A Proof to Beal's Conjecture

Raj C Thiagarajan, PhD

Technical and Managing Director, ATOA Scientific Technologies Pvt Ltd,

204 Regent Prime, 48 Whitefield Main Road, Bangalore 560066, India.

Raj@atoastech.com

Abstract

In this paper, we provide computational results and a proof for Beal's conjecture. We demonstrate that the common prime factor is intrinsic to this conjecture using the laws of powers. We show that the greatest common divisor is greater than 1 for the Beal's conjecture.

Keywords

Number Theory, Diophantine equations, conjecture, Beal's conjecture, proof, laws of powers, sum of perfect powers.

Introduction

All through the history of Number theory, Diophantine type higher order polynomial equation based theorem and conjecture are under constant exploration. For example, Pythagorean triples and

Fermat's Last Theorem received considerable exploration [1, 2]. This paper is related to the Beal's conjecture. This conjecture is concerned with the common prime factor for positive integers and their exponents greater than 2. If $A^x + B^y = C^z$, where A, B, C, x, y and z are positive integers and x, y and z are all greater than 2, then A, B and C must have a common prime factor [3].

Proof

The formal statement of Beal's conjecture is,

$$\forall_{\{A,B,C,x,y,z\}},$$

$$A^x + B^y = C^z, \tag{1}$$

$$(A, B, C, x, y, z) \in Z^+, (x, y, z) > 2, \ \gcd(A, B, C) > 1.$$

We performed a computational numerical search to discover a counter example. The search for the Beal's conjecture proof or counter example is related to finding the perfect power of the sum of the perfect powers and the gcd (greatest common divisor) of the bases. A set of perfect power table for a given range of integer and power was compiled. An algorithm was developed to lookup in the perfect power table for the sum of the perfect powers. Typical and selective solution results are shown in table 1. The data shown in table 1 are the output of the computational search algorithm, except the last row. The table shows that the gcd is greater than 1 and confirms the validity of Beal's conjecture for a wide range of numbers. The table result also highlights the inherent patterns of the Beal's conjecture.

Table 1.Computer Search Results of Beal's Conjecture Solution

A	x	+	B	y	=	C	z	gcd(A,B,C)
2	3	+	2	3	=	2	4	2
6	3	+	3	3	=	3	5	3
8	3	+	8	3	=	4	5	4
7	3	+	7	4	=	14	3	7
91	3	+	13	5	=	104	3	13
15	5	+	15	4	=	30	4	15
34	5	+	51	4	=	85	4	17
38	3	+	19	4	=	57	3	19
26	3	+	26	4	=	78	3	26
84	3	+	28	3	=	28	4	28
31	6	+	31	5	=	62	5	31
64	4	+	64	4	=	32	5	32
66	5	+	33	5	=	1089	3	33
54	5	+	54	5	=	972	3	54
63	4	+	63	3	=	252	3	63
260	3	+	65	3	=	65	4	65
80	12	+	80	13	=	1536000	4	80
124	12	+	124	13	=	1182106880	3	124
127	7	+	127	8	=	254	7	127
250	5	+	250	5	=	12500	3	250
1028	4	+	257	4	=	257	5	257
1705	3	+	2046	3	=	341	4	341
4104	3	+	513	3	=	513	4	513
10080	3	+	2016	3	=	1008	4	1008
2	99999	+	2	99999	=	2	100000	2

Without loss of generality, Let,

$$A = mR \qquad (2)$$

$$B = nR \qquad (3)$$

$$y = x + p \qquad (4)$$

$Where, (m, n, R) \in Z^+, (p) \in Z \; and \; y \geq x.$ It can be noted that equation 2 and 3 can be used to represent any exponent base as a distinct integer 'A' and 'B', respectively. Similarly, equation 4 can be used to represent any exponent power as a distinct integer 'y'.

Using equation 2 to 4, we can write, $A^x + B^y$, as follows,

$$= (mR)^x + (nR)^{x+p} \qquad (5)$$

$$= m^x R^x + n^x n^p R^x R^p$$

$$= (m^x + n^x n^p R^p) R^x \qquad (6)$$

$A^x + B^y = C^z$, Therefore,

$$C^z = (m^x + n^x n^p R^p) R^x \qquad (7)$$

By substituting $A^x, B^y \, and \; C^z$, we can write equation 1 as follows,

$$(mR)^x + (nR)^{x+p} = (m^x + n^x n^p R^p) R^x \qquad (8)$$

$$m^x \boldsymbol{R^x} + \; n^x n^p R^p \boldsymbol{R^x} = \; (m^x + n^x n^p R^p) \boldsymbol{R^x} \qquad (9)$$

$$gcd \; (m^x \boldsymbol{R^x}, n^x n^p R^p \boldsymbol{R^x}, (m^x + n^x n^p R^p) \boldsymbol{R^x}) = R \qquad (10)$$

Sum of any two integer powers will have base greater than 1 and hence R>1.

Therefore,

$$gcd(\ A\ ,B\ ,C)\ is$$

$$gcd\ (m^x R^x\ ,n^x n^p R^p R^x, (m^x + n^x n^p R^p) R^x) = R$$

$$\text{and}\ \ R > 1 \tag{11}$$

■

Concluding Remarks

For all integer solutions of Beal's equation, the gcd will be greater than 1. We demonstrated using the laws of powers that the gcd of $A^x + B^y = C^z$ is of multiples of R, the greatest common divisor and is greater than 1, where A, B, C, x, y and z are positive integers and x, y and z are all greater than 2. The sum of the perfect powers will have gcd greater than 1.

References

[1] Mordell, L. J, Diophantine equations, Series: Pure & Applied Mathematics, Academic Press Inc, July 1969, 312 pages.

[2] Chena, I and Siksekb, S, Perfect powers expressible as sums of two cubes, Journal of Algebra, Volume 322, Issue 3, 1 August 2009, Pages 638–656.

[3] Mauldin, R.D, A Generalization of Fermat's Last Theorem: The Beal Conjecture and Prize Problem, AMS Notices 44, No 11, December 1997, 1436–1437.