Scientia Magna
Vol. 5 (2009), No. 3, 87-90

A limit problem of the Smarandache dual function $S^{**}(n)$[1]

Qiuhong Zhao† and Yang Wang‡

†Department of Mathematics, Northwest University, Xi'an, Shaanxi, P.R.China
‡College of Mathematics and Statistics, Nanyang Normal University,
Nanyang, Henan, P.R.China

Abstract For any positive integer n, the Smarandache dual function $S^{**}(n)$ is defined as

$$S^{**}(n) = \begin{cases} \max\{2m:\ m \in N^*,\ (2m)!! \mid n\}, & 2 \mid n; \\ \max\{2m-1:\ m \in N^*,\ (2m-1)!! \mid n\}, & 2 \nmid n. \end{cases}$$

The main purpose of this paper is using the elementary methods to study the convergent properties of an infinity series involving $S^{**}(n)$, and give an interesting limit formula for it.

Keywords The Smarandache dual function, limit problem, elementary method.

§1. Introduction and Results

For any positive integer n, the Smarandache dual function $S^{**}(n)$ is defined as the greatest positive integer $2m-1$ such that $(2m-1)!!$ divide n, if n is an odd number; $S^{**}(n)$ is the greatest positive $2m$ such that $(2m)!!$ divides n, if n is an even number. From the definition of $S^{**}(n)$ we know that the first few values of $S^{**}(n)$ are: $S^{**}(1) = 1$, $S^{**}(2) = 2$, $S^{**}(3) = 3$, $S^{**}(4) = 2$, $S^{**}(5) = 1$, $S^{**}(6) = 2$, $S^{**}(7) = 1$, $S^{**}(8) = 4$, $\cdots$. About the elementary properties of $S^{**}(2)$, some authors had studied it, and obtained many interesting results. For example, Su Gou [1] proved that for any real number $s > 1$, the series $\sum_{n=1}^{\infty} \frac{S^{**}(n)}{n^s}$ is absolutely convergent, and

$$\sum_{n=1}^{\infty} \frac{S^{**}(n)}{n^s} = \zeta(s)\left(1-\frac{1}{2^s}\right)\left(1+\sum_{m=1}^{\infty}\frac{2}{((2m+1)!!)^s}\right) + \zeta(s)\left(\sum_{m=1}^{\infty}\frac{2}{((2m)!!)^s}\right),$$

where $\zeta(s)$ is the Riemann zeta-function.

Yanting Yang [2] studied the mean value estimate of $S^{**}(n)$, and gave an interesting asymptotic formula:

$$\sum_{n\le x} S^{**}(n) = x\left(2e^{\frac{1}{2}} - 3 + 2e^{\frac{1}{2}}\int_0^1 e^{-\frac{y^2}{2}}dy\right) + O(\ln^2 x),$$

where $e = 2.7182818284\cdots$ is a constant.

[1]This work is supported by the Shaanxi Provincial Education Department Foundation 08JK433.

Yang Wang [3] also studied the mean value properties of $S^{**}(n)^2$, and prove that

$$\sum_{n\leq x} S^{**}(n)^2 = \frac{13x}{2} + O\left(\left(\frac{\ln x}{\ln\ln x}\right)^3\right).$$

In this paper, we using the elementary method to study the convergent properties of the series

$$\sum_{n=1}^{\infty} \frac{S^{**}(n)^2}{n^s},$$

and give an interesting identity and limit theorem. That is, we shall prove the following:
Theorem. For any real number $s > 1$, we have the identity

$$\sum_{n=1}^{\infty} \frac{S^{**}(n)^2}{n^s} = \zeta(s)\left[1 - \frac{1}{2^s} + \left(1 - \frac{1}{2^s}\right)\sum_{m=1}^{\infty} \frac{8m}{((2m+1)!!)^s} + \sum_{m=1}^{\infty} \frac{8m-4}{((2m)!!)^s}\right],$$

where $\zeta(s)$ is the Riemann zeta-function.

From this Theorem we may immediately deduce the following limit formula:

Corollary. We have the limit

$$\lim_{s\to 1}(s-1)\left(\sum_{n=1}^{\infty} \frac{S^{**}(n)^2}{n^s}\right) = \frac{13}{2}.$$

§2. Proof of the theorem

In this section, we shall complete the proof of our theorem directly. It is clear that $S^{**}(n) \ll \ln n$, so if $s > 1$, then the series $\sum\limits_{n=1}^{\infty} \frac{S^{**}(n)^2}{n^s}$ is convergent absolutely, so we have

$$\sum_{n=1}^{\infty} \frac{S^{**}(n)^2}{n^s} = \sum_{\substack{n=1\\ 2\nmid n}}^{\infty} \frac{S^{**}(n)^2}{n^s} + \sum_{\substack{n=1\\ 2\mid n}}^{\infty} \frac{S^{**}(n)^2}{n^s} \equiv S_1 + S_2,$$

where

$$S_1 = \sum_{\substack{n=1\\ 2\nmid n}}^{\infty} \frac{S^{**}(n)^2}{n^s}, \quad S_2 = \sum_{\substack{n=1\\ 2\mid n}}^{\infty} \frac{S^{**}(n)^2}{n^s}.$$

From the definition of $S^{**}(n)$ we know that if $2 \nmid n$, we can assume that $S^{**}(n) = 2m-1$, then $(2m-1)!! \mid n$. Let $n = (2m-1)!!u$, $2m+1 \nmid u$. Note that the identity

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^s} = \sum_{n=1}^{\infty} \frac{1}{n^s} - \sum_{n=1}^{\infty} \frac{1}{(2n)^s} = \left(1 - \frac{1}{2^s}\right)\sum_{n=1}^{\infty} \frac{1}{n^s} = \left(1 - \frac{1}{2^s}\right)\zeta(s),$$

so from the definition of $S^{**}(n)$ we can deduce that ($s > 1$),

$$\begin{aligned}
S_1 &= \sum_{m=1}^{\infty} \sum_{\substack{u=1,\ 2\nmid u \\ 2m+1\nmid u}}^{\infty} \frac{(2m-1)^2}{((2m-1)!!)^s\, u^s} \\
&= \sum_{m=1}^{\infty} \frac{(2m-1)^2}{((2m-1)!!)^s} \sum_{\substack{u=1,\ 2\nmid u \\ 2m+1\nmid u}}^{\infty} \frac{1}{u^s} \\
&= \sum_{m=1}^{\infty} \frac{(2m-1)^2}{((2m-1)!!)^s} \left(\sum_{n=1}^{\infty} \frac{1}{(2n-1)^s} - \frac{1}{(2m+1)^s} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s} \right) \\
&= \zeta(s)\left(1-\frac{1}{2^s}\right) \left(\sum_{m=1}^{\infty} \frac{(2m-1)^2}{((2m-1)!!)^s} - \sum_{m=1}^{\infty} \frac{(2m-1)^2}{((2m+1)!!)^s} \right) \\
&= \zeta(s)\left(1-\frac{1}{2^s}\right) \left(1 + \sum_{m=1}^{\infty} \frac{(2m+1)^2-(2m-1)^2}{((2m+1)!!)^s} \right) \\
&= \zeta(s)\left(1-\frac{1}{2^s}\right) \left(1 + \sum_{m=1}^{\infty} \frac{8m}{((2m+1)!!)^s} \right).
\end{aligned}$$

For even number n, we assume that $S^{**}(n) = 2m$, then $(2m)!! \mid n$. Let $n = (2m)!!v$, $2m+2 \nmid v$. If $s > 1$, then we can deduce that

$$\begin{aligned}
S_2 &= \sum_{m=1}^{\infty} \sum_{\substack{v=1 \\ 2m+2\nmid v}}^{\infty} \frac{(2m)^2}{((2m)!!)^s v^s} \\
&= \sum_{m=1}^{\infty} \frac{(2m)^2}{((2m)!!)^s} \sum_{\substack{v=1 \\ (2m+2)\nmid v}}^{\infty} \frac{1}{v^s} \\
&= \sum_{m=1}^{\infty} \frac{(2m)^2}{((2m)!!)^s} \left(\sum_{n=1}^{\infty} \frac{1}{n^s} - \frac{1}{(2m+2)^s} \sum_{n=1}^{\infty} \frac{1}{n^s} \right) \\
&= \zeta(s) \left(\sum_{m=1}^{\infty} \frac{(2m)^2}{((2m)!!)^s} - \sum_{m=1}^{\infty} \frac{(2m)^2}{((2m+2)!!)^s} \right) \\
&= \zeta(s) \left(\frac{1}{2^{s-2}} + \sum_{m=1}^{\infty} \frac{(2m+2)^2-(2m)^2}{((2m+2)!!)^s} \right) \\
&= \zeta(s) \left(\frac{1}{2^{s-2}} + \sum_{m=1}^{\infty} \frac{8m+4}{((2m+2)!!)^s} \right) \\
&= 4\zeta(s) \sum_{m=1}^{\infty} \frac{2m-1}{((2m)!!)^s}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{S^{**}(n)^2}{n^s} = S_1 + S_2\\
= &\ \zeta(s)\left(1-\frac{1}{2^s}\right)\left(1+\sum_{m=1}^{\infty}\frac{8m}{((2m+1)!!)^s}\right)+4\zeta(s)\sum_{m=1}^{\infty}\frac{2m-1}{((2m)!!)^s}\\
= &\ \zeta(s)\left[1-\frac{1}{2^s}+\left(1-\frac{1}{2^s}\right)\sum_{m=1}^{\infty}\frac{8m}{((2m+1)!!)^s}+\sum_{m=1}^{\infty}\frac{8m-4}{((2m)!!)^s}\right].
\end{aligned}$$

This completes the proof of our Theorem.

Now we prove Corollary, note that

$$\begin{aligned}
&\frac{1}{2}+\sum_{m=1}^{\infty}\frac{4m}{(2m+1)!!}+\sum_{m=1}^{\infty}\frac{8m-4}{(2m)!!}\\
= &\ \frac{1}{2}+\sum_{m=1}^{\infty}\left(\frac{2}{(2m-1)!!}-\frac{2}{(2m+1)!!}\right)+\sum_{m=1}^{\infty}\left(\frac{4}{(2m-2)!!}-\frac{4}{(2m+2)!!}\right)\\
= &\ \frac{1}{2}+2+4=\frac{13}{2}
\end{aligned}$$

and

$$\lim_{s\to 1}(s-1)\zeta(s)=1,$$

from Theorem we may immediately deduce that

$$\begin{aligned}
&\lim_{s\to 1}(s-1)\left(\sum_{n=1}^{\infty}\frac{S^{**}(n)}{n^s}\right)\\
= &\ \lim_{s\to 1}(s-1)\zeta(s)\left[1-\frac{1}{2^s}+\left(1-\frac{1}{2^s}\right)\sum_{m=1}^{\infty}\frac{8m}{((2m+1)!!)^s}+\sum_{m=1}^{\infty}\frac{8m-4}{((2m)!!)^s}\right]\\
= &\ \frac{1}{2}+\sum_{m=1}^{\infty}\frac{4m}{(2m+1)!!}+\sum_{m=1}^{\infty}\frac{8m-4}{(2m)!!}=\frac{13}{2}.
\end{aligned}$$

This completes the proof of Corollary.

References

[1] Gou Su, On the Smarandache dual function, Pure and Applied Mathematics, **24**(2008), No. 1, 17-20.

[2] Yang Yanting, On the mean value of a number theoretic function, Journal of Natural Science of Heilongjiang University , **25**(2008), No. 3, 340-342.

[3] Wang Yang, On the quadratic mean value of the Smarandache dual function $S^{**}(n)$, Research on Number Theory and Smarandache Notions, Hexis, 2009, 109-115.

[4] Zhang Wenpeng, The elementary number theory (in Chinese), Shaanxi Normal University Press, Xi'an, 2007.

[5] Li Jianghua and Guo Yanchun, Research on Smarandache Unsolved Problems (in Chinese), High American Press, 2009.