Some Considerations Concerning the Sumatory Function Associated to Generalised Smarandache Function

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Let us denote by $\overset{d}{\vee}$ the least common multiple, by $\underset{d}{\wedge}$ the greatest common devisor and $\wedge = \min, \vee = \max$. It is known that $N_0 = (N^*, \wedge, \vee)$ and $N_d = \left(N^*, \underset{d}{\wedge}, \overset{d}{\vee}\right)$ are lattices. The order on the set N^* : $n_1 \leq n_2 \Leftrightarrow n_1 \wedge n_2 = n_1$. corresponding to the first of these lattices and it is known that this is a total order. But the order $\underset{d}{\leq}$ induced on the same set by $\underset{d}{\wedge}$ and $\overset{d}{\vee}$ and defined by: $n_1 \underset{d}{\leq} n_2 \Leftrightarrow n_1 \underset{d}{\wedge} n_2 = n_1 \Leftrightarrow n_1$ divides n_2 is only a partial order.

Let $\sigma : N_0 \to N_d$ (1) a sequence of positive integers defined on the set N^*.
The sequence (1) is said to be a multiplicatively convergent to zero sequence (mcz) if:

$$\forall n \in N^*, \exists m_n \in N^*, \forall m > m_n \Rightarrow n \underset{d}{\leq} \sigma(m) \qquad (2).$$

The sequence

$$\sigma : N_d \to N_d \qquad (3)$$

is said to be a divisibility sequence (ds) if: $n \underset{d}{\leq} m \Rightarrow \sigma(n) \underset{d}{\leq} \sigma(m)$ and it is said to be a strong divisibility sequence (sds) if:

$$\sigma\left(n \underset{d}{\wedge} m\right) = \sigma(n) \underset{d}{\wedge} \sigma(m) \text{ for every } n, m \in N^* \qquad (4).$$

Let the lattices N_0 and N_d. We'll use the following notations:

(a) a sequence $\sigma_{00} : N_0 \to N_0$ is a (oo) - sequences;
(b) a sequence $\sigma_{0d} : N_0 \to N_d$ is a (od) - sequences;
(c) a sequence $\sigma_{d0} : N_d \to N_0$ is a (do) - sequences;
(d) a sequence $\sigma_{dd} : N_d \to N_d$ is a (dd) - sequences.

Then A(do) - sequence σ_{d0} the monotonicity yields:

$$(m_{d0}) \forall n_1, n_2 \in N^*, n_1 \underset{d}{\leq} n_2 \Rightarrow \sigma_{d0}(n_1) \leq \sigma_{d0}(n_2) \qquad (5)$$

and the condition of convergence to inifinity is:

$$(c_{d0}) \forall n \in N^*, \exists m_n \in N^*, \forall m \underset{d}{\leq} m_n \Rightarrow \sigma_{d0}(m) \geq n \qquad (6).$$

Analogously, for a (dd) - sequence σ_{dd} the monotonicity yields:

$$(m_{dd}) \forall n_1, n_2 \in N^*, n_1 \underset{d}{\leq} n_2 \Rightarrow \sigma_{dd}(n_1) \underset{d}{\leq} \sigma_{dd}(n_2) \qquad (7)$$

and the convergence to zero is:

$$(c_{dd}) \forall n \in N^*, \exists m_n \in N^*, \forall m \underset{d}{\geq} m_n \Rightarrow \sigma_{dd}(m_n) \underset{d}{\geq} n \,. \qquad (8)$$

To each sequence σ_{ij}, with $i, j \in \{0, d\}$, satisfying the condition (c_{ij}), one may attach a sequence S_{ij} (a generalised Smarandache function) defined by:

$$S_{ij} = \min\{m_n : m_n \text{ is given by the condition } (c_{ij})\} \qquad (9).$$

For the properties the functions S_{ij}, see [2].

It is said that for every numerical function f it can be attashed the sumatory function:

$$F(n) = \sum_{d/n} f(d) \qquad (10)$$

The function f is expressed as:

$$f(n) = \sum_{uv=n} \mu(u) F(v) \qquad (11)$$

where μ is the Mobius function ($\mu(1) = 1, \mu(n) = 0$ if n is divisible by the square of a prime number, $\mu(n) = (-1)^k$ if n the product of k different prime numbers).

If f is the a generalised Smarandache function, S_{ij} then

$$F^s_{ij}(n) = \sum_{d/n} S_{ij}(d), i,j \in \{0,d\}. \qquad (12)$$

Now let us consider $n = p_1 p_2 \dots p_k$, with $p_1 < p_2 < \dots < p_k$ primes number and $S_{ij}(p_1) \le S_{ij}(p_2) \le \dots \le S_{ij}(p_k)$, for example. If i=0, j=d, then $S_{0d}\left(n_1 \overset{d}{\vee} n_2\right) = S_{0d}(n_1) \vee S_{0d}(n_2)$ and $F^S_{0d}(n) = S_{0d}(1) + \sum_{h=1}^{k} S_{0d}(p_h) + \sum_{h,t=1, h\ne t}^{k} S_{0d}(p_h p_t) + \sum_{h,t,q=1, h\ne t\ne q}^{k} S_{0d}(p_h p_t p_q) + \dots S_{0d}(n)$. It result:

$F^S_{0d}(1) = S_{0d}(1)$;
$F^S_{0d}(p_1) = S_{0d}(1) + S_{0d}(p_1) = F^S_{0d}(1) + 2^0 S_{0d}(p_1)$;
$F^S_{0d}(p_1 p_2) = S_{0d}(1) + S_{0d}(p_1) + S_{0d}(p_2) + S_{0d}(p_1 p_2) = S_{0d}(1) + S_{0d}(p_1) + 2S_{0d}(p_2) = F^S_{0d}(p_1) + 2S_{0d}(p_2)$;
$F^S_{0d}(p_1 p_2 p_3) = F^S_{0d}(p_1 p_2) + 2^2 S_{0d}(p_3)$;
$F^S_{0d}(p_1 p_2 p_3 p_4) = F^S_{0d}(p_1 p_2 p_3) + 2^3 S_{0d}(p_4)$;
$F^S_{0d}(p_1 p_2 \dots p_k) = F^S_{0d}(p_1 p_2 \dots p_{k-1}) + 2^{k-1} S_{0d}(p_k)$.

That is $F^S_{0d}(p_1 p_2 \dots p_k) = S_{0d}(1) + \sum_{i=1}^{k} 2^{i-1} S_{0d}(p_i)$.

The equality (11) becomes:

$$S_{0d}(p_k) = S(n) = \sum_{ab=n} \mu(n) F^S_{0d}(b) =$$
$$= F^S_{0d}(n) - \sum_i F^S_{0d}\left(\tfrac{n}{p_i}\right) + \sum_{i\ne j} F^S_{0d}\left(\tfrac{n}{p_i p_j}\right) + \dots$$

with $F^S_{0d}\left(\frac{n}{p_i}\right) = F^S_{0d}(p_1 p_2 \dots p_{i-1} p_{i+1} \dots p_k) = \sum_{j=1}^{i-1} 2^{j-1} S_{0d}(p_j) + \sum_{j=i+1}^{k} 2^{j-1} S_{0d}(p_j) =$

$$= F^S_{0d}(p_1 p_2 \dots p_{i-1}) + 2^{i-1} F^S_{0d}(p_{i+1} \dots p_k).$$

Analogously,

$$F^S_{0d}\left(\frac{n}{p_i p_j}\right) = F^S_{0d}(p_1 \dots p_{i-1}) + 2^{i-1} F^S_{0d}(p_{i+1} \dots p_{j-1}) + 2^{j-1} F^S_{0d}(p_{j+1} \dots p_k) =$$
$$= \sum_{h=1}^{i-1} 2^{h-1} S_{0d}(p_h) + \sum_{h=i+1}^{j-1} 2^{h-2} S_{0d}(p_h) + \sum_{h=j+1}^{k} 2^{h-3} S_{0d}(p_h).$$

In particulary, for $n = p^a$, p prime number, it result:

$$S_{0d}(p^a) = \sum_{u+q=a} \mu(p^u) F^S_{0d}(p^q) = F^S_{0d}(p^a) - F^S_{0d}(p^{a-1}).$$

If $n = p^a q^b$ with $\max\{S_{0d}(p), \dots, S_{0d}(p^a)\} \le \min\{S_{0d}(q), \dots, S_{0d}(q^b)\}$, then

$$F^S_{0d}(p^a q^b) = F^S_{0d}(p^a) + (a+1) F^S_{0d}(q^b).$$

If i=d, j=d and if σ_{dd} is a (sds) satisfying the condition (c_{dd}), then

$$S_{dd}\left(n_1 \overset{d}{\vee} n_2\right) = S_{dd}(n_1) \overset{d}{\vee} S_{dd}(n_2) \qquad (13)$$

and $F^S_{dd}(n) = S_{dd}(1) + \sum_{h=1}^{k} S_{dd}(p_h) + \sum_{h,t=1, h\ne t}^{k} \left[S_{dd}(p_h) \overset{d}{\vee} S_{dd}(p_t) \right] +$

$$+ \sum_{h,t,q=1, h\ne t\ne q}^{k} \left[S_{dd}(p_h) \overset{d}{\vee} S_{dd}(p_t) \overset{d}{\vee} S_{dd}(p_q) \right] + \dots + S_{dd}(n) \qquad (14)$$

$$S_{dd}(p^a) + F^S_{dd}(p^a) - F^S_{dd}(p^{a-1}). \qquad (15)$$

Example: The Fibonacci sequence $(F_n)_{n\in N^*}$ definited by $F_{n+1} = F_n + F_{n-1}$, with $F_1 = F_2 = 1$ ia s (sds), so for the generalised Smarandache function S_F attached to this sequence we have: $S_F\left(n_1 \overset{d}{\vee} n_2\right) = S_F(n_1) \overset{d}{\vee} S_F(n_2)$, and the calculus of $S_p(n)$ is reduced to the calculus of $S_F(p^a)$, with p a prime number. For instance:

n	$S_F(n)$	n	$S_F(n)$	n	$S_F(n)$
1	1	7	8	13	21
2	3	8	6	14	24
3	4	9	24	15	20
4	6	10	15	16	12
5	5	11	20	17	
6	12	12	12	18	

$F_{dd}^{S}(4) = 10, F_{dd}^{S}(8) = 16, F_{dd}^{S}(16) = 28, F_{dd}^{S}(15) = 30.$

References:

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