ABOUT THE SMARANDACHE COMPLEMENTARY PRIME FUNCTION

by

Marcela POPESCU and Vasile SELEACU

Let $c : N \to N$ be the function defined by the condition that $n + c(n) = p_i$, where p_i is the smallest prime number, $p_i \geq n$.

Example

$c(0) = 2$, $c(1) = 1$, $c(2) = 0$, $c(3) = 0$, $c(4) = 1$, $c(5) = 0$, $c(6) = 1$, $c(7) = 0$ and so on.

1) If p_k and p_{k-1} are two consecutive primes and $p_k < n \leq p_{k-1}$, then :

$c(n) \in \{ p_{k-1} - p_k - 1, p_{k-1} - p_k - 2, \ldots, 1, 0 \}$, because :

$c(p_k + 1) = p_{k-1} - p_k - 1$ and so on, $c(p_{k-1}) = 0$.

2) $c(p) = c(p-1) - 1 = 0$ for every p prime, because $c(p) = 0$ and $c(p-1) = 1$.

We also can observe that $c(n) \neq c(n+1)$ for every $n \in N$.

1. Property

The equation $c(n) = n$, $n > 1$ has no solutions.

Proof

If n is a prime it results $c(n) = 0 < n$.

It is wellknown that between n and 2n, $n > 1$ there exists at least a prime number. Let p_k be the smallest prime of them. Then if n is a composite number we have :

$c(n) = p_k - n < 2n - n = n$, therefore $c(n) < n$.

It results that for every $n \neq p$, where p is a prime, we have $\frac{1}{n} \leq \frac{c(n)}{n} < 1$, therefore $\sum_{\substack{n \neq p \\ p \text{ prime}}} \frac{c(n)}{n}$ diverges. Because for the primes $c(p)/p = 0$ we can say that $\sum_{n \geq 1} \frac{c(n)}{n}$ diverges.

2. Property

If n is a composite number, then $c(n) = c(n-1) - 1$.

Proof

Obviously.

It results that for n and $(n-1)$ composite numbers we have $\frac{c(n)}{c(n-1)} > 1$. Now, if $p_k < n < p_{k-1}$ where p_k and p_{k-1} are consecutive primes, then we have :

$c(n)\,c(n+1)\ldots c(p_{k-1}-1) = (p_{k-1}-n)!$

and if $n \leq p_k < p_{k-1}$ then $c(n)\,c(n+1)\ldots c(p_{k-1}-1) = 0$.

Of course, every $\prod_{n=k}^{r} c(n) = 0$ if there exists a prime number p, $k \leq p \leq r$.

If $n = p_k$ is any prime number, then $c(n) = 0$ and because $c(n+1) = p_{k-1} - n - 1$ it results that $|c(n) - c(n+1)| = 1$ if and only if n and $(n+2)$ are primes (friend prime numbers)

3. Property

For every k - th prime number p_k we have :

$c(p_k + 1) < (\log p_k)^2 - 1$.

Proof

Because $c(p_k + 1) = p_{k-1} - p_k - 1$ we have $p_{k-1} - p_k = c(p_k + 1) + 1$.

But, on the other hand we have $p_{k-1} - p_k < (\log p_k)^2$, then the assertion follows.

4. Property

$c(c(n)) < c(n)$ and $c^m(n) < c(n) < n$, for every $n > 1$ and $m \geq 2$.

Proof

If we denote $c(n) = r$ then we have :

$c(c(n)) = c(r) < r = c(n)$.

Then we suppose that the assertion is true for m : $c^m(n) < c(n) < n$, and we prove it for (m − 1), too :

$c^{m+1}(n) = c(c^{m}(n)) < c^{m}(n) < c(n) < n.$

5. Property

For every prime p we have $(c(p-1))^{n} \leq c((p-1)^{n})$.

Proof

$c(p-1) = 1 \Rightarrow (c(p-1))^{n} = 1$ while $(p-1)^{n}$ is a composite number, therefore $c((p-1)^{n}) \geq 1$.

6. Property

The following kind of Fibonacci equation :

$$c(n) + c(n+1) = c(n+2) \qquad (1)$$

has solutions.

Proof

If n and (n + 1) are both composite numbers, then $c(n) > c(n+1) \geq 1$. If (n + 2) is a prime, then $c(n+2) = 0$ and we have no solutions in this case. If (n + 2) is also a composite number, then :

$c(n) > c(n+1) > c(n+2) \geq 1$, therefore $c(n) + c(n+1) > c(n+2)$

and we have no solutions also in this case.

Therefore n and (n + 1) are not both composite numbers in the equality (1).

If n is a prime, then (n + 1) is a composite number and we must have :

$0 + c(n+1) = c(n+2)$, wich is not possible (see (2)).

We have only the case when (n + 1) is a prime; in this case we must have :

$1 + 0 = c(n+2)$ but this implies that (n + 3) is a prime number, so the only solutions are when (n + 1) and (n + 2) are friend prime numbers.

7. Property

The following equation:

$$\frac{c(n) + c(n+2)}{2} = c(n+1) \qquad (2)$$

has an infinite number of solutions.

Proof

Let p_k and p_{k+1} be two consecutive prime numbers, but not friend prime numbers.

Then, for every integer i between $p_k + 1$ and $p_{k+1} - 1$ we have:

$$\frac{c(i-1)+c(i+1)}{2} = \frac{(p_{k+1}-i+1)+(p_{k+1}-i-1)}{2} = p_{k+1}-i = c(i).$$

So, for the equation (2) all positive integer n between $p_k + 1$ and $p_{k+1} - 1$ is a solution.

If n is prime, the equation becomes $\frac{c(n+2)}{2} = c(n+1)$.

But (n + 1) is a composite number, therefore $c(n+1) \neq 0 \Rightarrow c(n+2)$ must be composite number. Because in this case $c(n+1) = c(n+2) + 1$ and the equation has the form $\frac{c(n+2)}{2} = c(n+2) + 1$, so we have no solutions.

If (n + 1) is prime, then we must have $\frac{c(n)+c(n+2)}{2} = 0$, where n and (n + 2) are composite numbers. So we have no solutions in this case, because $c(n) \geq 1$ and $c(n+2) \geq 1$.

If (n + 2) is a prime, the equation has the form $\frac{c(n)}{2} = c(n+1)$, where (n + 1) is a composite number, therefore $c(n+1) \neq 0$. From (2) it rezults that $c(n) \neq 0$, so n is also a composite number. This case is the same with the first considered case.

Therefore the only solutions are for $\overline{p_k, p_{k+1}-2}$, where p_k, p_{k+1} are consecutive primes, but not friend consecutive primes.

8. Property

The greatest common divisor of n and c (x) is 1 :

$(x, c(x)) = 1$, for every composite number x.

Proof

Taking into account of the definition of the function c, we have $x + c(x) = p$, where p is a prime number.

If there exists $d \neq 1$ so that d / x and d / c(x), then it implies that d / p. But p is a prime number, therefore d = p.

This is not possibile because $c(x) < p$.

If p is a prime number, then $(p, c(p)) = (p, 0) = p$.

9. Property

The equation $[x, y] = [c(x), c(y)]$, where $[x, y]$ is the least common multiple of x and y has no solutions for $x, y > 1$.

Proof

Let us suppose that $x = dk_1$ and $y = dk_2$, where $d = (x, y)$. Then we must have :

$[x, y] = dk_1k_2 = [c(x), c(y)]$.

But $(x, c(x)) = (dk_1, c(x)) = 1$, therefore dk_1 is given in the least common multiple $[c(x), c(y)]$ by $c(y)$.

But $(y, c(y)) = (dk_1, c(y)) = 1 \Rightarrow d = 1 \Rightarrow (x, y) = 1 \Rightarrow$

$\Rightarrow [x, y] = xy > c(x)c(y) \geq [c(x), c(y)]$, therefore the above equation has no solutions, for $x, y > 1$.

For $x = 1 = y$ we have $[x, y] = [c(x), c(y)] = 1$.

10. Property

The equation :

$$(x, y) = (c(x), c(y)) \qquad (3)$$

has an infinite number of solutions.

Proof

If $x = 1$ and $y = p - 1$ then $(x, y) = 1$ and $(c(x), c(y)) = (1, 1) = 1$, for an arbitrary prime p.

Easily we observe that every pair $(n, n + 1)$ of numbers is a solutions for the equation (3), if n is not a prime.

11. Property

The equation :

$$c(x) + x = c(y) + y \qquad (4)$$

has an infinite number of solutions.

Proof

From the definition of the function c it results that for every x and y satisfying

$p_k < x \le y \le p_{k-1}$ we have $c(x) + x = c(y) + y = p_{k-1}$. Therefore we have $(p_{k-1} - p_k)^2$ couples (x, y) as different solutions. Then, until the n-th prime p_n, we have $\sum_{k=1}^{n-1} (p_{k-1} - p_k)^2$ different solutions.

Remark

It seems that the equation $c(x) + y = c(y) + x$ has no solutions $x \ne y$, but it is not true.

Indeed, let p_k and p_{k-1} be consecutive primes such that $p_{k-1} - p_k = 6$ (is possibile : for example $29 - 23 = 6$, $37 - 31 = 6$, $53 - 47 = 6$ and so on) and $p_k - 2$ is not a prime.

Then $c(p_k - 2) = 2$, $c(p_k - 1) = 1$, $c(p_k) = 0$, $c(p_k + 1) = 5$, $c(p_k + 2) = 4$, $c(p_k + 3) = 3$ and we have :

1. $c(p_k + 1) - c(p_k - 2) = 5 - 2 = 3 = (p_k + 1) - (p_k - 2)$
2. $c(p_k + 2) - c(p_k - 1) = 3 = (p_k + 2) - (p_k - 1)$
3. $c(p_k + 3) - c(p_k) = 3 = (p_k + 3) - p_k$, thus

$c(x) - c(y) = x - y$ ($\Leftrightarrow c(x) + y = c(y) + x$) has the above solutions if $p_k - p_{k-1} > 3$

If $p_k - p_{k-1} = 2$ we have only the two last solutions.

In the general case, when $p_{k-1} - p_k = 2h$, $h \in N^*$, let $x = p_k - u$ and $y = p_k + v$, $u, v \in N$ be the solutions of the above equation.

Then $c(x) = c(p_k - u) = u$ and $c(y) = c(p_k + v) = 2h - v$.

The equation becomes:

$u + (p_k + v) = (2h - v) + (p_k - u)$, thus $u + v = h$.

Therefore, the solutions are $x = p_k - u$ and $y = p_k + h - u$, for every $u = \overline{0, h}$ if $p_k - p_{k-1} > h$ and $x = p_k - u$, $y = p_k + h - u$, for every $u = \overline{0, l}$ if $p_k - p_{k-1} = l + 1 \le h$.

Remark

$c(p_k + 1)$ is an odd number, because if p_k and p_{k-1} are consecutive primes, $p_k > 2$, then p_k and p_{k-1} are, of course, odd numbers; then $p_{k-1} - p_k - 1 = c(p_k + 1)$ are **always** odd.

12. Property

The sumatory function of c, $F_c(n) \overset{def}{=} \sum_{\substack{d \in N \\ d/n}} c(d)$ has the properties :

a) $F_c(2p) = 1 + c(2p)$

b) $F_c(pq) = 1 + c(pq)$, where p and q are prime numbers.

Proof

a) $F_c(2p) = c(1) + c(2) + c(p) + c(2p) = 1 + c(2p)$.

b) $F_c(pq) = c(1) + c(p) + c(q) + c(pq) = 1 + c(pq)$.

Remark

The function c is not multiplicative : $0 = c(2) \cdot c(p) < c(2p)$.

13. Property

$$c^k(p) = \begin{cases} 0 & \text{for } k \text{ odd number} \\ 2 & \text{for } k \text{ even number}, \quad k \geq 1 \end{cases}$$

Proof

We have :

$c^1(p) = 0$;

$c^2(p) = c(c(p)) = c(0) = 2$;

$c^3(p) = c(2) = 0$;

$c^4(p) = c(0) = 2$.

Using the complete mathematical induction, the property holds.

Consequences

1) We have $\dfrac{c^k(p) + c^{k+1}(p)}{2} = 1$ for every $k \geq 1$ and p prime number.

2) $\sum_{k=1}^{r} c^k(p) = \left[\frac{r}{2}\right] \cdot 2$, where $[x]$ is the integer part of x, and

$\sum_{\substack{k=2 \\ k \text{ even}}}^{r} \dfrac{1}{c^k(p)} = \left[\frac{r}{2}\right] \cdot \frac{1}{2}$, thus $\sum_{k \geq 1} c^k(p)$ and $\sum_{\substack{k \geq 2 \\ k \text{ even}}} \dfrac{1}{c^k(p)}$ are divergent series.

Remark

$c^k(p-1) = c^{k-1}(c(p-1)) = c^{k-1}(1) = 1$, for every prime $p > 3$ and $k \in N^*$, therefore $c^{k_1}(p_1 - 1) = c^{k_2}(p_2 - 1)$ for every primes $p_1, p_2 > 3$ and $k_1, k_2 \in N^*$.

14.Property

The equation :

$$c(x) + c(y) + c(z) = c(x)c(y)c(z) \qquad (5)$$

has an infinite number of solutions.

Proof

The only non-negative solutions for the diofantine equation $a + b + c = abc$ are $a = 1$, $b = 2$ and $c = 3$ and all circular permutations of $\{ 1, 2, 3 \}$.

Then :

$c(x) = 1 \Rightarrow x = p_k - 1$, p_k prime number, $p_k > 3$

$c(y) = 2 \Rightarrow y = p_r - 2$, where p_{r-1} and p_r are consecutive prime numbers such that $p_r - p_{r-1} \geq 3$

$c(z) = 3 \Rightarrow z = p_t - 3$, where p_{t-1} and p_t are consecutive prime numbers such that $p_t - p_{t-1} \geq 4$

and all circular permutations of the above values of x, y and z.

Of course, the equation $c(x) = c(y)$ has an infinite number of solutions.

Remark

We can consider $c^{\leftarrow}(y)$, for every $y \in N^*$, defined as $c^{\leftarrow}(y) = \{ x \in N \mid c(x) = y \}$. For example $c^{\leftarrow}(0)$ is the set of all primes, and $c^{\leftarrow}(1)$ is the set $\{ 1, p_{k-1} \}_{\substack{p_k \text{ prime} \\ p_k > 3}}$ and so on.

A study of these sets may be interesting.

Remark

If we have the equation :

$$c^k(x) = c(y), k \geq 2 \qquad (6)$$

then, using property 13, we have two cases.

If x is prime and k is odd, then $c^k(x) = 0$ and (5) implies that y is prime.

In the case when x is prime and k is even it results $c^k(x) = 2 = c(y)$, which implies that y is a prime, such that $y - 2$ is not prime.

If $x = p$, $y = q$, p and q primes, $p,q > 3$, then $(p - 1, q - 1)$ are also solutions, because $c^k(p - 1) = 1 = c(q - 1)$, so the above equation has an infinite number of couples as solutions.

Also a study of $(c^k(x))^{\leftarrow}$ seems to be interesting.

Remark

The equation :

$$c(n) + c(n+1) + c(n+2) = c(n-1) \qquad (7)$$

has solutions when $c(n-1) = 3$, $c(n) = 2$, $c(n+1) = 1$, $c(n+2) = 0$, so the solutions are $n = p - 2$ for every p prime number such that between $p - 4$ and p there is not another prime.

The equation :

$$c(n-2) + c(n-1) + c(n+1) + c(n+2) = 4c(n) \qquad (8)$$

has as solutions $n = p - 3$, where p is a prime such that between $p - 6$ and p there is not another prime, because $4c(n) = 12$ and $c(n-2) + c(n-1) + c(n+1) + c(n+2) = 12$.

For example $n = 29 - 3 = 26$ is a solution of the equation (7).

The equation :

$$c(n) + c(n-1) + c(n-2) + c(n-3) + c(n-4) = 2c(n-5) \qquad (9)$$

(see property 7) has as solution $n = p - 5$, where p is a prime, such that between $p - 6$ and p there is not another prime. Indeed we have $0 + 1 + 2 + 3 + 4 = 2 \cdot 5$.

Thus, using the properties of the function c we can decide if an equation, which has a similar form with the above equations, has or has not solutions.

But a difficult problem is : " For any even number a, can we find consecutive primes such that $p_{k-1} - p_k = a$? "

The answer is useful to find the solutions of the above kind of equations, but is also important to give the answer in order to solve another open problem :

" Can we get, as large as we want, but finite decreasing sequence k, k - 1, ..., 2, 1, 0 (**odd k**), included in the sequence of the values of c?"

If someone gives an answer to this problem, then it is easy to give the answer (it will be the same) at the similar following problem :

" Can we get, as large as we want, but finite decreasing sequence k, k - 1, ..., 2, 1, 0 (**even k**), included in the sequence of the values of c?"

We suppose the answer is negative.

In the same order of ideea, it is interesting to find $\max_n \frac{c(n)}{n}$.

It is wellknown (see [4], page 147) that $p_{n+1} - p_n < (\ln p_n)^2$, where p_n and p_{n+1} are two consecutive primes.

Moreover, $\frac{c(n)}{n}$, $p_k < n \le p_{k+1}$ reaches its maximum value for $n = p_k + 1$, where p_k is a prime.

So, in this case :

$$\frac{c(n)}{n} = \frac{p_{k+1} - p_k - 1}{p_k + 1} < \frac{(\ln p_k)^2 - 1}{p_k + 1} \overset{k\to\infty}{\rightarrow} 0$$

Using this result, we can find the maximum value of $\frac{c(n)}{n}$

For $p > 100$ we have $\frac{(\ln p)^2 - 1}{p + 1} < \frac{(\ln 100)^2 - 1}{101} < \frac{1}{4}$.

Using the computer, by a straight forward computation, it is easy to prove that $\max_{2\le n\le 100} \frac{c(n)}{n} = \frac{3}{8}$, wich is reached for $n = 8$.

Because $\frac{c(n)}{n} < \frac{1}{4}$ for every $n>100$ it results that $\max_{n\ge 2} \frac{c(n)}{n} = \frac{3}{8}$ reached for $n = 8$.

Remark

There exists an infinite number of finite sequences $\{ c(k_1), c(k_1 + 1), \ldots, c(k_2) \}$ such that $\sum_{k=k_1}^{k_2} c(k)$ is a three-cornered number for $k_1, k_2 \in N^*$ (the n-th three-cornered number is $T_n \overset{def}{=} \frac{n(n+1)}{2}$, $n \in N^*$).

For example, in the case $k_1 = p_k$ and $k_2 = p_{k+1}$, two consecutive primes, we have the finite sequence $\{ c(p_k), c(p_k + 1), \ldots, c(p_{k+1} - 1), c(p_{k+1}) \}$ and

$$\sum_{k=p_k}^{p_{k+1}} c(k) = 0 + (p_{k+1} - p_k - 1) + \ldots + 2 + 1 + 0 = \frac{(p_{k+1} - p_k - 1)(p_{k+1} - p_k)}{2} = T_{p_{k+1}-p_k-1}$$

Of course, we can define the function $c' : N \setminus \{0, 1\} \rightarrow N$, $c'(n) = n - k$, where k is the smallest natural number such that $n - k$ is a prime number, but we shall give some properties of this function in another paper.

References

[1] I. Cucurezeanu - " Probleme de aritmetica si teoria numerelor ",
Editura Tehnica, Bucuresti, 1976;

[2] P. Radovici - Marculescu - " Probleme de teoria elementara a numerelor ",
Editura Tehnica, Bucuresti, 1986;

[3] C. Popovici - " Teoria numerelor ",
Editura Didactica si Pedagogica, Bucuresti, 1973;

[4] W. Sierpinski - Elementary Theory of Numbers,
Warszawa, 1964;

[5] F. Smarandache - " Only problems, not solutions! "
Xiquan Publishing House, Phoenix - Chicago, 1990, 1991, 1993.

Current address

University of Craiova,
Department of Mathematics,
13, " A. I. Cuza " street,
Craiova - 1100,
ROMANIA