SMARANDACHE MULTIPLICATIVE FUNCTION

Liu Yanni
Department of Mathematics, Northwest University, Xi'an, Shaanxi, P.R.China

Gao Peng
School of Economics and Management, Northwest University, Xi'an, Shaanxi, P.R.China

Abstract The main purpose of this paper is using the elementary method to study the mean value properties of the Smarandache multiplicative function, and give an interesting asymptotic formula for it.

Keywords: Smarandache multiplicative function; Mean Value; Asymptotic formula.

§1. Introduction

For any positive integer n, we define $f(n)$ as a Smarandache multiplicative function, if $f(ab) = \max(f(a), f(b))$, $(a, b) = 1$. Now for any prime p and any positive integer α, we taking $f(p^\alpha) = \alpha p$. If $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ is the prime powers factorization of n, then

$$f(n) = \max_{1\le i\le k}\{f(p_i^{\alpha_i})\} = \max_{1\le i\le k}\{\alpha_i p_i\}.$$

Now we define $P_d(n)$ as another new arithmetical function. We let

$$P_d(n) = \prod_{d|n} d = n^{\frac{d(n)}{2}}, \qquad (1)$$

where $d(n) = \sum_{d|n} 1$ is the Dirichlet divisor function.

It is clear that $f(P_d(n))$ is a new Smarandache multiplicative function. About the arithmetical properties of $f(n)$, it seems that none had studied it before. This function is very important, because it has many similar properties with the Smarandache function $S(n)$. The main purpose of this paper is to study the mean value properties of $f(P_d(n))$, and obtain an interesting mean value formula for it. That is, we shall prove the following:

Theorem. *For any real number $x \ge 2$, we have the asymptotic formula*

$$\sum_{n\le x} f(P_d(n)) = \frac{\pi^4}{72}\frac{x^2}{\ln x} + C\cdot\frac{x^2}{\ln^2 x} + O\left(\frac{x^2}{\ln^3 x}\right),$$

where $C = \frac{5\pi^4}{288} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{d(n)\ln n}{n^2}$ *is a constant.*

§2. Proof of the Theorem

In this section, we shall complete the proof of the theorem. First we need following one simple Lemma. For convenience, let $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$ be the prime powers factorization of n, and $P(n)$ be the greatest prime factor of n, that is, $P(n) = \max_{1\le i\le k}\{p_i\}$. Then we have

Lemma. *For any positive integer n, if there exists $P(n)$ such that $P(n) > \sqrt{n}$, then we have the identity*

$$f(n) = P(n).$$

Proof. From the definition of $P(n)$ and the condition $P(n) > \sqrt{n}$, we get

$$f(P(n)) = P(n). \qquad (2)$$

For other prime divisors p_i of n ($1 \le i \le k \quad and \quad p_i \ne P(n)$), we have

$$f(p_i^{\alpha_i}) = \alpha_i p_i.$$

Now we will debate the upper bound of $f(p_i^{\alpha_i})$ in three cases:

(I) If $\alpha_i = 1$, then $f(p_i) = p_i \le \sqrt{n}$.

(II) If $\alpha_i = 2$, then $f(p_i^2) = 2p_i \le 2\cdot n^{\frac{1}{4}} \le \sqrt{n}$.

(III) If $\alpha_i \ge 3$, then $f(p_i^{\alpha_i}) = \alpha_i \cdot p_i \le \alpha_i \cdot n^{\frac{1}{2\alpha_i}} \le n^{\frac{1}{2\alpha_i}} \cdot \frac{\ln n}{\ln p_i} \le \sqrt{n}$, where we use the fact that $\alpha \le \frac{\ln n}{\ln p}$, if $p^\alpha | n$.

Combining (I)-(III), we can easily obtain

$$f(p_i^{\alpha_i}) \le \sqrt{n}. \qquad (3)$$

From (2) and (3), we deduce that

$$f(n) = \max_{1\le i\le k}\{f(p_i^{\alpha_i})\} = f(P(n)) = P(n).$$

This completes the proof of Lemma.

Now we use the above Lemma to complete the proof of Theorem. First we define two sets A and B as following:

$$A = \{n | n \le x, P(n) \le \sqrt{n}\} \quad and \quad B = \{n | n \le x, P(n) > \sqrt{n}\}.$$

Using the Euler summation formula, we may get

$$\sum_{n\in A} f(P_d(n)) \ll \sum_{n\in A} P(n)d(n) \ll \sum_{n\le x}\sqrt{x}d(n) \ll x^{\frac{3}{2}}\ln x. \qquad (4)$$

For another part of the summation, since $P(n) = p$, we can assume that $n = pl$, where $p > l$ and $(p, l) = 1$. Note that

$$P_d(n) = n^{\frac{d(n)}{2}} = (pl)^{\frac{d(lp)}{2}} = (pl)^{d(l)}$$

and

$$f(P_d(n)) = f((pl)^{d(l)}) = f(p^{d(l)}) = d(l)p,$$

we have

$$\begin{aligned}
& \sum_{n \in B} f(P_d(n) \\
= & \sum_{\substack{pl \le x \\ p > \sqrt{pl}}} d(l)p = \sum_{\substack{pl \le x \\ p > l}} d(l)p = \sum_{p \le x} p \sum_{\substack{l \le \frac{x}{p} \\ l < p}} d(l) \\
= & \sum_{\sqrt{x} \le p \le x} p \sum_{l < \frac{x}{p}} d(l) + \sum_{p \le \sqrt{x}} p \sum_{l < p} d(l) \\
= & \sum_{p \le x} p \sum_{l < \frac{x}{p}} d(l) + O\left(\sum_{p \le \sqrt{x}} p \sum_{l < p} d(l) \right) + O\left(\sum_{p \le \sqrt{x}} p \sum_{l < \frac{x}{p}} d(l) \right) \\
= & \sum_{p \le \sqrt{x}} p \sum_{l < \frac{x}{p}} d(l) + \sum_{l \le \sqrt{x}} d(l) \sum_{p < \frac{x}{l}} p - \left(\sum_{p \le \sqrt{x}} p \right) \left(\sum_{l \le \sqrt{x}} d(l) \right) \\
& + O\left(\sum_{p \le \sqrt{x}} p \sum_{l < \frac{x}{p}} d(l) \right) + O\left(\sum_{p \le \sqrt{x}} p \sum_{l < \frac{x}{p}} d(l) \right), \qquad (5)
\end{aligned}$$

where we have used Theorem 3.17 of [3]. Note that the asymptotic formula (see Theorem 3.3 of [3])

$$\sum_{n \le x} d(n) = x \ln x + (2\gamma - 1)x + O\left(\sqrt{x}\right) \ll x \ln x, \zeta(2) = \frac{\pi^2}{6}$$

(where γ is the Euler constant) and

$$\pi(x) = \frac{x}{\ln x} + \frac{x}{\ln^2 x} + \frac{2x}{\ln^3 x} + O\left(\frac{x}{\ln^4 x}\right),$$

we have

$$\sum_{p \le x} p \sum_{l < \frac{x}{p}} d(l) \quad = \quad \sum_{p \le x} p \left[\frac{x}{p} \ln \frac{x}{p} + (2\gamma - 1) \frac{x}{p} + O\left(\sqrt{\frac{x}{p}} \right) \right] \ll x^{\frac{3}{2}} \qquad (6)$$

$$\sum_{p\leq\sqrt{x}} p \sum_{l<p} d(l) \ll \sum_{p\leq\sqrt{x}} p^2 \ln p \ll x^{\frac{3}{2}} \qquad (7)$$

$$\sum_{p\leq\sqrt{x}} p \sum_{l<\frac{x}{p}} d(l) \ll \sum_{p\leq\sqrt{x}} p \times \frac{x}{p} \ln \frac{x}{p} \ll x^{\frac{3}{2}}. \qquad (8)$$

and

$$\left(\sum_{p\leq\sqrt{x}} p\right)\left(\sum_{l\leq\sqrt{x}} d(l)\right) \ll x^{\frac{3}{2}} \qquad (9)$$

Applying Abel's identity (Theorem 4.2 of [3]) we also have

$$\begin{aligned}\sum_{l\leq\sqrt{x}} d(l) \sum_{p<\frac{x}{l}} p &= \sum_{l\leq\sqrt{x}} d(l)\left[\frac{x}{l}\pi(\frac{x}{l}) - \int_2^{\frac{x}{l}} \pi(y)dy\right] \\ &= \sum_{l\leq\sqrt{x}} d(l)\left[\frac{1}{2}\frac{x^2}{l^2\ln\frac{x}{l}} + \frac{5}{8}\frac{x^2}{l^2\ln^2\frac{x}{l}} + O\left(\frac{x^2}{l^2\ln^3 x}\right)\right] \\ &= \frac{\pi^4}{72}\frac{x^2}{\ln x} + C\cdot\frac{x^2}{\ln^2 x} + O\left(\frac{x^2}{\ln^3 x}\right), \qquad (10)\end{aligned}$$

where $C = \frac{5\pi^4}{288} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{d(n)\ln n}{n^2}$ is a constant.

Combining (5), (6), (7),(8),(9) and (10) we may immediately deduce the asymptotic formula

$$\sum_{n\leq x} f(P_d(n)) = \frac{\pi^4}{72}\frac{x^2}{\ln x} + C\cdot\frac{x^2}{\ln^2 x} + O\left(\frac{x^2}{\ln^3 x}\right).$$

This completes the proof of Theorem.

Note. Substitute to

$$\sum_{n\leq x} d(n) = x\ln x + (2\gamma - 1)x + O\left(\sqrt{x}\right) \ll x\ln x, \zeta(2) = \frac{\pi^2}{6}$$

and

$$\pi(x) = \frac{x}{\ln x} + \frac{x}{\ln^2 x} + \frac{2x}{\ln^3 x} + O\left(\frac{x}{\ln^4 x}\right),$$

we can get a more accurate asymptotic formula for $\sum_{n\leq x} f(P_d(n))$.

References

[1] F. Smarandache, Only Problems, Not Solutions, Xiquan Publishing House, Chicago, 1993.

[2] Jozsef Sandor, On an generalization of the Smarandache function, Notes Numb. Th. Discr. Math. **5** (1999), 41-51.

[3] Tom M. Apostol, Introduction to Analytic Number Theory, New York, Springer-Verlag, 1976.