MORE RESULTS AND APPLICATIONS OF THE GENERALIZED SMARANDACHE STAR FUNCTION

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ABSTRCT: In [1] we define SMARANDACHE FACTOR PARTITION FUNCTION , as follows:

Let $\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_r$ be a set of r natural numbers and $p_1, p_2, p_3, \ldots p_r$ be arbitrarily chosen distinct primes then $F(\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_r)$ called the Smarandache Factor Partition of $(\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_r)$ is defined as the number of ways in which the number

$$N = p_1^{\alpha 1} p_2^{\alpha 2} p_3^{\alpha 3} \ldots p_r^{\alpha r}$$

could be expressed as the product of its' divisors. For simplicity , we denote $F(\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_r) = F'(N)$,where

$$N = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \ldots p_r^{\alpha_r} \ldots p_n^{\alpha_n}$$

and p_r is the r^{th} prime. $p_1 = 2$, $p_2 = 3$ etc.

Also for the case

$$\alpha_1 = \alpha_2 = \alpha_3 = \ldots = \alpha_r = \ldots = \alpha_n = 1$$

Let us denote

$$F(\underbrace{1, 1, 1, 1, 1 \ldots}_{\leftarrow\ n\text{ - ones}\ \rightarrow}) = F(1\#n)$$

In [2] we define **The Generalized Smarandache Star**

Function as follows:

Smarandache Star Function

(1) $F'^{*}(N) = \sum_{d/N} F'(d_r)$ where $d_r | N$

(2) $F'^{**}(N) = \sum_{d_r/N} F'^{*}(d_r)$

d_r ranges over all the divisors of N.

If N is a square free number with n prime factors, let us denote

$$F'^{**}(N) = F^{**}(1\#n)$$

Smarandache Generalised Star Function

(3) $F'^{n*}(N) = \sum_{d_r/N} F'^{(n-1)*}(d_r)$ $n > 1$

and d_r ranges over all the divisors of N.

For simplicity we denote

$$F'(Np_1p_2...p_n) = F'(N@1\#n), \text{ where}$$

$(N, p_i) = 1$ for i = 1 to n and each p_i is a prime.

F'(N@1#n) is nothing but the Smarandache factor partition of (a number N multiplied by n primes which are coprime to N).

In [3] I had derived a general result on the Smarandache Generalised Star Function. In the present note some more results and applications of Smarandache Generalised Star Function are explored and derived.

DISCUSSION:

THEOREM(4.1) :

$$F'^{n*}(p^{\alpha}) = \sum_{k=0}^{\alpha} {}^{n+k-1}C_{n-1}\, P(\alpha-k) \quad \text{------}(4.1)$$

Following proposition shall be applied in the proof of this

$$\sum_{k=0}^{\alpha} {}^{r+k-1}C_{r-1} = {}^{\alpha+r}C_r \quad \text{-------}(4.2)$$

Let the proposition (4.1) be true for $n = r$ to $n = 1$.

$$F'^{r*}(p^{\alpha}) = \sum_{k=0}^{\alpha} {}^{r+k-1}C_{r-1}\, P(\alpha-k) \quad \text{-------}(4.3)$$

$$F'^{(r+1)*}(p^{\alpha}) = \sum_{t=0}^{\alpha} F'^{r*}(p^{t})$$

(p ranges over all the divisors of p^{α} for $t = 0$ to α)

RHS = $F'^{r*}(p^{\alpha}) + F'^{r*}(p^{\alpha-1}) + F'^{r*}(p^{\alpha-2}) + \ldots + F'^{r*}(p) + F'^{r*}(1)$

from the proposition (4.3) we have

$$F'^{r*}(p^{\alpha}) = \sum_{k=0}^{\alpha} {}^{r+k-1}C_{r-1}\, P(\alpha-k)$$

expanding RHS from $k = 0$ to α

$$F'^{r*}(p^{\alpha}) = {}^{r+\alpha-1}C_{r-1}\, P(0) + {}^{r+\alpha-2}C_{r-1}\, P(1) + \ldots + {}^{r-1}C_{r-1}\, P(\alpha)$$

similarly

$$F'^{r*}(p^{\alpha-1}) = {}^{r+\alpha-2}C_{r-1}\, P(0) + {}^{r+\alpha-3}C_{r-1}\, P(1) + \ldots + {}^{r-1}C_{r-1}\, P(\alpha-1)$$

$$F'^{r*}(p^{\alpha-2}) = {}^{r+\alpha-3}C_{r-1}\, P(0) + {}^{r+\alpha-4}C_{r-1}\, P(1) + \ldots + {}^{r-1}C_{r-1}\, P(\alpha-2)$$

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$$F'^{r*}(p) = {}^{r}C_{r-1}\, P(0) + {}^{r-1}C_{r-1}\, P(1)$$

$$F'^{r*}(1) = {}^{r-1}C_{r-1}\, P(0)$$

summing up left and right sides separately we find that the

$$LHS = F'^{(r+1)*}(p^{\alpha})$$

The RHS contains $\alpha + 1$ terms in which P(0) occurs , α terms in which P(1) occurs etc.

$$RHS = [\sum_{k=0}^{\alpha} {}^{r+k-1}C_{r-1}].P(0) + \sum_{k=0}^{\alpha-1} {}^{r+k-1}C_{r-1}\, P(1) + \ldots + \sum_{k=0}^{1} {}^{r+k-1}C_{r-1}\, P(\alpha-1)$$

$$+ \sum_{k=0}^{0} {}^{r+k-1}C_{r-1}\, P(\alpha)$$

Applying proposition (4.2) to each of the Σ we get

$$RHS = {}^{r+\alpha}C_r\, P(0) + {}^{r+\alpha-1}C_r\, P(1) + {}^{r+\alpha-2}C_r\, P(2) + \ldots + {}^{r}C_r\, P(\alpha)$$

$$= \sum_{k=0}^{\alpha} {}^{r+k}C_r\, P(\alpha-k)$$

or

$$F'^{(r+1)*}(p^{\alpha}) = \sum_{k=0}^{\alpha} {}^{r+k}C_r\, P(\alpha-k)$$

The proposition is true for $n = r+1$, as we have

$$F'^{*}(p^{\alpha}) = \sum_{k=0}^{\alpha} P(\alpha-k) = \sum_{k=0}^{\alpha} {}^{k}C_0\, P(\alpha-k) = \sum_{k=0}^{\alpha} {}^{k+1-1}C_{1-1}\, P(\alpha-k)$$

The proposition is true for $n = 1$

Hence by induction the proposition is true for all n.

This completes the proof of theorem (4.1).

Following theorem shall be applied in the proof of theorem (4.3)

THEOREM (4.2)

n-r

$$\sum_{k=0} {}^{n}C_{r+k}\, {}^{r+k}C_r\, m^k = {}^{n}C_r\, (1+m)^{(n-r)}$$

PROOF:

$$LHS = \sum_{k=0}^{n-r} {}^{n}C_{r+k}\, {}^{r+k}C_r\, m^k$$

$$= \sum_{k=0}^{n-r} (n!)/\{(r+k)!.(n-r-k)!\} \;.(r+k)!/\{(k)!.(r)!\} \;.\; m^k$$

$$= \sum_{k=0}^{n-r} (n!)/\{(r)!.(n-r)!\} \;.(n-r)!/\{(k)!.(n-r-k)!\} \;.\; m^k$$

$$= {}^{n}C_r \sum_{k=0}^{n-r} {}^{n-r}C_k\, m^k$$

$$= {}^{n}C_r\, (1+m)^{(n-r)}$$

This completes the proof of theorem (4.2)

THEOREM(4.3):

$$F^{m*}(1\#n) = \sum_{r=0}^{n} {}^{n}C_r\, m^{n-r}\, F(1\#r)$$

Proof:

From theorem (2.4) (ref.[1] ne have

$$F^{*}(1\#n) = F(\,1\#\,(n+1)) = \sum_{r=0}^{n} {}^{n}C_r\, F(1\#r) = \sum_{r=0}^{n} {}^{n}C_r\, (1)^{n-r}\, F(1\#r)$$

hence the proposition is true for m = 1.

Let the proposition be true for m = s. Then we have

$$F^{s*}(1\#n) = \sum_{r=0}^{n} {}^{n}C_r\, s^{n-r}\, F(1\#r)$$

or

$$F^{s*}(1\#0) = \sum_{r=0}^{0} {}^{n}C_0\, s^{0-r}\, F(1\#0) \qquad F^{s*}(1\#1) = \sum_{r=0}^{1} {}^{n}C_1\, s^{1-r}\, F(1\#1)$$

$$F^{s*}(1\#2) = \sum_{r=0}^{2} {}^{n}C_2\, s^{2-r}\, F(1\#1) \qquad F^{s*}(1\#3) = \sum_{r=0}^{3} {}^{n}C_1\, s^{3-r}\, F(1\#3)$$

$F^{s*}(1\#0) = {}^0C_0\, F(1\#0)$ ----(0)

$F^{s*}(1\#1) = {}^1C_0\, s^1\, F(1\#0) + {}^1C_1\, s^0 F(1\#1)$ ----(1)

$F^{s*}(1\#2) = {}^2C_0\, s^2\, F(1\#0) + {}^2C_1\, s^1 F(1\#1) + {}^2C_2\, s^0 F(1\#2)$ ----(2)

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$F^{s*}(1\#r) = {}^rC_0\, s^r\, F(1\#0) + {}^rC_1\, s^1 F(1\#1) + \ldots + {}^rC_r\, s^0 F(1\#r)$ ----(r)

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$F^{s*}(1\#n) = {}^nC_0\, s^r\, F(1\#0) + {}^nC_1\, s^1 F(1\#1) + \ldots + {}^nC_n\, s^0 F(1\#r)$ ----(n)

multiplying the r^{th} equation with ${}^nC_r$ and then summing up we get the RHS as

$=[{}^nC_0{}^0C_0\, s^0 + {}^nC_1{}^1C_0\, s^1 + {}^nC_2{}^2C_0\, s^2 + \ldots + {}^nC_k{}^kC_0\, s^k + \ldots + {}^nC_n{}^nC_0\, s^n]F(1\#0)$

$[{}^nC_1{}^1C_1\, s^0 + {}^nC_2{}^2C_1\, s^1 + {}^nC_3{}^3C_1\, s^2 + \ldots + {}^nC_k{}^kC_1\, s^k + \ldots + {}^nC_n{}^nC_1\, s^n]F(1\#1)$

. . .

$[{}^nC_r{}^rC_r\, s^0 + {}^nC_{r+1}{}^{r+1}C_r\, s^1 + \ldots + {}^nC_{r+k}{}^{r+k}C_r\, s^k + \ldots + {}^nC_n{}^nC_r\, s^n]F(1\#r)$

$+ {}^nC_n{}^nC_n\, s^0]F(1\#n)$

$$= \sum_{r=0}^{n} \left\{ \sum_{k=0}^{n-r} {}^nC_{r+k}\, {}^{r+k}C_r\, s^k \right\} F(1\#r)$$

$$= \sum_{r=0}^{n} {}^nC_r\, (1+s)^{n-r}\, F(1\#n) \qquad \text{, by theorem (4.2)}$$

$$\text{LHS} = \sum_{r=0}^{n} {}^nC_r\, F^{s*}(1\#r)$$

Let $N = p_1p_2p_3 \ldots p_n$. Then there are ${}^nC_r$ divisors of N containing exactly r primes . Then LHS = the sum of the s^{th} Smarandache star functions of all the divisors of N. $= F'^{(s+1)*}(N) = F^{(s+1)*}(1\#n)$.

Hence we have

$$F^{(s+1)*}(1\#n) = \sum_{r=0}^{n} {}^nC_r\, (1+s)^{n-r}\, F(1\#n)$$

$F^{s*}(1\#0) = {}^{0}C_0\, F(1\#0)$ ----(0)

$F^{s*}(1\#1) = {}^{1}C_0\, s^1\, F(1\#0) + {}^{1}C_1\, s^0 F(1\#1)$ ----(1)

$F^{s*}(1\#2) = {}^{2}C_0\, s^2\, F(1\#0) + {}^{2}C_1\, s^1 F(1\#1) + {}^{2}C_2\, s^0 F(1\#2)$ ----(2)

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$F^{s*}(1\#r) = {}^{r}C_0\, s^r\, F(1\#0) + {}^{r}C_1\, s^1 F(1\#1) + \ldots + {}^{r}C_r\, s^0 F(1\#r)$ ----(r)

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$F^{s*}(1\#n) = {}^{n}C_0\, s^r\, F(1\#0) + {}^{n}C_1\, s^1 F(1\#1) + \ldots + {}^{n}C_n\, s^0 F(1\#r)$ ----(n)

multiplying the r^{th} equation with ${}^{n}C_r$ and then summing up we get the RHS as

$=[{}^{n}C_0{}^{0}C_0\, s^0 + {}^{n}C_1{}^{1}C_0\, s^1 + {}^{n}C_2{}^{2}C_0\, s^2 + \ldots + {}^{n}C_k{}^{k}C_0\, s^k + \ldots + {}^{n}C_n{}^{n}C_0\, s^n]F(1\#0)$

$[{}^{n}C_1{}^{1}C_1\, s^0 + {}^{n}C_2{}^{2}C_1\, s^1 + {}^{n}C_3{}^{3}C_1\, s^2 + \ldots + {}^{n}C_k{}^{k}C_1\, s^k + \ldots + {}^{n}C_n{}^{n}C_1\, s^n]F(1\#1)$

. . .

$[{}^{n}C_r{}^{r}C_r\, s^0 + {}^{n}C_{r+1}{}^{r+1}C_r\, s^1 + \ldots + {}^{n}C_{r+k}{}^{r+k}C_r\, s^k + \ldots + {}^{n}C_n{}^{n}C_r\, s^n]F(1\#r)$

$+ {}^{n}C_n{}^{n}C_n\, s^0]F(1\#n)$

$$= \sum_{r=0}^{n} \left\{ \sum_{k=0}^{n-r} {}^{n}C_{r+k}\, {}^{r+k}C_r\, s^k \right\} F(1\#r)$$

$$= \sum_{r=0}^{n} {}^{n}C_r\, (1+s)^{n-r}\, F(1\#n) \quad \text{, by theorem (4.2)}$$

$$\text{LHS} = \sum_{r=0}^{n} {}^{n}C_r\, F^{s*}(1\#r)$$

Let $N = p_1 p_2 p_3 \ldots p_n$. Then there are ${}^{n}C_r$ divisors of N containing exactly r primes . Then LHS = the sum of the s^{th} Smarandache star functions of all the divisors of N. $= F'^{(s+1)*}(N) = F^{(s+1)*}(1\#n)$.

Hence we have

$$F^{(s+1)*}(1\#n) = \sum_{r=0}^{n} {}^{n}C_r\, (1+s)^{n-r}\, F(1\#n)$$

which takes the same format

$P(s) \Rightarrow P(s+1)$

and it has been verified that the proposition is true for m = 1

hence by induction the proposition is true for all m.

$$F^{m*}(1\#n) = \sum_{r=0}^{n} {}^{n}C_r \, m^{n-r} \, F(1\#r)$$

This completes the proof of theorem (4.3)

NOTE:

From theorem (3.1) we have

$$F'(N@1\#n) = F'(Np_1p_2...p_n) = \sum_{m=0}^{n} a_{(n,m)} F'^{m*}(N)$$

where

$$a_{(n,m)} = (1/m!) \sum_{k=1}^{m} (-1)^{m-k} \, .{}^{m}C_k \, .k^n$$

If $N = p_1p_2...p_k$ Then we get

$$F(1\#(k+n) = \sum_{m=0}^{n} [\, a_{(n,m)} \sum_{t=0}^{k} {}^{k}C_t \, m^{k-t} \, F(1\#t) \,] \qquad \text{------(4.4)}$$

The above result provides us with a formula to express B_n in terms of smaller Bell numbers. It is in a way generalisation of theorem (2.4) in Ref [5].

THEOREM(4.4):

$$F(\alpha,1\#(n+1)) = \sum_{k=0}^{\alpha} \sum_{r=0}^{n} {}^{n}C_r \, F(k,1\#r)$$

PROOF: LHS = $F(\alpha,1\#(n+1)) = F'(p^{\alpha} p_1p_2p_3...p_{n+1}) = F'^{*}(p^{\alpha}p_1p_2p_3...p_n) + \sum F'$ (all the divisors containing only p^0) + $\sum F'$ (all the divisors containing only p^1) + $\sum F'$ (all the divisors containing only p^2) +. . .+ $\sum F'$ (all the divisors containing only p^r) +. . .+ $\sum F'$ (all the divisors containing only p^α)

$$= \sum_{r=0}^{n} {}^nC_r\, F(0,1\#r) + \sum_{r=0}^{n} {}^nC_r\, F(1,1\#r) + \sum_{r=0}^{n} {}^nC_r\, F(2,1\#r) + \sum_{r=0}^{n} {}^nC_r\, F(3,1\#r)$$

$$+ \ldots + \sum_{r=0}^{n} {}^nC_r\, F(k,1\#r) + \ldots + \sum_{r=0}^{n} {}^nC_r\, F(\alpha,1\#r)$$

$$= \sum_{k=0}^{\alpha} \sum_{r=0}^{n} {}^nC_r\, F(k,1\#r)$$

This is a reduction formula for $F(\alpha,1\#(n+1))$

A Result of significance

From theorem (3.1) of Ref.: [2] , we have

$$\mathbf{F'(p^\alpha@1\#(n+1)) = F(\alpha,1\#(n+1)) = \sum_{m=0}^{n} a_{(n+1,m)}\, F'^{m*}(N)}$$

where

$$\mathbf{a_{(n+1,m)} = (1/m!) \sum_{k=1}^{m} (-1)^{m-k} . {}^mC_k . k^{n+1}}$$

and

$$\mathbf{F'^{m*}(p^\alpha) = \sum_{k=0}^{\alpha} {}^{m+k-1}C_{m-1}\, P(\alpha-k)}$$

This is the first result of some substance , giving a formula for evaluating the number of Smarandache Factor Partitions of numbers representable in a (one of the most simple) particular canonical form. The complexity is evident. The challenging task ahead for the readers is to derive similar expressions for other canonical forms.

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