Lattices of Smarandache Groupoid

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Abstract :

Smarandache groupoid (Zp, Δ) is not partly ordered under Smarandache inclusion relation but it contains some partly ordered sets, which are lattices under Smarandache union and intersection. We propose to establish the complemented and distributive lattices of Smarandache groupoid. Some properties of these lattices are discussed here.

1. Preliminaries :

The following definitions and properties are recalled to introduce complemented and distributive lattices of Smarandache groupoid .

Definition 1.1

A set S is partly ordered with respect to a binary relation R if this relation on S is reflexive, antisymmetric and transitive.

Definition 1.2

Two partly ordered sets S_1 and S_2 are isomorphic if there exists a one - one correspondence T between S_1 and S_2 such that for $x \in S_1$ and $y \in S_1$

$T(x) \subseteq T(y)$ iff $x \subseteq y$

Definition 1.3

A lattice is a partly ordered set in which any two elements x and y have a greatest lower bound or infimum denoted by $x \cap y$ and a least upper bound or supremum denoted by $x \cup y$.

Definition 1.4

If every element of lattice has a complement , then it is called complemented lattice.

Definition 1.5

A lattice L is called distributive if identically

$$x \cap (y \cup z) = (x \cap y) \cup (x \cap z), \forall x, y, z \in L.$$

Definition 1.6

If a lattice L is distributive and complemented then it is called a Boolean lattice.

2. Lattices of Smarandache groupiod :

We introduce some definitions to establish the lattices of Smarandache groupoid.

Definition 2.1

i) Two integer $r = (a_{n-1} a_{n-2} \ldots\ldots a_1 a_0)_m$ and $s = (b_{n-1} b_{n-2} \ldots\ldots b_1 b_0)_m$ are said to be equal and written as $r = s$ if $a_i = b_i$ for $i = 0, 1, 2, \ldots\ldots, n - 1$.

ii) The integer $r = (a_{n-1} a_{n-2} \ldots\ldots a_1 a_0)_m$ is contained in the integer $s = (b_{n-1} b_{n-2} \ldots b_1 b_0)_m$ and written as $r \subseteq s$ if $a_i \leq b_i$ for $i = 0, 1, \ldots, n - 1$. This relation is called Smarandache inclusion relation.

iii) The Smarandache union of two integers r and s is denoted by $r \cup s$ and defined as

$$r \cup s = (a_{n-1} a_{n-2} \ldots a_1 a_0) \cup (b_{n-1} b_{n-2} \ldots b_1 b_0) = (c_{n-1} c_{n-2} \ldots c_1 c_0)$$

where $c_i = \max\{a_i, b_i\}$ for $i = 0, 1, \ldots, n - 1$.

iv) The Smarandache intersection of two integers r and s is denoted by $r \cap s$ and defined as-

$$r \cap s = (a_{n-1} a_{n-2} \ldots a_1 a_0) \cap (b_{n-1} b_{n-2} \ldots b_1 b_0) = (d_{n-1} d_{n-2} \ldots d_1 d_0)$$

Where $d_i = \min\{a_i, b_i\}$ for $i = 0, 1, 2, \ldots\ldots, n - 1$.

v) The complement of the integer $r = (a_{n-1} a_{n-2} \ldots\ldots a_1 a_0)_m$ is redefined as -

$$C(r) = (e_{n-1} e_{n-2} \ldots \ldots e_1 e_0)_m$$

Where $e_i = 1 - a_i$ for $i = 0, 1, \ldots\ldots, n - 1$.

Proposition 2.2

The Smarandache groupoid (Zp, Δ) with two operations Smarandache union and intersection satisfies the following properties for x, y, z ∈ Zp.

i) Idempotency : $x \cup x = x$ and $x \cap x = x$
ii) Commutativity : $x \cup y = y \cup x$ and $x \cap y = y \cap x$
iii) Associativity : $(x \cup y) \cup z = x \cup (y \cup z)$ and $(x \cap y) \cap z = x \cap (y \cap z)$.
iv) Absorption : $x \cup (x \cap y) = x = x \cap (x \cup y)$ if $x \subseteq y$.

But (Zp, Δ) is not partly ordered with respect to Smarandache inclusion relation and this groupoid consists of some partly ordered sets . Any two elements x and y of any partly ordered set of (Zp, Δ) have infimum $x \cap y$ and supremum $x \cup y$. So these partly ordered sets are lattices of Smarandache groupoid (Zp, Δ). This can be verified with an example of Smarandache groupoid.

Example - 2. 3

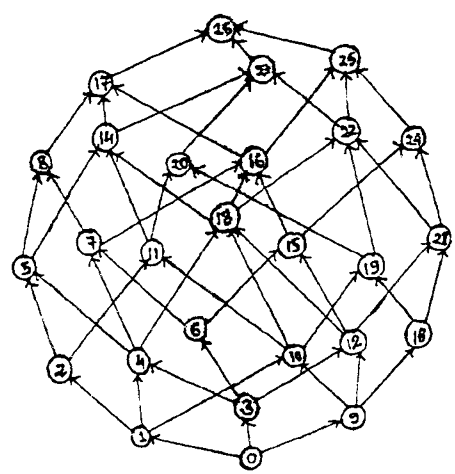
The Smarandache groupoid (Z_{27}, Δ) is taken for verification.

Here $Z_{27} = \{0, 1, 2, \ldots\ldots, 26\}$. For all x, y ∈ Z_{27}, x is not contained and equal to y under Smarandache inclusion relation. For example

$$11 = (1\ 0\ 2)_3 \text{ and } 13 = (1\ 1\ 1)_3 \in Z_{27}$$

But $11 \not\subseteq 13$ under Smarandache inclusion relation. All the elements Z_{27} are not related. so reflexive, antisymmetric and transitive laws are not satisfied . Z_{27} is not treated as lattice under Smarandache inclusion relation.

Under this inclusion relation, some partly ordered sets are contained in Z_{27}. About 87 partly ordered sets of seven elements are determined in the Smarandache groupoid (Z_{27}, Δ). A diagram of the above 87 partly ordered sets are given below :



Consider a partly ordered set L, given by $0 \subseteq 1 \subseteq 2 \subseteq 5 \subseteq 8 \subseteq 17 \subseteq 26$ of Smarandache groupoid (Z_{27}, Δ). The Smarandache intersection and union tables of this partly ordered set are given below :

$\cap$	0	1	2	5	8	17	26
0	0	0	0	0	0	0	0
1	0	1	1	1	1	1	1
2	0	1	2	2	2	2	2
5	0	1	2	5	5	5	5
8	0	1	2	5	8	8	8
17	0	1	2	5	8	17	17
26	0	1	2	5	8	17	26

Table -1

$\cup$	0	1	2	5	8	17	26
0	0	1	2	5	8	17	26
1	1	1	2	5	8	17	26
2	2	2	2	5	8	17	26
5	5	5	5	5	8	17	26
8	8	8	8	8	8	17	26
17	17	17	17	17	17	17	26
26	26	26	26	26	26	26	26

Table -2

The system $(L, \subseteq, \cap, \cup)$ in which any two elements a and b have an infimum $a \cap b$ and a supremum $a \cup b$ is a lattice. Similarly, taking the other 86 partly ordered sets, we can show that they are lattices of the Smarandache groupoid (Z_{27}, Δ). If we take the complement of every element of the lattice L, we get the following function.

L =	0	1	2	5	8	17	26
C(L) =	26	25	24	21	18	9	0

Here $L \neq C(L)$. But the system $(C(L), \subseteq, \cap, \cup)$ is a lattice. If $L = C(L)$, then the lattice $(L, \subseteq, \cap, \cup)$ is called complemented. The complemented lattices of seven elements belonging to (Z_{27}, Δ) are given below :

$0 \subseteq 1 \subseteq 4 \subseteq 13 \subseteq 22 \subseteq 25 \subseteq 26$
$0 \subseteq 1 \subseteq 10 \subseteq 13 \subseteq 16 \subseteq 25 \subseteq 26$
$0 \subseteq 3 \subseteq 4 \subseteq 13 \subseteq 22 \subseteq 23 \subseteq 26$
$0 \subseteq 3 \subseteq 12 \subseteq 13 \subseteq 14 \subseteq 23 \subseteq 26$
$0 \subseteq 9 \subseteq 10 \subseteq 13 \subseteq 16 \subseteq 17 \subseteq 26$
$0 \subseteq 9 \subseteq 12 \subseteq 13 \subseteq 14 \subseteq 17 \subseteq 26$

From table 1 and table 2, it is clear that $a \cup (b \cap c) = (a \cup b) \cap (a \cup c)$ and $a \cap (b \cup c) = (a \cap b) \cup (a \cap c)$ $\forall$ a, b, c, $\in$ L.

Hence the lattice $(L, \subseteq, \cap, \cup)$ is a distributive lattice. Similarly we can show that the other lattices of (Z_{27}, Δ) are distributive. The above six complemented lattices are distributive and they are called Boolean lattices.

Remark - i) The ordinary intersection of two lattices is a lattice .
ii) The ordinary union of two lattices is not a lattice .

Proposition 2.4

Every Smarandache groupoid has a lattice.

Proof :

Let (Zp, Δ) be a Smarandache groupoid. A partly ordered set L_1 of Zp is determined with respect to the Smarandache inclusion relation.

Let $L_1 = \{0 = I_{10} \subseteq I_{11} \subseteq I_{12} \subseteq \ldots\ldots\ldots \subseteq I_{1p} = m^n-1\}$

For $I_{1i}, I_{ij} \in L_1$ we get
$I_{1i} \subseteq I_{1j}$ or $I_{1i} \supseteq I_{1j}$

case I : If $I_{1i} \subseteq I_{1j}$, then
$I_{1i} \cap I_{1j} = I_{1i} \in L_1$ and $I_{1i} \cup I_{1j} = I_{1j} \in L_1$

Hence L_1 is a lattice of Zp .

Case II : If $I_{1i} \supseteq I_{1j}$, then
$I_{1i} \cap I_{1j} = I_{1i} \in L_1$ and $I_{1i} \cup I_{1j} = I_{1j} \in L_1$

Hence L_1 is a lattice of Zp .

Proposition 2.5

Every distributive lattice is modular.

Proof : A modular lattice is defined as a lattice in which
$z \subseteq x$ implies $x \cup (y \cap z) = (x \cup y) \cap z$

Let $(L_1, \subseteq, \cap, \cup)$ be a distributive lattice, in which $I_{1i} \subseteq I_{1k}$,

$$\text{then} \quad I_{1i} \cup (I_{1j} \cap I_{1k}) = (I_{1i} \cup I_{1j}) \cap (I_{1i} \cup I_{1k}) = (I_{1i} \cup I_{1j}) \cap I_{1k}$$

Hence $(L_1, \subseteq, \cap, \cup)$ is modular.

3. Isomorphic lattices :

Let L_1 and L_2 be two lattices of a Smarandache groupoid (Zp, Δ). A one - one mapping T from L_1 onto L_2 is said to be isomorphism if -

$$T(x \cup y) = T(x) \cup T(y) \text{ and}$$
$$T(x \cap y) = T(x) \cap T(Y) \text{ for } x, y \in L_1.$$

Proposition . 3. 1

Two lattices having same number of elements of a smarandache groupoid (Zp, Δ) are isomorphic to each other.

Proof : Let $L_1 = \{ I_{10} \subseteq I_{11} \subseteq I_{12} \subseteq \ldots\ldots\ldots\ldots \subseteq I_{1p} \}$
and $L_2 = \{ I_{20} \subseteq I_{21} \subseteq I_{22} \subseteq \ldots\ldots\ldots\ldots \subseteq I_{2p} \}$

where $I_{10} = I_{20} = 0$ and $I_{1p} = I_{2p} = m^n - 1$ be two lattices of (Zp, Δ).

A one - one onto mapping $T : L_1 \rightarrow L_2$ is defined such that $T(I_{1i}) = I_{2i}$ for all $I_{1i} \in L_1$

For $I_{1i} \subseteq I_{1j} \in L_1$,

$I_{1i} \cup I_{1j} = I_{1j}$ and $I_{1i} \cap I_{1j} = I_{1i}$.

For $I_{2i} \subseteq I_{2j} \in L_2$,

$I_{2i} \cup I_{2j} = I_{2j}$ and $I_{2i} \cap I_{2j} = I_{2i}$.

Again $T(I_{1i}) = I_{2i}$ and $T(I_{1j}) = I_{2j}$

Now $T(I_{1i}) \cup T(I_{1j}) = I_{2i} \cup I_{2j} = I_{2j}$ and
$T(I_{1i}) \cap T(I_{1j}) = I_{2i} \cap I_{2j} = I_{2i}$.

Here $T(I_{1i} \cup I_{1j}) = T(I_{1j}) = I_{2j} = T(I_{1i}) \cup T(I_{1j})$ and
$T(I_{1i} \cap I_{1j}) = T(I_{1i}) = I_{2i} = T(I_{1i}) \cap T(I_{1j})$

Hence the lattices L_1 and L_2 are isomorphic to each other.

Proposition 3.2

Let L and C(L) be two lattices of Smarandache groupoid (Zp, Δ). If T be the mapping from L to C(L), defined by T(x) = C(x) ∀ x ∈ L, then

$$T(x \cup y) = T(x) \cap T(y) \text{ and}$$
$$T(x \cap y) = T(x) \cup T(y) \ \forall\, x, y \in L.$$

Proof : For $x \subseteq y \in L$, $x \cup y = y$ and $x \cap y = x$

Again $C(y) \subseteq C(x) \in C(L)$, $C(x) \cup C(y) = C(x)$ and $C(x) \cap C(y) = C(y)$

Here $T(x) = C(x)$ and $T(y) = C(y)$

Now $T(x) \cup T(y) = C(x) \cup C(y) = C(x)$ and
$T(x) \cap T(y) = C(x) \cap C(y) = C(y)$.

Again $T(x \cup y) = T(y) = C(y) = T(x) \cap T(y)$ and
$T(x \cap y) = T(x) = C(x) = T(x) \cup T(y)$

Proposition - 3. 3

Let L be a complemented lattice of (Zp, Δ). If the mapping T from L to L, defined by $T(x) = C(x) \quad \forall \quad x \in L$, then

$$T(x \cup y) = T(x) \cap T(y) \quad \text{and}$$
$$T(x \cap y) = T(x) \cup T(y) \quad \forall \quad x, y \in L.$$

Proof is similar to proposition 3.2

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