ON A LIMIT OF A SEQUENCE OF THE NUMERICAL FUNCTION

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In this paper is studied the limit of the following sequence:

$$T(n) = 1 - \log \sigma_S(n) + \sum_{i=1}^{n} \sum_{k=1}^{n} \frac{1}{\sigma_S(p_i^K)}$$

We shall demonstrate that $\lim\limits_{n\to\infty} T(n) = -\infty$.

We shal consider define the sequence $p_1 = 2, p_2 = 3, \ldots, p_n =$ the nth prime number and the function $\sigma_S : \mathbf{N}^* \to \mathbf{N}$, $\sigma_S(x) = \sum\limits_{\substack{d|x \\ d>0}} S(d)$, where S is Smarandache Function.

For example: $\sigma_S(18) = S(1) + S(2) + S(3) + S(6) + S(9) + S(18) =$ 0+2+3+3+6+6=20

We consider the natural number p_m^n, where p_m is a prime number. It is known that $(p-1)r + 1 \le S(p^r) \le pr$ so $S(p^r) > (p-1)r$.

Next, we can write $\sigma_S(p^r) = \sum\limits_{s=0}^{r} S(p^s) > \sum\limits_{s=0}^{r} (p-1)s = (p-1)\dfrac{r(r+1)}{2}$

$$\sigma_S(p_i^k) > (p_i - 1)\frac{k(k+1)}{2}, \quad \forall i \in \{1, \ldots, m\}, \ \forall k \in \{1, \ldots, n\}.$$

$$\frac{1}{\sigma_S(p_i^k)} < \frac{2}{(p_i - 1)k(k+1)}$$

This involves that:

$$\sum_{i=1}^{m} \sum_{k=1}^{n} \frac{1}{\sigma_S(p_i^k)} < \sum_{i=1}^{m} \sum_{k=1}^{n} \frac{2}{(p_i - 1)k(k+1)} = \left(\sum_{i=1}^{m} \frac{1}{p_i - 1} \right) \cdot \left(\sum_{k=1}^{n} \frac{2}{k(k-1)} \right)$$

$\sigma_S(k) > 0$, $\forall k \ge 2$ and $p_a^b \le p_m^n$ if $a \le m$ and $b \le n$ and $p_a^b = p_c^d$ if $a = c$ and $b = d$.

But $\sigma_S(p_m^n) > (p_m - 1)\dfrac{n(n+1)}{2}$ implies that $-\log \sigma_S(p_m^n) < -\log(p_m - 1)\dfrac{n(n+1)}{2}$ because $\log x$ is strictly increasing from 2 to $+\infty$.

Next, using inequality (1) we obtain

$$T(p_m^n) = 1 - \log \sigma_S(p_m^n) + \sum_{i=1}^{m} \sum_{k=1}^{n} \frac{1}{\sigma_S(p_i^k)} < 1 - \log(p_m - 1)\frac{n(n+1)}{2} +$$

$$+\left(\sum_{k=1}^{m}\frac{1}{p_k-1}\right)\cdot\left(\sum_{k=1}^{p_m}\frac{2}{k(k+1)}\right)$$

But $\sum_{k=1}^{p_m}\frac{2}{k(k+1)}=\frac{2p_m}{p_m+1} \Rightarrow T(p_m^{p_m})<1+\log 2-2\log p_m-\log(p_m-1)+$

$$+\frac{2p_m}{p_m+1}\sum_{k=1}^{m}\frac{1}{p_k-1}$$

$$T(p_m^{p_m}) < 1+\log 2+2\left(-\log p_m+\sum_{k=1}^{p_m}\frac{1}{k}\right)+\frac{2p_m}{p_m+1}\sum_{k=1}^{m}\frac{1}{p_k-1}-2\sum_{k=1}^{p_m}\frac{1}{k}-\log(p_m-1)$$

We have $\sum_{k=1}^{m}\frac{1}{p_k-1} \le \sum_{k=1}^{p_m}\frac{1}{k}$.

So: $T(p_m^{p_m})<1+\log 2+2\left(-\log p_m+\sum_{k=1}^{p_m}\frac{1}{k}\right)+2\sum_{k=1}^{p_m}\frac{1}{k}\left(\frac{p_m}{p_m+1}-1\right)-\log(p_m-1)$

And then $\lim_{m\to\infty} T(p_m^{p_m}) \le 1+\log 2+2\lim_{m\to\infty}(-\log p_m+\sum_{k=1}^{p_m}\frac{1}{k})-\lim_{m\to\infty}\left[2\left(\sum_{k=1}^{p_m}\frac{1}{k}\right)\frac{1}{p_m+1}\right]-$

$$-\lim_{m\to\infty}\log(p_m-1)=1+\log 2+2\lim_{p_m\to\infty}(-\log p_m+\sum_{k=1}^{p_m}\frac{1}{k})-\lim_{p_m\to\infty}\left[\frac{2}{p_m+1}\left(\sum_{k=1}^{p_m}\frac{1}{k}\right)\right]-$$

$$-\lim_{p_m\to\infty}\log(p_m-1)=1+\log 2+2\gamma-0-\infty=-\infty.$$

It is known that $\lim_{p_m\to\infty}\left(-\log p_m+\sum_{k=1}^{\infty}\frac{1}{k}\right)=\gamma$ (Euler's constant) and $\lim_{p_m\to\infty}\left(\frac{2}{p_m+1}\cdot\sum_{k=1}^{p_m}\frac{1}{k}\right)=0$.

In conclusion $\lim_{n\to\infty} T(n)=-\infty$.

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