# Recent results on modular convergence theorems, rates of approximation and Korovkin theorems for filter convergence

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**Abstract**

We give a survey on recent results about the problem of approximating a real-valued function $f$ by means of suitable families of sampling type operators, which include both discrete and integral ones, and about the order of approximation, and abstract Korovkin-type theorems with respect to different types of test functions, in the context of filter convergence. We give a unified approach, by means of which it is possible to consider several kinds of classical operators, for instance Urysohn integral operators, in particular Mellin-type convolution integrals, and generalized sampling series. We obtain proper extensions of classical results.

We consider the problem of approximating a real-valued function $f$ by a net of operators of the form

$$(T_w f)(s) = \int_{H_w} K_w(s, t, f(t))\, d\mu_w(t), \quad w \in W, \quad s \in G,$$

where $W \subset \mathbb{R}$ is a suitable directed set, $(H_w)_w$ is a net of nonempty closed subsets of $G$ with $G = \overline{\bigcup_{w \in W} H_w}$, $\mu_w$ is a regular measure defined on the Borel $\sigma$-algebra $\mathcal{B}_w$ of $H_w$ and $f$ belongs to the domain of the operators $T_w$ for each $w \in W$.

These kinds of operators, give a unifying approach for the treatment of both integral and discrete operators, by specifying the subspaces $H_w$ and the measures $\mu_w$. So, they represent a powerful tool for a general study of the approximation properties in various functional spaces, and include several classical discrete operators and integral operators of Urysohn type. We study the approximation properties of $T_w$ in modular spaces, a class of function spaces which includes $L^p$, Orlicz and Musielak-Orlicz spaces, and we determine a subspace $Y$ of the modular space involved, for which the modular convergence holds for every $f \in Y$. We consider an abstract version of the modular convergence, associated with filter convergence.

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We extend to the filter setting some results about convergence theorems and rates of approximations for generalized sampling type operators, associated to real-valued functions (signals) defined on the real line or on a multidimensional Euclidean space. This corresponds to the choice $G = \mathbb{R}$ or $\mathbb{R}^d$ endowed with the Lebesgue measure $\mu$, $H_w = \dfrac{1}{w}\mathbb{Z}$ or $H_w = \dfrac{1}{w}\mathbb{Z}^d$ with the counting measure $\mu_w$. These operators represent fundamental tools in signal processing, images and video reconstruction

Another kind of theorem of Approximation Theory, studied in this framework, is the Korovkin theorem. In the classical Korovkin theorem the uniform convergence in $C([a,b])$, the space of all continuous real-valued functions defined on the compact interval $[a,b]$, is proved for a sequence of positive linear operators, assuming the convergence only on the test functions 1, $x$, $x^2$. There are also trigonometric versions of the Korovkin theorem, using the test functions 1, $\sin x$, $\cos x$. There are several possibilities of using test functions. We present some Korovkin-type theorems in the setting of modular spaces and filter convergence, dealing with different classes of test functions. Also the case of not necessarily positive operators is treated.

Let $G = (G, +)$ be a locally compact abelian Hausdorff topological group with neutral element $\theta$, and suppose that $\mathcal{U} \subset \mathcal{P}(G \times G)$ is a uniform structure which generates the topology of $G$. For every $U \in \mathcal{U}$ and $s \in G$, we set $U_s = \{t \in G : (s,t) \in U\}$. The family $\{U_s : U \in \mathcal{U}\}$ represents the class of the neighborhoods of $s \in G$ in the uniform topology. Let $\mathcal{B}$ be the $\sigma$-algebra of all Borel subsets of $G$, $\mu : \mathcal{B} \to \mathbb{R}$ be a positive $\sigma$-finite regular measure, and $\mathcal{U}$ be a base of $\mu$-measurable symmetric neighborhoods of $\theta$.

A nonempty class $\mathcal{F}$ of subsets of an infinite set $W$ is a *filter* of $W$ iff $\emptyset \notin \mathcal{F}$, $A \cap B \in \mathcal{F}$ whenever $A$, $B \in \mathcal{F}$ and for each $A \in \mathcal{F}$ and $B \supset A$ we get $B \in \mathcal{F}$. If $W = (W, \geq)$ is a directed set, then a filter $\mathcal{F}$ of $W$ is *free* iff it contains the sets of type $M_w := \{v \in W : v \geq w\}$ for every $w \in W$.

Let $\mathcal{F}$ be a free filter of $W$. A net $x_w$, $w \in W$, in $\mathbb{R}$ is said to be $\mathcal{F}$*-bounded* iff there exists an $M > 0$ such that $\{w \in W : |x_w| \leq M\} \in \mathcal{F}$.

A net $x_w$, $w \in W$, of elements of $G$ is said to be $\mathcal{F}$*-convergent* to $x \in G$ (and we write $x = (\mathcal{F}) \lim_{w \in W} x_w$) iff $\{w \in W : (x_w, x) \in U\} \in \mathcal{F}$ for every $U \in \mathcal{U}$. If $\underline{\mathbf{x}} = x_w$, $w \in W$, is a net in $\mathbb{R}$ and $A_{\underline{\mathbf{x}}} = \{a \in \mathbb{R} : \{w \in W : x_w \geq a\} \notin \mathcal{F}\}$, $B_{\underline{\mathbf{x}}} = \{b \in \mathbb{R} : \{w \in W : x_w \leq b\} \notin \mathcal{F}\}$, then the $\mathcal{F}$*-limit superior* and the $\mathcal{F}$*-limit inferior* of $(x_w)_w$ are defined by

$$(\mathcal{F}) \limsup_w x_w = \begin{cases} \sup B_{\underline{\mathbf{x}}}, & \text{if } B_{\underline{\mathbf{x}}} \neq \emptyset, \\ -\infty, & \text{if } B_{\underline{\mathbf{x}}} = \emptyset, \end{cases} \qquad (\mathcal{F}) \liminf_w x_w = \begin{cases} \inf A_{\underline{\mathbf{x}}}, & \text{if } A_{\underline{\mathbf{x}}} \neq \emptyset, \\ +\infty, & \text{if } A_{\underline{\mathbf{x}}} = \emptyset, \end{cases}$$

respectively. A net $f_w : G \to \mathbb{R}$, $w \in W$, is said to $\mathcal{F}$*-converge uniformly* (resp. *in measure*) to $f$ on $G$ iff $(\mathcal{F}) \lim_w [\sup_{t \in G} |f_w(t) - f(t)|] = 0$ (resp. $(\mathcal{F}) \lim_w \mu(\{t \in G : |f_w(t) - f(t)| > \varepsilon\}) = 0$ for each $\varepsilon > 0$).

A free filter $\mathcal{F}$ of $\mathbb{N}$ is said to be a *Lebesgue filter* iff for every measure space $(G, \mathcal{B}, \mu)$, with $\mu$ finite and positive, we have $(\mathcal{F}) \lim_n \int_G f_n \, d\mu = 0$ whenever $f_n : G \to \mathbb{R}$, $n \in \mathbb{N}$, is a sequence, pointwise $\mathcal{F}$-convergent to 0 and such that there is a non-negative function $h \in L^1(G, \mathcal{B}, \mu)$, with $|f_n(t)| \leq h(t)$ for all $t \in G$. Note that the filter of all subsets of $\mathbb{N}$ with asymptotic density one is a Lebesgue filter.

Given two functions $f_1$, $f_2 : W \to \mathbb{R}$ and a filter $\mathcal{F}$ of $W$, we say that $f_1(w) = O(f_2(w))$ *with* *respect to* $\mathcal{F}$ iff there exists a $D>0$ with $\{w \in W : |f_1(w)| \leq D\,|f_2(w)|\} \in \mathcal{F}$.

From now on we suppose that $\mathcal{F}$ is a free filter of $W$. Some examples used frequently in the literature are $W=(\mathbb{N},\geq)$, or $W \subset [a,w_0[\subset \mathbb{R}$ endowed with the usual order, where $w_0 \in \mathbb{R}\cup\{+\infty\}$ is a limit point of $W$. We also consider the above set $G$ endowed with the filter $\mathcal{H}_\theta$ of all neighborhoods of its neutral element $\theta$.

A net $f_w : G \to \mathbb{R}$, $w \in W$, is said to be $\mathcal{F}$-*exhaustive at* $s \in G$ iff for every $\varepsilon>0$ there exist a neighborhood $U_s$ of $s$ and $A \in \mathcal{F}$ with $|f_w(z)-f_w(s)| \leq \varepsilon$, whenever $w \in A$ and $z \in U_s$.

For each $w \in W$, let $H_w$ be a nonempty closed set of $\mathcal{B}$, with $\overline{\bigcup_{w\in W} H_w} = G$, and $\mu_w$ be a regular measure defined on the Borel $\sigma$-algebra $\mathcal{B}_w$ generated by the family $\{A \cap H_w : A$ is an open subset of $G\}$. For every $w \in W$ let $\mathcal{L}_w$ be the set of all measurable non-negative functions $L_w : G\times G \to \mathbb{R}$ such that the sections $L(\cdot,t)$ and $L(s,\cdot)$ belong to $L^1(G)$, $L^1(H_w)$ for all $s$, $t \in G$, where $L^1(G)$ and $L^1(H_w)$ are the spaces of Lebesgue integrable functions with respect to $\mu$ and $\mu_w$ respectively, and suppose that $L_w$ is $\mathcal{F}$-*homogeneous uniformly with respect to* $w \in W$, namely there is a set $F^* \in \mathcal{F}$ with $L_w(\sigma+s,u+s)=L_w(\sigma,u)$ for every $\sigma,s,u \in G$ and $w \in F^*$.

Let $\mathbb{R}_0^+$ be the set of all non-negative real numbers and $\Psi$ be the class of all functions $\psi : G\times\mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that $\psi(t,\cdot)$ is continuous, nondecreasing, $\psi(t,0)=0$ and $\psi(t,u)>0$, for every $t\in G$ and $u>0$. We consider a family $(\psi_w)_w \subset \Psi$, with the property that there exist two constants $E_1$, $E_2 \geq 1$ and measurable functions $\phi_w : G\times G \to \mathbb{R}_0^+$, $w \in W$, with $\psi_w(t,u) \leq E_1\psi_w(t-s,E_2\,u)+\phi_w(t,s-t)$ for all $u \in \mathbb{R}_0^+$, $s$, $t \in G, w \in F^*$. Let $\widetilde{\Psi}$ be the class of all functions $\psi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that $\psi$ is continuous, nondecreasing, $\psi(0)=0$ and $\psi(u)>0$ for all $u>0$. Let $\Gamma=(\psi_w)_w \subset \widetilde{\Psi}$ be a net, $\mathcal{F}$-exhaustive at 0 and such that for every $u>0$ the net $(\psi_w(u))_w$ is $\mathcal{F}$-bounded.

Let $\mathcal{K}$ (resp. $\widetilde{\mathcal{K}_\Gamma}$) be the class of the families of functions $K_w : G\times H_w\times\mathbb{R} \to \mathbb{R}$, $w\in W$, such that $K_w(\cdot,\cdot,u)$ is measurable on $G\times H_w$ for each $w \in W$ and $u \in \mathbb{R}$, $K_w(s,t,0)=0$ for every $w \in W$, $s \in G$ and $t \in H_w$, and for each $w \in W$ there are $L_w \in \mathcal{L}_w$ and $\psi_w \in \Psi$ (resp. $\psi_w \in \widetilde{\Psi}$), with

$$\begin{aligned} |K_w(s,t,u)-K_w(s,t,v)| &\leq L_w(s,t)\psi_w(t,|u-v|) \\ \text{(resp. } |K_w(s,t,u)-K_w(s,t,v)| &\leq L_w(s,t)\psi_w(|u-v|)) \end{aligned} \tag{1}$$

for all $s \in G$, $t \in H_w$, $u$, $v \in \mathbb{R}$. Let $\mathbb{K}=(K_w)_w \in \mathcal{K}$ and $(T_w)_{w\in W}$ be a net of operators defined by

$$(T_w f)(s) = \int_{H_w} K_w(s,t,f(t))\,d\mu_w(t), \quad s \in G, \tag{2}$$

where for any $w \in W$, $T_w$ is defined in a suitable subset of $L^0(G)$. For $s \in G$, $w \in W$ and $L_w \in \mathcal{L}_w$, set $l_w(s) := L_w(\theta,s)$, suppose that $l_w$ is a $\mu$-measurable function with $l_w(\cdot-s) \in L^1(H_w)$ for every $s \in G$, that there are $D^*>0$ and $\overline{F} \in \mathcal{F}$ with $\int_{H_w} l_w(t-s)\,d\mu_w(t) \leq D^*$ for each $s \in G$ and $w \in \overline{F}$.

Let $\mathbb{K} \in \mathcal{K}_\Gamma$. We say that $\mathbb{K}$ is $\mathcal{F}$-*singular* iff

- there is a $D_1>0$ with $\Lambda = \Big\{w \in W : \int_{H_w} L_w(s,t)\,d\mu_w(t) \leq D_1$ for all $s \in G\Big\} \in \mathcal{F}$;

- for every $s \in G$ and for each neighborhood $U_s \subset G$ we get

$$(\mathcal{F}) \lim_w \int_{H_w \setminus U_s} L_w(s,t)\, d\mu_w(t) = 0;$$

- for every $s \in G$ and $u \in \mathbb{R}$ we have $(\mathcal{F}) \lim_w \int_{H_w} K_w(s,t,u)\, d\mu_w(t) = u.$

Analogously as above, it is possible to formulate the concepts of strong filter singularity and singularity with respect to filter convergence in measure or uniform.

Let $\Xi$ be the class of all functions $\xi : W \to \mathbb{R}_0^+$ such that $(\mathcal{F}) \lim_{w \in W} \xi(w) = 0$, let $\xi \in \Xi$, $\mathbb{K} \in \mathcal{K}$, $l_w$ be as before and $\pi_w : G \to \mathbb{R}_0^+$, $w \in W$, be $\mu$-measurable functions. We say that $\mathbb{K}$ is $(\mathcal{F}, \xi)$-*singular with respect to* $l_w$ *and* $\pi_w$ iff:

(1) $\displaystyle\int_{G \setminus U} l_w(s)\, (\pi_w(s) + 1) d\mu(s) = O(\xi(w))$ with respect to $\mathcal{F}$ for each $U \in \mathcal{U}$;

(2) if $r^w(s) := \displaystyle\sup_{u \in \mathbb{R} \setminus \{0\}} \left| \frac{1}{u} \int_{H_w} K_w(s,t,u)\, d\mu_w(t) - 1 \right|$, $s \in G$, then $\displaystyle\sup_{s \in G} r^w(s) = O(\xi(w))$ with respect to $\mathcal{F}$;

(3) there exist $F^* \in \mathcal{F}$ and $D' > 0$ such that for every $s \in G$ and $w \in F^*$ we get $r^w(s) \le D'$ and $\displaystyle\int_G l_w(s)\, d\mu(s) \le D'$.

A family $m_w : G \times \mathcal{B}_w \to \mathbb{R}_0^+$, $w \in W$, is said to be $\mathcal{F}$-*regular* iff it is of the type

$$m_w(s, A) = \int_A \gamma_w(s,t)\, d\mu_w(t), \quad s \in G,\, w \in W,\, A \in \mathcal{B}_w,$$

where $\gamma_w : G \times G \to \mathbb{R}$ is measurable and the following properties are fulfilled:

(a) there is a constant $D_1 > 0$ such that, if $b_w^*(s) := m_w(s, H_w)$ for any $w \in W$ and $s \in G$, then

$$\Big\{ w \in W : 0 < b_w^*(s) \le D_1 \text{ for all } s \in G \Big\} \in \mathcal{F};$$

(b) putting $\omega_w^t(A) := \displaystyle\int_A \gamma_w(t, s+t)\, d\mu(s)$, $w \in W$, $t \in H_w$, $A \in \mathcal{B}_w$ there is a family of measures $\omega_w$, $w \in W$, such that $\{w \in W \colon \omega_w^t(A) \le \omega_w(A)$ for all $t \in H_w$ and $A \in \mathcal{B}(G)\} \in \mathcal{F}$.

Note that the family $\gamma_w(s,t) = l_w(t-s)$, $w \in W$, $s \in G$, $t \in H_w$, generates a family $(m_w)_w$ of $\mathcal{F}$-regular measures.

Let $\Phi$ (resp. $\widetilde{\Phi}$) be the set of all continuous non-decreasing (resp. convex) functions $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ with $\varphi(0) = 0$, $\varphi(u) > 0$ for any $u > 0$ and $\displaystyle\lim_{u \to +\infty} \varphi(u) = +\infty$. For each $\varphi \in \Phi$ (resp. $\widetilde{\Phi}$), set

$$\rho^\varphi(f) = \int_G \varphi(|f(s)|)\, d\mu(s), \quad f \in L^0(G). \tag{3}$$

The functional $\rho^\varphi$ is a (resp. convex) modular on $L^0(G)$, satisfying the given properties of the modulars. The subspace $L^\varphi(G) = \{f \in L^0(G) : \rho^\varphi(\lambda f) < +\infty$ for some $\lambda > 0\}$ is the Orlicz space generated by $\varphi$.

A net $(f_w)_w$ of functions in $L^\rho(G)$ is $\mathcal{F}$-*modularly convergent* (resp. $\mathcal{F}$-*strongly convergent*) to $f \in L^\rho(G)$ iff there is a $\lambda > 0$ with $(\mathcal{F}) \lim_w \rho(\lambda(f_w - f)) = 0$ for some (resp. for each) $\lambda > 0$.

For $w \in W$, let $\rho_w$, $\eta_w$ be modulars on $L^0(H_w, \mathcal{B}_w, \mu_w) = L^0(H_w)$. We denote by $L^{\rho_w}(H_w)$, $L^{\eta_w}(H_w)$ the spaces of all functions $f \in L^0(G)$, whose restriction $f_{|H_w}$ belongs to the modular spaces generated by $\rho_w$, $\eta_w$ respectively.

An $\mathcal{F}$-regular family $(m_w)_w$ is *$\mathcal{F}$-compatible* with the pair $(\rho, \rho_w)$ with respect to a net $(b_w)_w$ in $\mathbb{R}$ iff there are two positive real numbers $N$, $Q$ and a set $F_1 \in \mathcal{F}$ with

$$\rho\Big(\int_{H_w} g(t,\cdot)\, dm_w^{(\cdot)}(t)\Big) \leq Q \int_G \rho_w(N\, g(\cdot, s+\cdot))\, d\omega_w(s) + b_w \tag{4}$$

for every measurable function $g : G \times G \to \mathbb{R}_0^+$ and for each $w \in F_1$.

Let $\Gamma = (\psi_w)_w \subset \Psi$ (resp. $\widetilde{\Psi}$). The triple $(\rho_w, \psi_w, \eta_w)$, $w \in W$, is said to be *$\mathcal{F}$-properly directed* with respect to a net $(c_w)_w$ in $\mathbb{R}$ with $(\mathcal{F}) \lim\limits_w c_w = 0$, iff for every $\lambda \in (0,1)$ there are $C_\lambda \in (0,1)$ and $F_2 \in \mathcal{F}$ with $\rho_w(C_\lambda \psi_w(g(\cdot))) \leq \eta_w(\lambda\, g(\cdot)) + c_w$ (resp. $\rho_w(C_\lambda \psi_w(s, g(\cdot))) \leq \eta_w(\lambda\, g(\cdot)) + c_w$) whenever $w \in W$, $s \in G$, $0 \leq g \in L^0(G)$.

Let $\mathcal{T}$ be the class of all measurable functions $\tau : G \to \mathbb{R}_0^+$, continuous at $\theta$, with $\tau(\theta) = 0$ and $\tau(t) > 0$ for all $t \neq \theta$. For a fixed $\tau \in \mathcal{T}$, let $Lip(\tau)$ be the class of all functions $f \in L^0(G)$ such that there are $\lambda > 0$ and $\widetilde{F} \in \mathcal{F}$ with $\sup\limits_{w \in \widetilde{F}} [\eta_w(\lambda\, |f(\cdot) - f(\cdot + t)|)] = O(\tau(t))$ with respect to the filter $\mathcal{H}_\theta$ of all neighborhoods of $\theta$.

A family of modulars $\eta_w$, $w \in W$, is *$\mathcal{F}$-subbounded* iff there are $C \geq 1$, $\pi_w : G \to \mathbb{R}_0^+$, $\widetilde{F} \in \mathcal{F}$ and a non-trivial linear subspace $Y_\eta$ of $L^0(G)$, with

$$\eta_w(f(s+\cdot)) \leq \eta_w(C\, f) + \pi_w(s) \quad \text{for all } f \in Y_\eta,\ s \in G \text{ and } w \in \widetilde{F}. \tag{5}$$

We say that $f \in L^{\eta_w}(H_w)$ *$\mathcal{F}$-uniformly with respect to* $w \in W$ iff there are $R^* > 0$ and $\nu > 0$ with $\{w \in W : \eta_w(\nu\, f) \leq R^*\} \in \mathcal{F}$. Let now $\phi_w$, $w \in W$, $E_1$, $E_2$ be as above and $\tau \in \mathcal{T}$. We say that $(\phi_w)_w$ *satisfies property* $(*)$ iff there exist $E_3 > 0$, $\lambda' > 0$ and $\underline{F} \in \mathcal{F}$ with

$$\rho_w(\lambda'\, \phi_w(\cdot, s)) \leq E_3 \text{ for each } s \in G \text{ and } \sup_{w \in \underline{F}} \rho_w(\lambda' \phi_w(\cdot, s)) = O(\tau(s)) \tag{6}$$

with respect to the filter $\mathcal{H}_\theta$ of all neighborhoods of $\theta$.

The family $(L_w(\cdot, t))_{w \in W, t \in H_w}$ is said to fulfil *property* $\rho$-$(*)$ (with respect to the modular $\rho$) iff for every $\varepsilon$, $\lambda > 0$ and for each compact set $C \subset G$ there exists a compact set $B \subset G$ such that

$$\Lambda_* := \Big\{ w \in W : \rho\Big(\lambda \int_{H_w \cap C} L_w(\cdot, t)\, d\mu_w(t) \chi_{G \setminus B}(\cdot)\Big) \leq \varepsilon \Big\} \in \mathcal{F}.$$

Let $\eta$ be a modular on $L^0(G)$, $(\eta_w)_w$ be a net of modulars on $L^0(H_w)$, and let us denote by $Y_\eta$ the set of all functions $f \in L^\eta(G)$ with the property that there are a $P > 0$ and a bounded net $(\gamma_w)_w$ of positive real numbers with $(\mathcal{F}) \limsup\limits_w \gamma_w\, \eta_w(\lambda\, f) \leq P\, \eta(\lambda\, f)$ for every $\lambda > 0$. We assume that $Y_\eta$ contains a subspace $\mathcal{G} \subset C_c(G)$, and denote by $\overline{\mathcal{G}}_\eta$ the modular sequential closure of $\mathcal{G}$ in the space $L^\eta(G)$. This assumption allows us to consider several types of abstract integral operators, in which the $H_w$'s are *proper* subspaces of $G$: this is the case of the discrete operators, for example sampling, Bernstein, Szász-Mirak'jan and Baskakov operators. Note that even the nonlinear Urysohn operators can be viewed as particular cases of these kinds of operators, setting $H_w = G$ for each $w \in W$. In this case we have $Y_\eta = L^\eta(G)$. The following result holds.

**Theorem 0.1** *(a) Let $\mathcal{F}$ be a free filter of $\mathbb{N}$, $f \in L^\infty(G, \mathcal{B}, \mu)$ and $\mathbb{K} \in \mathcal{K}_\Gamma$ be $\mathcal{F}$-singular. Then for every continuity point $s \in G$ of $f$ we get $(\mathcal{F}) \lim\limits_n T_n f(s) = f(s)$.*

*(b) If $\mathbb{K}$ is strongly $\mathcal{F}$-singular, then for every $s \in G$ and $f \in C(G)$ the sequence $(T_n f)_n$ is $\mathcal{F}$-exhaustive at $s$.*

*(c) If $f \in \mathbb{R}^G$ is uniformly continuous and bounded on $G$ and $\mathbb{K} \in \mathcal{K}_\Gamma$ is $\mathcal{F}$-singular in measure (resp. $\mathcal{F}$-uniformly singular), then $(T_n f)_n$ $\mathcal{F}$-converges in measure (resp. uniformly) to $f$.*

A modular $\rho$ is *absolutely continuous* iff there is an $a > 0$ such that, for all $f \in L^0(G)$ with $\rho(f) < +\infty$, we get: for every $\varepsilon > 0$ there is a set $A \in \mathcal{B}$ with $\mu(A) < +\infty$ and $\rho(af\chi_{G\setminus A}) \leq \varepsilon$; and moreover for every $\varepsilon > 0$ there is $\delta > 0$ with $\rho(af\,\chi_B) \leq \varepsilon$ whenever $B \in \mathcal{B}$ with $\mu(B) < \delta$.

Given a modular $\rho$ and a free filter $\mathcal{F}$, we say that a net $f_w : G \to \mathbb{R}$, $w \in W$, is *$\rho$-$\mathcal{F}$-equi-absolutely continuous* iff there is an $a > 0$ satisfying the following conditions:

- for every $\varepsilon > 0$ there are $A \in \mathcal{B}$ with $\mu(A) < +\infty$ and $\Lambda_0 \in \mathcal{F}$ with $\rho(af_w\chi_{G\setminus A}) \leq \varepsilon$ whenever $w \in \Lambda_0$;

- for every $\varepsilon > 0$ there are $\delta > 0$ and $\Lambda \in \mathcal{F}$ with $\rho(af_w\chi_B) \leq \varepsilon$ for every $w \in \Lambda$ and whenever $B \in \mathcal{B}$ with $\mu(B) < \delta$.

We state a modular version of the Vitali theorem.

**Theorem 0.2** *(a) Let $\mathcal{F}$ be any free filter of $\mathbb{N}$, and $\rho$ be a monotone and finite modular on $L^0(G)$. Let $(f_n)_n$ be a sequence of functions in $L^0(G)$, $\rho$-$\mathcal{F}$-equi-absolutely continuous. Moreover, assume that either $(f_n)_n$ $\mathcal{F}$-converges in measure to $0$ or that for every $A \in \mathcal{B}$ with $\mu(A) < +\infty$ and $\varepsilon > 0$ there is $A' \in \mathcal{B}$, $A' \subset A$, with $\mu(A') < \varepsilon$ and $(\mathcal{F}) \lim_n [\sup_{t \in A\setminus A'} |f_n(t)|] = 0$.*

*Then there exists a positive real number $a$ with $(\mathcal{F}) \lim\limits_n \rho(a\, f_n) = 0$.*

*(b) Let $\mathcal{F}$, $\rho$, $(f_n)_n$ be as in (a). Suppose that $(f_n)_n$ is $\mathcal{F}$-exhaustive at every $s \in G$ and $(\mathcal{F}) \lim\limits_n f_n(t) = 0$ for every $t \in G$. Then there is an $a > 0$ with $(\mathcal{F}) \lim\limits_n \rho(a\, f_n) = 0$.*

The next result is a sufficient condition for $\rho$-$\mathcal{F}$-equi-absolute continuity of the sequence $(T_n f)_n$.

**Theorem 0.3** *Let $\mathcal{F}$ be any free filter of $\mathbb{N}$, $\rho$ be a finite, monotone and absolutely finite modular, and let $\mathbb{K}$ be $\mathcal{F}$-singular. Suppose that $f$ is a bounded function with compact support $C$. If the family $(L_n(\cdot, t))_{t \in G, n \in \mathbb{N}}$ satisfies property $\rho$-$(*)$, then there is an $a > 0$, independent of $f$, such that:*

*i) for every $\varepsilon > 0$ there is a compact set $B \subset G$ with $\Big\{n \in \mathbb{N} : \rho(a(T_n f)\chi_{G\setminus B}) \leq \varepsilon\Big\} \in \mathcal{F}$;*

*ii) there is a set $\Lambda \in \mathcal{F}$ (depending only on $f$) such that for every $\varepsilon > 0$ there exists a $\delta > 0$ with $\rho(a(T_n f)\chi_B) \leq \varepsilon$ whenever $n \in \Lambda$ and $B \in \mathcal{B}$ with $\mu(B) < \delta$.*

We now turn to the main theorems on modular filter convergence. Let Dom $\mathbf{T}$ be the intersection of the domains of $T_w$ as $w$ varies in $W$.

**Theorem 0.4** *Let $\mathcal{F}$ be a free filter, $\rho$, $\eta$ be two monotone, absolutely finite and absolutely continuous modulars on $L^0(G)$. Let $\rho_n, \eta_n$ be modulars on $L^0(H_n)$ and $\Gamma = (\psi_n)_n \subset \Psi$ such that the triple $(\rho_n, \psi_n, \eta_n)$ is $\mathcal{F}$-properly directed. Suppose that $\mathbb{K}$ is strongly $\mathcal{F}$-singular, or $\mathcal{F}$-singular in measure. Furthermore, assume that the sequence*

$$\nu_n^{(\cdot)}(A) := \int_A L_n(\cdot, t)\, d\mu_n(t), \quad n \in \mathbb{N}, \quad A \in \mathcal{B}_n$$

*is $\mathcal{F}$-compatible with the modulars $(\rho, \rho_n)$, and the family $(L_n(\cdot, t))_{t \in G, n \in \mathbb{N}}$ satisfies property $\rho$-$(*)$.*

*Then for every $f \in \overline{\mathcal{G}}_{\rho+\eta} \cap$ Dom $\boldsymbol{T}$ with $f - \mathcal{G} \subset Y_\eta$ there is an $a > 0$ with $(\mathcal{F}) \lim_n \rho[a(T_n f - f)] = 0$.*

When $W = \mathbb{N}$, $H_n = G$ for every $n \in \mathbb{N}$ and the modular involved generate Orlicz spaces, $\mathcal{F}$ is a Lebesgue filter of $\mathbb{N}$, we get also the following result.

**Theorem 0.5** *Let $\mathcal{F}$ be a Lebesgue filter of $\mathbb{N}$, $\varphi \in \widetilde{\Phi}$, $\eta \in \Phi$ and $\Xi = (\psi_n)_n \subset \Psi$ be such that $(\rho^\varphi, \psi_n, \rho^\eta)$ is $\mathcal{F}$-properly directed. Let $\mathbb{K} = (K_n)_n$ be $\mathcal{F}$-singular. Then for every $f \in L^{\varphi+\eta}(G) \cap$ Dom $\boldsymbol{T}$ there exists a positive real number $a$ such that $(\mathcal{F}) \lim\limits_n \rho^\varphi[a(T_n f - f)\chi_S] = 0$ whenever $S \in \mathcal{B}$ with $\mu(S) < +\infty$.*

We state our main theorem about rates of approximation with respect to filter convergence.

**Theorem 0.6** *Let $\rho$ be a quasi convex and monotone modular on $L^0(G)$, $\rho_w$, $\eta_w$, $w \in W$, be monotone modulars on $L^0(H_w)$, such that the triple $(\rho_w, \psi_w, \eta_w)$ is $\mathcal{F}$-properly directed with respect to a net $(c_w)_w$ in $\mathbb{R}$, where $c_w = O(\xi(w))$ with respect to $\mathcal{F}$.*

*Let $K_w$, $L_w$, $l_w$ satisfy the above assumptions. Let $\xi \in \Xi$ and $\tau \in \mathcal{T}$ be fixed.*

*Assume that $\mathbb{K}$ is $(\mathcal{F}, \xi)$-singular with respect to $l_w$ and $\pi_w$, $\eta_w$ is $\mathcal{F}$-subbounded, $f \in L^\rho(G) \cap$ Lip$(\tau) \cap Y_\eta$, where $Y_\eta$ is as in (5), and $f \in L^{\eta_w}(H_w)$ $\mathcal{F}$-uniformly with respect to $w \in W$.*

*Suppose that the family $\gamma_w(s,t) = l_w(t-s)$, $s \in G$, $t \in H_w$, generates a family of measures $(m_w)_w$, $\mathcal{F}$-compatible with the pair ($\rho$, $\rho_w$) with respect to a net $(b_w)_w$, with $b_w = O(\xi(w))$ with respect to $\mathcal{F}$, and let $(\phi_w)_w$ satisfy property $(*)$ as in (6).*

*Finally, assume that there is a neighborhood $U$ of $\theta$ with*

$$\int_U l_w(s)\tau(s)\, d\mu(s) = O(\xi(w)) \quad \text{with respect to } \mathcal{F}. \tag{7}$$

*Then there is a constant $c > 0$ with $\rho(c(T_w f - f)) = O(\xi(w))$ with respect to $\mathcal{F}$.*

We now turn to Korovkin-type theorems in the setting of modular filter convergence. For a sake of simplicity, assume $W = \mathbb{N}$. Let $\mathbf{T}$ be a sequence of linear operators $T_n : \mathcal{D} \to L^0(G)$, $n \in \mathbb{N}$, with $C_b(G) \subset \mathcal{D} \subset L^0(G)$. Here the set $\mathcal{D}$ is the domain of the operators $T_n$.

We say that the sequence $\mathbf{T}$, together with the modular $\rho$, satisfies the *property* $(\rho)$-$(+)$ iff there exist a subset $X_{\mathbf{T}} \subset \mathcal{D} \cap L^\rho(G)$ with $C_b(G) \subset X_{\mathbf{T}}$ and a positive real constant $N$ with $T_n f \in L^\rho(G)$ for all $f \in X_{\mathbf{T}}$ and $n \in \mathbb{N}$, and $(\mathcal{F}) \limsup\limits_n \rho(\tau(T_n f)) \leq N \rho(\tau f)$ for every $f \in X_{\mathbf{T}}$ and $\tau > 0$.

Set $e_0(t) \equiv 1$ for all $t \in G$, let $e_i$, $i = 1, \ldots, m$ and $a_i$, $i = 0, \ldots, m$ be functions in $C_b(G)$. Put $P_s(t) := \sum_{i=0}^{m} a_i(s) e_i(t)$, $s, t \in G$, and suppose that $P_s(t)$, $s, t \in G$, satisfies the following property:

(P1) $P_s(s) = 0$ for all $s \in G$ and for every neighborhood $U \in \mathcal{U}$ there is $\eta > 0$ with $P_s(t) \geq \eta$ whenever $s,\, t \in G$, $(s, t) \notin U$.

Let $G = I^m$ be endowed with the usual norm $\| \cdot \|_2$, where $I \subset \mathbb{R}$ is a connected set, $\phi : I \to \mathbb{R}$ be monotone and such that $\phi^{-1}$ is uniformly continuous on $I$. Examples of such functions are $\phi(t) = t$ or $\phi(t) = e^t$ when $I$ is a bounded interval. For every $t = (t_1, \ldots, t_m) \in G$ set $e_i(t) := \phi(t_i)$, $i = 1, \ldots, m$, and $e_{m+1}(t) := \sum_{i=1}^{m} [\phi(t_i)]^2$. For all $s = (s_1, \ldots, s_m) \in G$ put $a_0(s) := \sum_{i=1}^{m} [\phi(s_i)]^2$, $a_i(s) = -2\phi(s_i)$, $i = 1, \ldots, m$, and $a_{m+1}(s) \equiv 1$. It is not difficult to check that (P1) is fulfilled. Moreover, if $G = [0, a]$ with $0 < a < \pi/2$,

$$e_1(t) = \cos t,\, e_2(t) = \sin t,\, t \in G,\, a_0(s) \equiv 1,\, a_1(s) = -\cos s\, a_2(s) = -\sin s\, s \in G, \tag{8}$$

then it is not difficult to see that (P1) is satisfied.

We now give the following Korovkin-type theorem.

**Theorem 0.7** *Let $\rho$ be a monotone, strongly finite, absolutely continuous and $Q$-quasi semiconvex modular on $L^0(G)$, and $T_n$, $n \in \mathbb{N}$ be a sequence of positive linear operators satisfying property $(\rho)$-$(+)$. If $T_n e_i$ is $\mathcal{F}$-strongly convergent to $e_i$, $i = 0, \ldots, m$ in $L^\rho(G)$, then $T_n f$ is $\mathcal{F}$-modularly convergent to $f$ in $L^\rho(G)$ for all $f \in L^\rho(G) \cap \mathcal{D}$ with $f - C_b(G) \subset X_{\boldsymbol{T}}$, where $\mathcal{D}$ and $X_{\boldsymbol{T}}$ are as before.*

One can ask, whether it is possible, in the Korovkin theorems, to relax the positivity condition on the linear operators involved. We now give a positive answer in this direction. Let $\mathcal{F}$ be any fixed free filter of $\mathbb{N}$, $I$ be a bounded interval of $\mathbb{R}$, $C^2(I)$ (resp. $C^2_b(I)$) be the space of all functions defined on $I$, (resp. bounded and) continuous together with their first and second derivatives, $\mathcal{C}_+ := \{f \in C^2_b(I) : f \geq 0\}$, $\mathcal{C}^2_+ := \{f \in C^2_b(I) : f'' \geq 0\}$.

Let $e_i$, $i = 1, \ldots, m$ and $a_i$, $i = 0, \ldots, m$ be functions in $C^2_b(I)$, and suppose that $P_s(t)$, $s, t \in I$, satisfies (P1) and

(P2) there is a positive real constant $C_0$ with $P''_s(t) \geq C_0$ for all $s,\, t \in I$.

It is not difficult to see that (P2) is satisfied when $P_s(t) = (s - t)^2$, when $I = \left[0, \log \frac{3}{2}\right]$, and $P_s(t) = (e^s - e^t)^2$, $s,\, t \in I$, and when $I = [0, a]$ with $0 < a < 2\pi$ and $e_i$, $a_i$ as in (8).

We now give the following Korovkin-type theorem for not necessarily positive linear operators.

**Theorem 0.8** *Let $\mathcal{F}$ be any free filter of $\mathbb{N}$, $\rho$ be a monotone, strongly finite absolutely continuous and $Q$-quasi semiconvex modular on $L^0(G)$, and assume that $e_i$, $a_i$, $i = 0, \ldots, m$ and $P_s(t)$, $s,\, t \in I$, satisfy properties (P1) and (P2). Let $T_n$, $n \in \mathbb{N}$ be a sequence of linear operators, satisfying property $(\rho)$-$(+)$ with respect to $\mathcal{F}$-convergence. Suppose that $\{n \in \mathbb{N} : T_n(\mathcal{C}_+ \cap \mathcal{C}^2_+) \subset \mathcal{C}_+\} \in \mathcal{F}$. If $T_n e_i$ is $\mathcal{F}$-modularly convergent to $e_i$, $i = 0, \ldots, m$ in $L^\rho(I)$, then $T_n f$ is $\mathcal{F}$-modularly convergent to $f$ in $L^\rho(I)$, for every $f \in C^2_b(I)$.*

*If $T_n e_i$ is $\mathcal{F}$-strongly convergent to $e_i$, $i = 0, \ldots, m$ in $L^\rho(I)$, then $T_n f$ is $\mathcal{F}$-strongly convergent to $f$ in $L^\rho(I)$, for every $f \in C_b^2(I)$.*

*Furthermore, if $\rho$ is absolutely continuous and $T_n e_i$ is $\mathcal{F}$-strongly convergent to $e_i$, $i = 0, \ldots, m$ in $L^\rho(I)$, then $T_n f$ is $\mathcal{F}$-modularly convergent to $f$ in $L^\rho(I)$ for every $f \in L^\rho(I) \cap \mathcal{D}$ with $f - C_b(I) \subset X_{\boldsymbol{T}}$.*