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On the F.Smarandache LCM function and its mean value

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Abstract For any positive integer n, the F.Smarandache LCM function $SL(n)$ defined as the smallest positive integer k such that $n \mid [1,\ 2,\ \cdots,\ k]$, where $[1,\ 2,\ \cdots,\ k]$ denotes the least common multiple of $1,\ 2, \cdots, k$. The main purpose of this paper is to use the elementary methods to study the mean value of the F.Smarandache LCM function $SL(n)$, and give a sharper asymptotic formula for it.

Keywords F.Smarandache LCM function, mean value, asymptotic formula.

§1. Introduction and results

For any positive integer n, the famous F.Smarandache LCM function $SL(n)$ defined as the smallest positive integer k such that $n \mid [1,\ 2,\ \cdots,\ k]$, where $[1,\ 2,\ \cdots,\ k]$ denotes the least common multiple of $1,\ 2,\ \cdots,\ k$. For example, the first few values of $SL(n)$ are $SL(1) = 1$, $SL(2) = 2$, $SL(3) = 3$, $SL(4) = 4$, $SL(5) = 5$, $SL(6) = 3$, $SL(7) = 7$, $SL(8) = 8$, $SL(9) = 9$, $SL(10) = 5$, $SL(11) = 11$, $SL(12) = 4$, $SL(13) = 13$, $SL(14) = 7$, $SL(15) = 5$, $\cdots$. About the elementary properties of $SL(n)$, some authors had studied it, and obtained some interesting results, see reference [3] and [4]. For example, Murthy [3] showed that if n is a prime, then $SL(n) = S(n)$, where $S(n)$ denotes the Smarandache function, i.e., $S(n) = \min\{m :\ n|m!,\ m \in N\}$. Simultaneously, Murthy [3] also proposed the following problem:

$$SL(n) = S(n), \quad S(n) \neq n\ ? \tag{1}$$

Le Maohua [4] completely solved this problem, and proved the following conclusion:

Every positive integer n satisfying (1) can be expressed as

$$n = 12 \ \ \texttt{or} \ \ n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} p,$$

where $p_1,\ p_2,\ \cdots,\ p_r,\ p$ are distinct primes, and $\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_r$ are positive integers satisfying $p > p_i^{\alpha_i}$, $i = 1,\ 2,\ \cdots,\ r$.

The main purpose of this paper is to use the elementary methods to study the mean value properties of $SL(n)$, and obtain a sharper asymptotic formula for it. That is, we shall prove the following conclusion:

Theorem. Let $k \geq 2$ be a fixed integer. Then for any real number $x > 1$, we have the asymptotic formula

$$\sum_{n\leq x} SL(n) = \frac{\pi^2}{12}\cdot\frac{x^2}{\ln x} + \sum_{i=2}^{k}\frac{c_i\cdot x^2}{\ln^i x} + O\left(\frac{x^2}{\ln^{k+1} x}\right),$$

where c_i $(i = 2,\ 3, \cdots, k)$ are computable constants.

From our Theorem we may immediately deduce the following:

Corollary. For any real number $x > 1$, we have the asymptotic formula

$$\sum_{n\leq x} SL(n) = \frac{\pi^2}{12}\cdot\frac{x^2}{\ln x} + O\left(\frac{x^2}{\ln^2 x}\right).$$

§2. Proof of the theorems

In this section, we shall prove our theorem directly. In fact for any positive integer $n > 1$, let $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_s^{\alpha_s}$ be the factorization of n, then from [3] we know that

$$SL(n) = \max\{p_1^{\alpha_1},\ p_2^{\alpha_2},\ \cdots,\ p_s^{\alpha_s}\}. \tag{2}$$

Now we consider the summation

$$\sum_{n\leq x} SL(n) = \sum_{n\in A} SL(n) + \sum_{n\in B} SL(n), \tag{3}$$

where we have divided the interval $[1,\ x]$ into two sets A and B. A denotes the set involving all integers $n \in [1,\ x]$ such that there exists a prime p with $p|n$ and $p > \sqrt{n}$. And B denotes the set involving all integers $n \in [1,\ x]$ with $n \notin A$. From (2) and the definition of A we have

$$\sum_{n\in A} SL(n) = \sum_{\substack{n\leq x\\ p|n,\ \sqrt{n}<p}} SL(n) = \sum_{\substack{pn\leq x\\ n<p}} SL(pn) = \sum_{\substack{pn\leq x\\ n<p}} p = \sum_{n\leq\sqrt{x}}\sum_{n<p\leq\frac{x}{n}} p. \tag{4}$$

By Abel's summation formula (See Theorem 4.2 of [5]) and the Prime Theorem (See Theorem 3.2 of [6]):

$$\pi(x) = \sum_{i=1}^{k}\frac{a_i\cdot x}{\ln^i x} + O\left(\frac{x}{\ln^{k+1} x}\right),$$

where a_i $(i = 1,\ 2, \cdots, k)$ are constants and $a_1 = 1$.

We have

$$\begin{aligned}\sum_{n<p\leq\frac{x}{n}} p &= \frac{x}{n}\cdot\pi\left(\frac{x}{n}\right) - n\cdot\pi(n) - \int_n^{\frac{x}{n}}\pi(y)dy\\ &= \frac{x^2}{2n^2\ln x} + \sum_{i=2}^{k}\frac{b_i\cdot x^2\cdot\ln^i n}{n^2\cdot\ln^i x} + O\left(\frac{x^2}{n^2\cdot\ln^{k+1} x}\right),\end{aligned} \tag{5}$$

where we have used the estimate $n \leq \sqrt{x}$, and all b_i are computable constants.

Note that $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$, and $\sum_{n=1}^{\infty}\frac{\ln^i n}{n^2}$ is convergent for all $i=2,\ 3,\ \cdots,\ k$. From (4) and (5) we have

$$\begin{aligned}\sum_{n\in A}SL(n) &= \sum_{n\le\sqrt{x}}\left(\frac{x^2}{2n^2\ln x}+\sum_{i=2}^{k}\frac{b_i\cdot x^2\cdot\ln^i n}{n^2\cdot\ln^i x}+O\left(\frac{x^2}{n^2\cdot\ln^{k+1}x}\right)\right)\\ &= \frac{\pi^2}{12}\cdot\frac{x^2}{\ln x}+\sum_{i=2}^{k}\frac{c_i\cdot x^2}{\ln^i x}+O\left(\frac{x^2}{\ln^{k+1}x}\right), \qquad (6)\end{aligned}$$

where c_i $(i=2,\ 3,\cdots,k)$ are computable constants.

Now we estimate the summation in set B. Note that for any positive integer α, the series $\sum_{n=1}^{\infty}\frac{1}{n^{\frac{\alpha+1}{\alpha}}}$ is convergent, so from (2) and the definition of B we have

$$\begin{aligned}\sum_{n\in B}SL(n) &= \sum_{\substack{n\le x\\ SL(n)=p,\ p\le\sqrt{n}}} p+\sum_{\substack{n\le x\\ SL(n)=p^\alpha,\ \alpha>1}} p^\alpha\\ &\ll \sum_{\substack{n\le x\\ p|n,\ p\le\sqrt{n}}} p+\sum_{2\le\alpha\le\ln x}\sum_{p\le x}\sum_{np^\alpha\le x}p^\alpha\\ &\ll \sum_{n\le x}\sum_{p\le\min\{n,\ \frac{x}{n}\}}p+\sum_{2\le\alpha\le\ln x}\sum_{n\le x}\sum_{p\le\left(\frac{x}{n}\right)^{\frac{1}{\alpha}}}p^\alpha\\ &\ll \frac{x^{\frac{3}{2}}}{\ln x}+\frac{x^{\frac{3}{2}}}{\ln x}\cdot\ln x\ll x^{\frac{3}{2}}. \qquad (7)\end{aligned}$$

Combining (3), (6) and (7) we may immediately deduce that

$$\sum_{n\le x}SL(n)=\frac{\pi^2}{12}\cdot\frac{x^2}{\ln x}+\sum_{i=2}^{k}\frac{c_i\cdot x^2}{\ln^i x}+O\left(\frac{x^2}{\ln^{k+1}x}\right),$$

where c_i $(i=2,\ 3,\cdots,k)$ are computable constants.

This completes the proof of Theorem.

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