

ON THE CALCULUS OF SMARANDACHE FUNCTION

by

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Introduction. The Smarandache function $S : \mathbb{N}^* \rightarrow \mathbb{N}^*$ is defined [5] by the condition that $S(n)$ is the smallest integer m such that $m!$ is divisible by n . So, we have $S(1) = 1, S(2^{12}) = 16$.

Considering on the set \mathbb{N}^* two lattical structures $\mathcal{N} = (\mathbb{N}^*, \wedge, \vee)$ and $\mathcal{N}_d = (\mathbb{N}^*, \wedge_d, \vee_d)$, where $\wedge = \min, \vee = \max, \wedge_d$ = the greatest common divisor, \vee_d = the smallest common multiple, it results that S has the followings properties:

$$\begin{aligned} (s_1) \quad S(n_1 \vee_d n_2) &= S(n_1) \vee S(n_2) \\ (s_2) \quad n_1 \leq_d n_2 &\implies S(n_1) \leq S(n_2) \end{aligned}$$

where \leq is the order in the lattice \mathcal{N} and \leq_d is the order in the lattice \mathcal{N}_d . It is said that

$$n_1 \leq_d n_2 \iff n_1 \text{ divides } n_2$$

From these properties we deduce that in fact on must consider

$$S : \mathcal{N}_d \rightarrow \mathcal{N}$$

Methods for the calculus of S. If

$$n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \dots \cdot p_t^{\alpha_t} \tag{1}$$

is the decomposition of n into primes, from (s_1) it results

$$S(n) = \vee S(p_i^{\alpha_i})$$

so the calculus of $S(n)$ is reduced to the calculus of $S(p^\alpha)$.

If $e_p(\alpha)$ is the exponent of the prime p in the decomposition into primes of $n!$:

$$n! = \prod_{j=1}^l p_j^{e_p(n)}$$

by Legendre's formula it is said that

$$e_p(n) = \sum_{i \geq 1} \left[\frac{n}{p^i} \right]$$

Also we have

$$e_p(n) = \frac{n - \sigma_{(p)}(n)}{p - 1} \quad (2)$$

where $[x]$ is the integer part of x and $\sigma_{(p)}(n)$ is the sum of digits of n in the numerical scale

$$(p) : 1, p, p^2, \dots, p^i \dots$$

For the calculus of $S(p^\alpha)$ we need to consider in addition a generalised numerical scale $[p]$ given by:

$$[p] : a_1(p), a_2(p), \dots, a_i(p), \dots$$

where $a_i(p) = (p^i - 1)/(p - 1)$. Then in [3] it is showed that

$$S(p^\alpha) = p(\alpha_{[p]})_{(p)} \quad (3)$$

that is the value of $S(p^\alpha)$ is obtained multiplying p by the number obtained writing the exponent α in the generalised scale $[p]$ and "reading" it in the usual scale (p) .

Let us observe that the calculus in the generalised scale $[p]$ is essentially different from the calculus in the scale (p) . That is because if we note

$$b_n(p) = p^n$$

then for the usual scale (p) it results the recurrence relation

$$b_{n+1}(p) = p \cdot b_n(p)$$

and for the generalised scale $[p]$ we have

$$a_{n+1}(p) = p \cdot a_n(p) + 1$$

For this, to add some numbers in the scale $[p]$ we do as follows:

1) We start to add from the digits of "decimals", that is from the column corresponding to $a_2(p)$.

2) If adding some digits it is obtained $pa_2(p)$, then we utilise an unit from the class of "units" (the column corresponding to $a_1(p)$) to obtain $p \cdot a_2(p) + 1 = a_3(p)$. Continuing to add, if again it is obtained $p \cdot a_2(p)$, then a new unit must be used from the class of units, etc.

Example. If

$$m_{[6]} = 442 = 4a_3(5) + 4a_2(5) + 2a_1(5), \quad n_{[6]} = 412, \quad r_{[6]} = 44$$

then

$$\begin{array}{r} m + n + r = 442 + \\ 412 \\ 44 \\ \text{dcba} \end{array}$$

To find the digits a, b, c, d we start to add from the column corresponding to $a_2(5)$:

$$4a_2(5) + a_2(5) + 4a_2(5) = 5a_2(5) + 4a_2(5)$$

Now, if we take an unit from the first column we get:

$$5a_2(5) + 4a_2(5) + 1 = a_3(5) + 4a_2(5)$$

so $b = 4$.

Continuing the addition we have:

$$4a_3(5) + 4a_3(5) + a_3(5) = 5a_3(5) + 4a_3(5)$$

and using a new unit (from the first column) it results:

$$4a_3(5) + 4a_3(5) + a_3(5) + 1 = a_4(5) + 4a_3(5)$$

so $c = 4$ and $d = 1$.

Finally, adding the remained units:

$$4a_1(5) + 2a_1(5) = 5a_1(5) + a_1(5) = 5a_1(5) + 1 = a_2(5)$$

it results that the digit $b = 4$ must be changed in $b = 5$ and $a = 0$.

So

$$m_{[5]} + n_{[5]} + r_{[5]} = 1450_{[5]} = a_4(5) + 4a_3(5) + 5a_2(5)$$

Remarque. As it is showed in [5], writing a positive integer α in the scale $[p]$ we may find the first non-zero digit on the right equals to p . Of course, that is no possible in the standard scale (p) .

Let us return now to the presentation of the formulae for the calculus of the Smarandache function. For this we express the exponent α in both the scales (p) and $[p]$:

$$\alpha_{(p)} = c_u p^u + c_{u-1} p^{u-1} + \dots + c_1 p + c_0 = \sum_{i=0}^u c_i p^i \quad (4)$$

and

$$\begin{aligned} \alpha_{[p]} &= k_v a_v(p) + k_{v-1} a_{v-1}(p) + \dots + k_1 a_1(p) = \sum_{j=1}^v k_j a_j(p) = \\ &= \sum_{j=1}^v k_j \frac{p^j - 1}{p - 1} \end{aligned}$$

It results

$$(p - 1)\alpha = \sum_{j=1}^v k_j p^j - \sum_{j=1}^v k_j \quad (5)$$

so, because $\sum_{j=1}^v k_j p^j = p(\alpha_{[p]})_{(p)}$, we get:

$$S(p^\alpha) = (p - 1)\alpha + \sigma_{[p]}(\alpha) \quad (6)$$

From (4) we deduce

$$p\alpha = \sum_{i=0}^u c_i (p^{i+1} - 1) + \sum_{i=0}^u c_i$$

and

$$\frac{p}{p-1}\alpha = \sum_{i=0}^u c_i a_{i+1}(p) + \frac{1}{p-1}\sigma_{(p)}(\alpha)$$

Consequently

$$\alpha = \frac{p-1}{p}(\alpha_{(p)})_{[p]} + \frac{1}{p}\sigma_{(p)}(\alpha) \quad (7)$$

Replacing this expression of α in (6) we get:

$$S(p^\alpha) = \frac{(p-1)^2}{p}(\alpha_{(p)})_{[p]} + \frac{p-1}{p}\sigma_{(p)}(\alpha) + \sigma_{[p]}(\alpha) \quad (8)$$

Example. To find $S(3^{89})$ we shall utilise the equality (3). For this we have:

$$\begin{aligned} (3) &: 1, 3, 9, 27, 81, \dots \\ [3] &: 1, 4, 13, 40, 121, \dots \end{aligned}$$

and $89_{[3]} = 2021$, so $S(3^{89}) = 3(2021)_{(3)} = 183$. That is 183! is divisible by 3^{89} and it is the smallest factorial with this property.

We shall use now the equality (6) to calculate the same value $S(3^{89})$. For this we observe that $\sigma_{[3]}(89) = 5$ and, so $S(3^{89}) = 2 \cdot 89 + 5 = 183$.

Using (8) we get $89_{(3)} = 10022$ and :

$$S(3^{89}) = \frac{4}{3}(10022)_{[3]} + \frac{2}{3} \cdot 5 + 5 = 183$$

It is possible to express $S(p^\alpha)$ by mins of the exponent $e_p(\alpha)$ in the following way: from (2) and (7) it results

$$e_p(\alpha) = (\alpha_{(p)})_{[p]} - \alpha \quad (9)$$

and then from (8) and (9) it results

$$S(p^\alpha) = \frac{(p-1)^2}{p}(e_p(\alpha) + \alpha) + \frac{p-1}{p}\sigma_{(p)}(\alpha) + \sigma_{[p]}(\alpha) \quad (10)$$

Remarque. From (3) and (8) we deduce a connection between the integer α written in the scale $[p]$ and readed in the scale (p) and the same integer written in the scale (p) and readed in the scale $[p]$. Namely:

$$p^2(\alpha_{[p]})_{(p)} - (p-1)^2(\alpha_{(p)})_{[p]} = p\sigma_{[p]}(\alpha) + (p-1)\sigma_{(p)}(\alpha) \quad (11)$$

The function $i_p(\alpha)$. In the followings let we note $S(p^\alpha) = S_p(\alpha)$. Then from Legendre's formula it results:

$$(p-1)\alpha < S_p(\alpha) \leq p\alpha$$

that is $S(p^\alpha) = (p-1)\alpha + z = p\alpha - y$.

From (6) it results that $z = \sigma_{[p]}(\alpha)$ and to find y let us write $S_p(\alpha)$ under the forme

$$S_p(\alpha) = p(\alpha - i_p(\alpha)) \quad (12)$$

As it is showed in [4] we have $0 \leq i_p(\alpha) \leq [\frac{\alpha-1}{p}]$.

Then it results that for each function S_p there exists a function i_p so that we have the linear combination

$$\frac{1}{p}S_p(\alpha) + i_p(\alpha) = \alpha \quad (13)$$

In [1] it is proved that

$$i_p(\alpha) = \frac{\alpha - \sigma_{[p]}(\alpha)}{p} \quad (14)$$

and so it is an evident analogy between the expression of $e_p(\alpha)$ given by the equality (2) and the expression of $i_p(\alpha)$ in (14).

In [1] it is also showed that

$$\alpha = (\alpha_{[p]})_{(p)} + [\frac{\alpha}{p}] - [\frac{\sigma_{[p]}(\alpha)}{p}] = (\alpha_{[p]})_{(p)} + \frac{\alpha - \sigma_{[p]}(\alpha)}{p}$$

and so

$$S(p^\alpha) = p(\alpha - [\frac{\alpha}{p}] + [\frac{\sigma_{[p]}(\alpha)}{p}]) \quad (15)$$

Finally, let us observe that from the definition of Smarandache function it results that

$$(S_p \circ e_p)(\alpha) = p[\frac{\alpha}{p}] = \alpha - \alpha_p$$

where α_p is the remainder of α modulus p . Also we have

$$(e_p \circ S_p)(\alpha) \geq \alpha \text{ and } e_p(S_p(\alpha) - 1) < \alpha$$

so using (2) it results

$$\frac{S_p(\alpha) - \sigma_{(p)}(S_p(\alpha))}{p-1} \geq \alpha \text{ and } \frac{S_p(\alpha) - 1 - \sigma_{(p)}(S_p(\alpha) - 1)}{p-1} < \alpha$$

Using (6) we obtains that $S(p^\alpha)$ is the unique solution of the system

$$\sigma_{(p)}(x) \leq \sigma_{[p]}(x) \leq \sigma_{(p)}(x-1) + 1$$

The calculus of $\text{card}(S^{-1}(n))$. Let q_1, q_2, \dots, q_k be all the prime integers smallest then n and non dividing n . Let also denote shortly $e_{q_j}(n) = f_j$. A solution x_0 of the equation

$$S(x) = n$$

has the property that x_0 divides $n!$ and non divides $(n-1)!$. Now, if $d(n)$ is the number of positive divisors of n , from the inclusion

$$\{m / m \text{ divides } (n-1)!\} \subset \{m / m \text{ divides } n!\}$$

and using the definition of Smarandache function it results that

$$\text{card}(S^{-1}(n)) = d(n!) - d((n-1)!) \quad (16)$$

Example. In [6] A. Stuparu and D. W. Sharpe has proved that if p is a given prime, the equation

$$S(x) = p$$

has just $d((p-1)!)$ solutions (all of them in between p and $p!$). Let us observe that $e_p(p!) = 1$ and $e_p((p-1)!) = 0$, so because

$$\begin{aligned} d(p!) &= (e_p(p!) + 1)(f_1 + 1)(f_2 + 1)\dots(f_k + 1) = 2(f_1 + 1)(f_2 + 1)\dots(f_k + 1) \\ d((p-1)!) &= (f_1 + 1)(f_2 + 1)\dots(f_k + 1) \end{aligned}$$

it results

$$\text{card}(S^{-1}(p!)) = d(p!) - d((p-1)!) = d((p-1)!)$$

References

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