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On the mean value of the Pseudo-Smarandache function

Lin Cheng

Department of Mathematics, Northwest University
Xi'an, Shaanxi, P.R.China

Abstract For any positive integer n, the Pseudo-Smarandache function $Z(n)$ is defined as the smallest positive integer k such that $n \mid \frac{k(k+1)}{2}$. That is, $Z(n) = \min\left\{k :\ n|\frac{k(k+1)}{2}\right\}$. The main purpose of this paper is using the elementary methods to study the mean value properties of $\frac{p(n)}{Z(n)}$, and give a sharper asymptotic formula for it, where $p(n)$ denotes the smallest prime divisor of n.

Keywords Pseudo-Smarandache function, mean value, asymptotic formula.

§1. Introduction and Results

For any positive integer n, the Pseudo-Smarandache function $Z(n)$ is defined as the smallest positive integer k such that $n \mid \frac{k(k+1)}{2}$. That is, $Z(n) = \min\left\{k :\ n|\frac{k(k+1)}{2},\ n \in N\right\}$, where N denotes the set of all positive integers. For example, the first few values of $Z(n)$ are $Z(1) = 1$, $Z(2) = 3$, $Z(3) = 2$, $Z(4) = 7$, $Z(5) = 4$, $Z(6) = 3$, $Z(7) = 6$, $Z(8) = 15$, $Z(9) = 8$, $Z(10) = 4$, $Z(11) = 10$, $Z(12) = 8$, $Z(13) = 12$, $Z(14) = 7$, $Z(15) = 5$, $\cdots$. About the elementary properties of $Z(n)$, some authors had studied it, and obtained many valuable results. For example, Richard Pinch [3] proved that for any given $L > 0$, there are infinitely many values of n such that

$$\frac{Z(n+1)}{Z(n)} > L.$$

Simultaneously, Maohua Le [4] proved that if n is an even perfect number, then n satisfies

$$S(n) = Z(n).$$

The main purpose of this paper is using the elementary methods to study the mean value properties of $\frac{p(n)}{Z(n)}$, and give a sharper asymptotic formula for it. That is, we shall prove the following conclusion:

Theorem. Let k be any fixed positive integer. Then for any real number $x > 1$, we have the asymptotic formula

$$\sum_{n \le x} \frac{p(n)}{Z(n)} = \frac{x}{\ln x} + \sum_{i=2}^{k} \frac{a_i x}{\ln^i x} + O\left(\frac{x}{\ln^{k+1} x}\right),$$

where $p(n)$ denotes the smallest prime divisor of n, and a_i $(i = 2,\ 3,\ \cdots, k)$ are computable constants.

§2. Proof of the theorem

In order to complete the proof of the theorem, we need the following several useful lemmas.

Lemma 1. For any prime $p \geq 3$, we have identity $Z(p) = p - 1$.

Proof. See reference [5].

Lemma 2. For any prime $p \geq 3$ and any $k \in N$, we have $Z\left(p^k\right) = p^k - 1$.

Proof. See reference [5].

Lemma 3. For any positive n, $Z(n) \geq \sqrt{n}$.

Proof. See reference [3].

Now, we shall use these lemmas to complete the proof of our theorem. We separate all integer n in the interval $[1, x]$ into four subsets A, B, C and D as follows:

A: $\Omega(n) = 0$, this time $n = 1$;

B: $\Omega(n) = 1$, then $n = p$, a prime;

C: $\Omega(n) = 2$, then $n = p^2$ or $n = p_1 p_2$, where p_i $(i = 1,\ 2)$ are two different primes with $p_1 < p_2$;

D: $\Omega(n) \geq 3$. This time, $p(n) \leq n^{\frac{1}{3}}$, where $\Omega(n) = \Omega\left(p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}\right) = \alpha_1 + \alpha_2 + \cdots + \alpha_s$. In fact in this case, we have $p^3(n) \leq p^{\Omega(n)}(n) \leq n$ and thus $p(n) \leq n^{\frac{1}{3}}$.

Let $p(n)$ denotes the smallest prime divisor of n, then we have $p(1) = 0$, $Z(1) = 1$ and

$$\sum_{n \in A} \frac{p(n)}{Z(n)} = 0.$$

So we have

$$\sum_{n \leq x} \frac{p(n)}{Z(n)} = \sum_{n \in B} \frac{p(n)}{Z(n)} + \sum_{n \in C} \frac{p(n)}{Z(n)} + \sum_{n \in D} \frac{p(n)}{Z(n)}. \tag{1}$$

From Lemma 1 we know that if $n \in B$, then we have $Z(2) = 3$ and $Z(p) = p - 1$ with $p > 2$. Therefore, by the Abel's summation formula (See Theorem 4.2 of [8]) and the Prime Theorem (See Theorem 3.2 of [9]):

$$\pi(x) = \sum_{i=1}^{k} \frac{a_i \cdot x}{\ln^i x} + O\left(\frac{x}{\ln^{k+1} x}\right),$$

where k be any fixed positive integer, a_i $(i = 1,\ 2,\ \ldots,\ k)$ are computable constants and $a_1 = 1$.

We have

$$\begin{aligned}\sum_{n\in B}\frac{p(n)}{Z(n)} &= \sum_{p\le x}\frac{p}{Z(p)}=\frac{2}{3}+\sum_{\substack{p\le x\\ p\ge 3}}\frac{p}{Z(p)}\\ &= \sum_{p\le x}\frac{p}{p-1}+O(1)\\ &= \sum_{p\le x}1+\sum_{p\le x}\frac{1}{p-1}+O(1)\\ &= \frac{x}{\ln x}+\sum_{i=2}^{k}\frac{a_i x}{\ln^i x}+O\left(\frac{x}{\ln^{k+1}x}\right),\end{aligned} \tag{2}$$

where a_i $(i=2,\ 3,\ \cdots,\ k)$ are computable constants.

Now we estimate the error terms in set D. From the definition of $\Omega(n)$ we know that $p(n)\le n^{\frac{1}{3}}$ if $n\in D$. From Lemma 3 we know that $Z(n)\ge\sqrt{n}$, so we have the estimate

$$\sum_{n\in D}\frac{p(n)}{Z(n)}\le\sum_{n\le x}\frac{n^{\frac{1}{3}}}{\sqrt{n}}=x^{\frac{5}{6}}. \tag{3}$$

Finally, we estimate the error terms in set C. For any integer $n\in C$, we have $n=p^2$ or $n=p_1p_2$. If $n=p^2$, then from Lemma 2 we have

$$\sum_{p^2\le x}\frac{p}{Z(p^2)}=\frac{2}{Z(4)}+\sum_{p^2\le x}\frac{p}{p^2-1}\ll\ln\ln x. \tag{4}$$

If $n=p_1p_2$, let $Z\left(p_1p_2\right)=k$, then from the definition of $Z(n)$ we have $p_1p_2\mid\dfrac{k(k+1)}{2}$.
If $p_1p_2\mid k$, then

$$\sum_{\substack{p_1p_2\le x\\ Z(p_1p_2)=k,\ p_1p_2\mid k}}\frac{p\left(p_1p_2\right)}{Z\left(p_1p_2\right)}\ll\sum_{p_1\le\sqrt{x}}\sum_{p_1<p_2\le\frac{x}{p_1}}\frac{p_1}{p_1p_2}\ll\sqrt{x}\cdot\ln\ln x. \tag{5}$$

If $p_1p_2\mid k+1$, then we also have the same estimate as in (5).
If $p_1\mid k+1$ and $p_2\mid k$, let $k=tp_1-1$, where $t\in N$, then we have

$$\sum_{p_1p_2\le x}\frac{p\left(p_1p_2\right)}{Z\left(p_1p_2\right)}\ll\sum_{p_1\le\sqrt{x}}\sum_{t\le x}\frac{p_1}{tp_1-1}+\sqrt{x}\cdot\ln\ln x\ll\sqrt{x}\cdot\ln\ln x. \tag{6}$$

If $p_1\mid k$ and $p_2\mid k+1$, then we can also obtain the same estimate as in (6).
From (4), (5) and (6) we have the estimate

$$\sum_{n\in C}\frac{p(n)}{Z(n)}\ll\sqrt{x}\cdot\ln\ln x. \tag{7}$$

Combining (1), (2), (3) and (7) we may immediately deduce the asymptotic formula

$$\begin{aligned}\sum_{n\le x}\frac{p(n)}{Z(n)} &= \sum_{n\in A}\frac{p(n)}{Z(n)}+\sum_{n\in B}\frac{p(n)}{Z(n)}+\sum_{n\in C}\frac{p(n)}{Z(n)}+\sum_{n\in D}\frac{p(n)}{Z(n)}\\ &= \frac{x}{\ln x}+\sum_{i=2}^{k}\frac{a_i x}{\ln^i x}+O\left(\frac{x}{\ln^{k+1}x}\right),\end{aligned}$$

where a_i $(i = 2,\ 3,\ \cdots, k)$ are computable constants.

This completes the proof of Theorem.

Some notes:

For any real number $x > 1$, whether there exist an asymptotic formula for the mean values

$$\sum_{n\leq x}\frac{P(n)}{Z(n)} \quad \text{and} \quad \sum_{n\leq x}\frac{Z(n)}{P(n)}$$

are two open problems, where $P(n)$ denotes the largest prime divisor of n.

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