+\beta}{\sqrt{1+\delta^2\sin^2\gamma}}] \equiv F_{0,\delta}(\gamma) - F_{\beta,\delta}(\gamma). \tag{26}$$

[Remark. The difference of the two inverse tangent functions in the J-kernel might be combined as [13, 4.4.34]

$$J_{\beta,\delta}(\gamma) = -\int d\gamma \frac{\cos\gamma}{(1+\delta^2\sin^2\gamma)^{3/2}} \arctan\frac{\beta\sqrt{1+\delta^2\sin^2\gamma}}{1+\delta^2+\delta\beta\cos\gamma}. \tag{27}$$

We do not use this representation because this combined inverse tangent cannot be merely represented by its principal value; there may be phase jumps as a function of γ when $1+\delta^2+\delta\beta\cos\gamma$ passes through zero and the argument of the arctan passes from $+\infty$ to $-\infty$. These difficulties do not occur with the differences of the two arctan values on the principal branch in (26).]

The solutions are in (B2) and (A11). Because these representations as indefinite integrals are odd functions of γ, $I_{\beta,\delta}(\gamma) = -I_{\beta,\delta}(-\gamma)$ and $F_{\beta,\delta}(\gamma) = -F_{\beta,\delta}(-\gamma)$, the values vanish at $\gamma = 0$, so the associated two terms with zero argument in (24) vanish as well.

IV. SUMMARY

We have written the solid angle Ω of the circle sector as a function of the scaled lengths δ and β defined in (18) and of the angle α in terms of auxiliary integrals (24) represented by (B2) and (A11).

Appendix A: Angular Integral over Arctan Terms

The integrals (26)

$$F_{\beta,\delta}(\gamma) = \int d\gamma \frac{\cos\gamma}{(1+\delta^2\sin^2\gamma)^{3/2}} \arctan\frac{\delta\cos\gamma+\beta}{\sqrt{1+\delta^2\sin^2\gamma}} \tag{A1}$$

will by solved by partial integration, integrating the first factor and differentiating the arctan factor. The integral of the algebraic term of the sine and cosine is handled by the substitution

$$\cos\gamma = x, \quad dx = -d\gamma\sin\gamma, \tag{A2}$$

followed by the substitution

$$x^2 = t, \quad dt = 2xdx: \tag{A3}$$

$$\begin{aligned} &\int d\gamma \frac{\cos\gamma}{(1+\delta^2\sin^2\gamma)^{3/2}} = -\int \frac{dx}{\sqrt{1-x^2}} \frac{x}{(1+\delta^2-\delta^2x^2)^{3/2}} \\ &= -\frac{1}{2}\int \frac{dt}{\sqrt{1-t}} \frac{1}{(1+\delta^2-\delta^2t)^{3/2}} = \frac{\sqrt{1-t}}{\sqrt{1+\delta^2-\delta^2t}} = \frac{\sqrt{1-x^2}}{\sqrt{1+\delta^2-\delta^2x^2}} = \frac{\sin\gamma}{\sqrt{1+\delta^2\sin^2\gamma}}. \end{aligned} \tag{A4}$$

The derivative of the arctan factor is

$$\frac{d}{d\gamma}\arctan\frac{\delta\cos\gamma+\beta}{\sqrt{1+\delta^2\sin^2\gamma}} = -\delta\frac{\sin\gamma(1+\delta^2+\delta\beta\cos\gamma)}{\sqrt{1+\delta^2\sin^2\gamma}(1+\delta^2+\beta^2+2\beta\delta\cos\gamma)}. \tag{A5}$$

The partial integration results in

$$
\begin{aligned}
F_{\beta,\delta}(\gamma) &= \frac{\sin\gamma}{\sqrt{1+\delta^2\sin^2\gamma}}\arctan\frac{\delta\cos\gamma+\beta}{\sqrt{1+\delta^2\sin^2\gamma}} + \int d\gamma \frac{\sin\gamma}{\sqrt{1+\delta^2\sin^2\gamma}}\delta\frac{\sin\gamma(1+\delta^2+\delta\beta\cos\gamma)}{\sqrt{1+\delta^2\sin^2\gamma}(1+\delta^2+\beta^2+2\beta\delta\cos\gamma)} \\
&= \frac{\sin\gamma}{\sqrt{1+\delta^2\sin^2\gamma}}\arctan\frac{\delta\cos\gamma+\beta}{\sqrt{1+\delta^2\sin^2\gamma}} + \delta\int d\gamma\frac{\sin^2\gamma(1+\delta^2+\delta\beta\cos\gamma)}{(1+\delta^2\sin^2\gamma)(1+\delta^2+\beta^2+2\beta\delta\cos\gamma)}. \qquad \text{(A6)}
\end{aligned}
$$

In the integral kernel, $\sin^2\gamma = 1-\cos^2\gamma$ is substituted at both places. The fraction is decomposed in partial fractions of the main variable $\cos\gamma$; the three dots represent the pre-integrated arctan term:

$$
\begin{aligned}
&= \ldots + \int d\gamma[\frac{1}{2\delta} + \frac{1+2(\delta^2+\beta^2)+(\delta^2-\beta^2)^2}{2\delta(1+\delta^2-\beta^2)}\frac{1}{1+\delta^2+\beta^2+2\delta\beta\cos\gamma} - \frac{1}{\delta(1+\delta^2-\beta^2)}\frac{1+\delta^2-\delta\beta\cos\gamma}{1+\delta^2-\delta^2\cos^2\gamma}] \\
&= \ldots + \int d\gamma[\frac{1}{2\delta} + \frac{[1+(\delta+\beta)^2][1+(\delta-\beta)^2]}{2\delta(1+\delta^2-\beta^2)}\frac{1}{1+\delta^2+\beta^2+2\delta\beta\cos\gamma} - \frac{1}{\delta(1+\delta^2-\beta^2)}\frac{1+\delta^2-\delta\beta\cos\gamma}{1+\delta^2\sin^2\gamma}]. \qquad \text{(A7)}
\end{aligned}
$$

The integral over the constant $1/(2\delta)$ is trivial. The second integral is solved by [12, 2.553.3]

$$
\begin{aligned}
&\int d\gamma\frac{1}{1+\delta^2+\beta^2+2\delta\beta\cos\gamma} \\
&= \frac{2}{\sqrt{[1+(\delta+\beta)^2][1+(\delta-\beta)^2]}}\arctan\left[\sqrt{\frac{1+(\delta-\beta)^2}{1+(\delta+\beta)^2}}\tan(\gamma/2)\right]. \qquad \text{(A8)}
\end{aligned}
$$

In the third integral the component that does not depend on γ in the numerator is replaced by [12, 2.562.1]

$$
\int d\gamma\frac{1}{1+\delta^2\sin^2\gamma} = \begin{cases} \frac{1}{\sqrt{1+\delta^2}}\arctan(\sqrt{1+\delta^2}\tan\gamma); & 0\le\gamma<\pi/2; \\ \frac{1}{\sqrt{1+\delta^2}}[\pi+\arctan(\sqrt{1+\delta^2}\tan\gamma)]; & \pi/2<\gamma<\pi. \end{cases} \qquad \text{(A9)}
$$

The component that is linear in $\cos\gamma$ in the numerator is easily solved via (A2):

$$
\int d\gamma\frac{\cos\gamma}{1+\delta^2\sin^2\gamma} = \frac{1}{\delta}\arctan[\delta\sin\gamma]. \qquad \text{(A10)}
$$

The previous three equations are inserted in (A7) which solves (A6):

$$
\begin{aligned}
F_{\beta,\delta}(\gamma) = \; & \frac{\sin\gamma}{\sqrt{1+\delta^2\sin^2\gamma}}\arctan\frac{\delta\cos\gamma+\beta}{\sqrt{1+\delta^2\sin^2\gamma}} + \frac{\gamma}{2\delta} \\
& + \frac{1}{\delta(1+\delta^2-\beta^2)}\Big\{\sqrt{[1+(\beta-\delta)^2][1+(\beta+\delta)^2]}\arctan\left[\sqrt{\frac{1+(\beta-\delta)^2}{1+(\beta+\delta)^2}}\tan(\gamma/2)\right] \\
& - \sqrt{1+\delta^2}\,\mathrm{Arctan}[\sqrt{1+\delta^2}\tan\gamma] + \beta\arctan(\delta\sin\gamma)\Big\}. \qquad \text{(A11)}
\end{aligned}
$$

The Arctan with a capital A denotes the inverse tangent function as defined in (A9) that switches branches where $\tan\gamma$ switches sign. The term $\gamma/(2\delta)$ on the right hand side can effectively be removed because it cancels as (26) deals only in differences between F-values.

Appendix B: Angular Integral over Rational Function

The integral (25) is started by substituting $\sin^2\gamma = 1-\cos^2\gamma$ in the denominator and expanding the rational function of $\cos\gamma$ in partial fractions:

$$
\begin{aligned}
I_{\beta,\delta}(\gamma) &\equiv \int d\gamma\frac{\delta\cos\gamma+\beta}{(1+\delta^2\sin^2\gamma)(1+\delta^2+2\delta\beta\cos\gamma+\beta^2)} \\
&= \int d\gamma[\frac{2\beta}{1+\delta^2-\beta^2}\frac{1}{1+\delta^2+\beta^2+2\delta\beta\cos\gamma} - \frac{1}{1+\delta^2-\beta^2}\frac{\beta-\delta\cos\gamma}{1+\delta^2\sin^2\gamma}]. \qquad \text{(B1)}
\end{aligned}
$$

The three representations (A8)–(A10) are plugged into this expression to yield

$$I_{\beta,\delta}(\gamma) = \frac{1}{1+\delta^2-\beta^2}\Big\{ \quad \frac{4\beta}{\sqrt{[1+(\delta+\beta)^2][1+(\delta-\beta)^2]}} \arctan\left[\sqrt{\frac{1+(\delta-\beta)^2}{1+(\delta+\beta)^2}}\tan(\gamma/2)\right] \\ -\frac{\beta}{\sqrt{1+\delta^2}}\,\mathrm{Arctan}[\sqrt{1+\delta^2}\tan\gamma] + \arctan[\delta\sin\gamma]\Big\}. \tag{B2}$$

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