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Linear Isometries on Pseudo-Euclidean Space $(\mathbb{R}^n, \mu)$

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Abstract: A *pseudo-Euclidean space* $(\mathbb{R}^n, \mu)$ is such a Euclidean space $\mathbb{R}^n$ associated with a mapping $\mu : \overrightarrow{V}_{\overline{x}} \to {}_{\overline{x}}\overrightarrow{V}$ for $\overline{x} \in \mathbb{R}^n$, and a linear isometry $T : (\mathbb{R}^n, \mu) \to (\mathbb{R}^n, \mu)$ is such a linear isometry $T : \mathbb{R}^n \to \mathbb{R}^n$ that $T\mu = \mu T$. In this paper, we characterize curvature of s-line, particularly, Smarandachely embedded graphs and determine linear isometries on $(\mathbb{R}^n, \mu)$.

Key Words: Smarandachely denied axiom, Smarandache geometry, s-line, pseudo-Euclidean space, isometry, Smarandachely map, Smarandachely embedded graph.

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§1. Introduction

As we known, a Smarandache geometry is defined following.

Definition 1.1 *A rule $R \in \mathcal{R}$ in a mathematical system $(\Sigma; \mathcal{R})$ is said to be Smarandachely denied if it behaves in at least two different ways within the same set Σ, i.e., validated and invalided, or only invalided but in multiple distinct ways.*

Definition 1.2 *A Smarandache geometry is such a geometry in which there are at least one Smarandachely denied ruler and a Smarandache manifold $(M; \mathcal{A})$ is an n-dimensional manifold M that support a Smarandache geometry by Smarandachely denied axioms in $\mathcal{A}$. A line in a Smarandache geometry is called an s-line.*

Applying the structure of a Euclidean space $\mathbb{R}^n$, we are easily construct a special Smarandache geometry, called pseudo-Euclidean space([5]-[6]) following. Let $\mathbb{R}^n = \{(x_1, x_2, \cdots, x_n)\}$ be a Euclidean space of dimensional n with a normal basis $\overline{\epsilon}_1 = (1, 0, \cdots, 0)$, $\overline{\epsilon}_2 = (0, 1, \cdots, 0)$, $\cdots$, $\overline{\epsilon}_n = (0, 0, \cdots, 1)$, $\overline{x} \in \mathbb{R}^n$ and $\overrightarrow{V}_{\overline{x}}$, ${}_{\overline{x}}\overrightarrow{V}$ two vectors with end or initial point at $\overline{x}$, respectively. A *pseudo-Euclidean space* $(\mathbb{R}^n, \mu)$ is such a Euclidean space $\mathbb{R}^n$ associated with a mapping $\mu : \overrightarrow{V}_{\overline{x}} \to {}_{\overline{x}}\overrightarrow{V}$ for $\overline{x} \in \mathbb{R}^n$, such as those shown in Fig.1,

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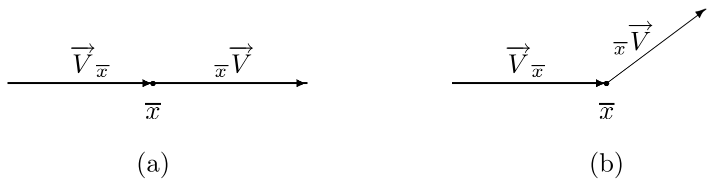

Fig.1

where $\overrightarrow{V}_{\overline{x}}$ and $_{\overline{x}}\overrightarrow{V}$ are in the same orientation in case (a), but not in case (b). Such points in case (a) are called *Euclidean* and in case (b) *non-Euclidean.* A pseudo-Euclidean $(\mathbb{R}^n, \mu)$ is *finite* if it only has finite non-Euclidean points, otherwise, *infinite.*

By definition, a Smarandachely denied axiom $A \in \mathcal{A}$ can be considered as an action of A on subsets $S \subset M$, denoted by S^A. If $(M_1; \mathcal{A}_1)$ and $(M_2; \mathcal{A}_2)$ are two Smarandache manifolds, where $\mathcal{A}_1$, $\mathcal{A}_2$ are the Smarandachely denied axioms on manifolds M_1 and M_2, respectively. They are said to be *isomorphic* if there is $1-1$ mappings $\tau : M_1 \to M_2$ and $\sigma : \mathcal{A}_1 \to \mathcal{A}_2$ such that $\tau(S^A) = \tau(S)^{\sigma(\mathcal{A})}$ for $\forall S \subset M_1$ and $A \in \mathcal{A}_1$. Such a pair (τ, σ) is called an isomorphism between $(M_1; \mathcal{A}_1)$ and $(M_2; \mathcal{A}_2)$. Particularly, if $M_1 = M_2 = M$ and $\mathcal{A}_1 = \mathcal{A}_2 = \mathcal{A}$, such an isomorphism (τ, σ) is called a *Smarandachely automorphism* of $(M, \mathcal{A})$. Clearly, all such automorphisms of $(M, \mathcal{A})$ form an group under the composition operation on τ for a given σ. Denoted by $\mathrm{Aut}(M, \mathcal{A})$. A special Smarandachely automorphism, i.e., linear isomorphism on a pseudo-Euclidean space $(\mathbb{R}^n, \mu)$ is defined following.

Definition 1.3 *Let $(\mathbb{R}^n, \mu)$ be a pseudo-Euclidean space with normal basis $\{\overline{\epsilon}_1, \overline{\epsilon}_2, \cdots, \overline{\epsilon}_n\}$. A linear isometry $T : (\mathbb{R}^n, \mu) \to (\mathbb{R}^n, \mu)$ is such a transformation that*

$$T(c_1\overline{e}_1 + c_2\overline{e}_2) = c_1T(\overline{e}_1) + c_2T(\overline{e}_2), \quad \langle T(\overline{e}_1), T(\overline{e}_2)\rangle = \langle \overline{e}_1, \overline{e}_2\rangle \quad \textit{and} \quad T\mu = \mu T$$

for $\overline{e}_1,\ \overline{e}_2 \in \mathbf{E}$ and $c_1,\ c_2 \in \mathscr{F}$.

Denoted by $\mathrm{Isom}(\mathbb{R}^n, \mu)$ the set of all linear isometries of $(\mathbb{R}^n, \mu)$. Clearly, $\mathrm{Isom}(\mathbb{R}^n, \mu)$ is a subgroup of $\mathrm{Aut}(M, \mathcal{A})$.

By definition, determining automorphisms of a Smarandache geometry is dependent on the structure of manifold M and axioms $\mathcal{A}$. So it is hard in general even for a manifold. The main purpose of this paper is to determine linear isometries and characterize the behavior of s-lines, particularly, Smarandachely embedded graphs in pseudo-Euclidean spaces $(\mathbb{R}^n, \mu)$. For terminologies and notations not defined in this paper, we follow references [1] for permutation group, [2]-[4] and [7]-[8] for graph, map and Smarandache geometry.

§2. Smarandachely Embedded Graphs in $(\mathbb{R}^n, \mu)$

2.1 Smarandachely Planar Maps

Let L be an s-line in a Smarandache plane $(\mathbf{R}^2, \mu)$ with non-Euclisedn points $A_1, A_2, \cdots, A_m$ for an integer $m \geq 0$. Its *curvature* $R(L)$ is defined by

$$R(L) = \sum_{i=1}^{m} (\pi - \mu(A_i)).$$

An s-line L is called *Euclidean* or *non-Euclidean* if $R(L) = \pm 2\pi$ or $\neq \pm 2\pi$. The following result characterizes s-lines on $(\mathbf{R}^2, \mu)$.

Theorem 2.1 *An s-line without self-intersections is closed if and only if it is Euclidean.*

Proof Let $(\mathbf{R}^2, \mu)$ be a Smarandache plane and let L be a closed s-line without self-intersections on $(\mathbf{R}^2, \mu)$ with vertices $A_1,\ A_2, \cdots, A_m$. From the Euclid geometry on plane, we know that the angle sum of an m-polygon is $(m-2)\pi$. Whence, the curvature $R(L)$ of s-line L is $\pm 2\pi$ by definition, i.e., L is Euclidean.

Now if an s-line L is Euclidean, then $R(L) = \pm 2\pi$ by definition. Thus there exist non-Euclidean points $B_1, B_2, \cdots, B_m$ such that

$$\sum_{i=1}^{m} (\pi - \mu(B_i)) = \pm 2\pi.$$

Whence, L is nothing but an n-polygon with vertices $B_1, B_2, \cdots, B_m$ on $\mathbf{R}^2$. Therefore, L is closed without self-intersection. □

A planar map is a 2-cell embedding of a graph G on Euclidean plane $\mathbb{R}^2$. It is called *Smarandachely* on $(\mathbb{R}^2, \mu)$ if all of its vertices are elliptic (hyperbolic). Notice that these pendent vertices is not important because it can be always Euclidean or non-Euclidean. We concentrate our attention to non-separated maps. Such maps always exist circuit-decompositions. The following result characterizes Smarandachely planar maps.

Theorem 2.2 *A non-separated planar map M is Smarandachely if and only if there exist a directed circuit-decomposition*

$$E_{\frac{1}{2}}(M) = \bigoplus_{i=1}^{s} E(\overrightarrow{C}_i)$$

of M such that one of the linear systems of equations

$$\sum_{v \in V(\overrightarrow{C}_i)} (\pi - x_v) = 2\pi, \quad \text{or} \quad \sum_{v \in V(\overrightarrow{C}_i)} (\pi - x_v) = -2\pi, \qquad 1 \leq i \leq s$$

is solvable, where $E_{\frac{1}{2}}(M)$ denotes the set of semi-arcs of M.

Proof If M is Smarandachely, then each vertex $v \in V(M)$ is non-Euclidean, i.e., $\mu(v) \neq \pi$. Whence, there exists a directed circuit-decomposition

$$E_{\frac{1}{2}}(M) = \bigoplus_{i=1}^{s} E(\overrightarrow{C}_i)$$

of semi-arcs in M such that each of them is an s-line in $(\mathbf{R}^2, \mu)$. Applying Theorem 9.3.5, we know that

$$\sum_{v \in V(\overrightarrow{C}_i)} (\pi - \mu(v)) = 2\pi \quad \text{or} \quad \sum_{v \in V(\overrightarrow{C}_i)} (\pi - \mu(v)) = -2\pi$$

for each circuit C_i, $1 \le i \le s$. Thus one of the linear systems of equations

$$\sum_{v \in V(\overrightarrow{C}_i)} (\pi - x_v) = 2\pi, \quad 1 \le i \le s \quad \text{or} \quad \sum_{v \in V(\overrightarrow{C}_i)} (\pi - x_v) = -2\pi, \quad 1 \le i \le s$$

is solvable.

Conversely, if one of the linear systems of equations

$$\sum_{v \in V(\overrightarrow{C}_i)} (\pi - x_v) = 2\pi, \quad 1 \le i \le s \quad \text{or} \quad \sum_{v \in V(\overrightarrow{C}_i)} (\pi - x_v) = -2\pi, \quad 1 \le i \le s$$

is solvable, define a mapping $\mu : \mathbf{R}^2 \to [0, 4\pi)$ by

$$\mu(x) = \begin{cases} x_v & \text{if } x = v \in V(M), \\ \pi & \text{if } x \notin v(M). \end{cases}$$

Then M is a Smarandachely map on $(\mathbf{R}^2, \mu)$. This completes the proof. □

In Fig.2, we present an example of a Smarandachely planar maps with μ defined by numbers on vertices.

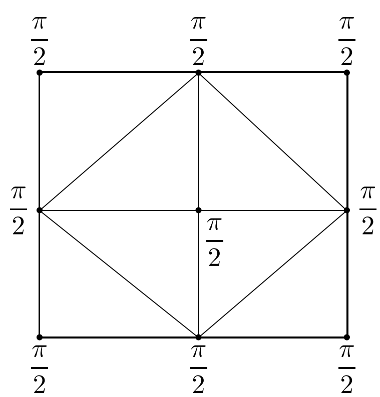

Fig.2

Let $\omega_0 \in (0, \pi)$. An s-line L is called *non-Euclidean of type* ω_0 if $R(L) = \pm 2\pi \pm \omega_0$. Similar to Theorem 2.2, we can get the following result.

Theorem 2.3 *A non-separated map M is Smarandachely if and only if there exist a directed circuit-decomposition*

$$E_{\frac{1}{2}}(M) = \bigoplus_{i=1}^{s} E(\overrightarrow{C}_i)$$

of M into s-lines of type ω_0, $\omega_0 \in (0, \pi)$ for integers $1 \le i \le s$ such that one of the linear *systems of equations*

$$\sum_{v\in V(\overrightarrow{C}_i)} (\pi - x_v) = 2\pi - \omega_0, \qquad 1 \le i \le s,$$
$$\sum_{v\in V(\overrightarrow{C}_i)} (\pi - x_v) = -2\pi - \omega_0, \qquad 1 \le i \le s,$$
$$\sum_{v\in V(\overrightarrow{C}_i)} (\pi - x_v) = 2\pi + \omega_0, \qquad 1 \le i \le s,$$
$$\sum_{v\in V(\overrightarrow{C}_i)} (\pi - x_v) = -2\pi + \omega_0, \qquad 1 \le i \le s$$

is solvable.

2.2 Smarandachely Embedded Graphs in $(\mathbb{R}^n, \mu)$

Generally, we define the *curvature* $R(L)$ of an s-line L passing through non-Euclidean points $\overline{x}_1, \overline{x}_2, \cdots, \overline{x}_m \in \mathbf{R}^n$ for $m \ge 0$ in $(\mathbf{R}^n, \mu)$ to be a matrix determined by

$$R(L) = \prod_{i=1}^{m} \mu(\overline{x}_i)$$

and *Euclidean* if $R(L) = I_{n\times n}$, otherwise, *non-Euclidean*. It is obvious that a point in a Euclidean space $\mathbf{R}^n$ is indeed Euclidean by this definition. Furthermore, we immediately get the following result for Euclidean s-lines in $(\mathbf{R}^n, \mu)$.

Theorem 2.4 *Let $(\mathbf{R}^n, \mu)$ be a pseudo-Euclidean space and L an s-line in $(\mathbf{R}^n, \mu)$ passing through non-Euclidean points $\overline{x}_1, \overline{x}_2, \cdots, \overline{x}_m \in \mathbf{R}^n$. Then L is closed if and only if L is Euclidean.*

Proof If L is a closed s-line, then L is consisted of vectors $\overrightarrow{\overline{x}_1\overline{x}_2}$, $\overrightarrow{\overline{x}_2\overline{x}_3}$, $\cdots$, $\overrightarrow{\overline{x}_n\overline{x}_1}$. By definition,

$$\frac{\overrightarrow{\overline{x}_{i+1}\overline{x}_i}}{\left|\overrightarrow{\overline{x}_{i+1}\overline{x}_i}\right|} = \frac{\overrightarrow{\overline{x}_{i-1}\overline{x}_i}}{\left|\overrightarrow{\overline{x}_{i-1}\overline{x}_i}\right|}\ \mu(\overline{x}_i)$$

for integers $1 \le i \le m$, where $i+1 \equiv (\bmod m)$. Consequently,

$$\overrightarrow{\overline{x}_1\overline{x}_2} = \overrightarrow{\overline{x}_1\overline{x}_2} \prod_{i=1}^{m} \mu(\overline{x}_i).$$

Thus $\prod\limits_{i=1}^{m} \mu(\overline{x}_i) = I_{n\times n}$, i.e., L is Euclidean.

Conversely, let L be Euclidean, i.e., $\prod\limits_{i=1}^{m} \mu(\overline{x}_i) = I_{n\times n}$. By definition, we know that

$$\frac{\overrightarrow{\overline{x}_{i+1}\overline{x}_i}}{\left|\overrightarrow{\overline{x}_{i+1}\overline{x}_i}\right|} = \frac{\overrightarrow{\overline{x}_{i-1}\overline{x}_i}}{\left|\overrightarrow{\overline{x}_{i-1}\overline{x}_i}\right|}\ \mu(\overline{x}_i), \quad \text{i.e.,} \quad \overrightarrow{\overline{x}_{i+1}\overline{x}_i} = \frac{\left|\overrightarrow{\overline{x}_{i+1}\overline{x}_i}\right|}{\left|\overrightarrow{\overline{x}_{i-1}\overline{x}_i}\right|}\overrightarrow{\overline{x}_{i-1}\overline{x}_i}\ \mu(\overline{x}_i)$$

for integers $1 \leq i \leq m$, where $i+1 \equiv (\mathrm{mod} m)$. Whence, if $\prod_{i=1}^{m} \mu(\overline{x}_i) = I_{n\times n}$, then there must be

$$\overrightarrow{\overline{x}_1\overline{x}_2} = \overrightarrow{\overline{x}_1\overline{x}_2} \prod_{i=1}^{m} \mu(\overline{x}_i).$$

Thus L consisted of vectors $\overrightarrow{\overline{x}_1\overline{x}_2}$, $\overrightarrow{\overline{x}_2\overline{x}_3}$, $\cdots$, $\overrightarrow{\overline{x}_n\overline{x}_1}$ is a closed s-line in $(\mathbf{R}^n, \mu)$. $\square$

Now we consider the pseudo-Euclidean space $(\mathbf{R}^2, \mu)$ and find the rotation matrix $\mu(\overline{x})$ for points $\overline{x} \in \mathbf{R}^2$. Let $\theta_{\overline{x}}$ be the angle form $\overline{\epsilon}_1$ to $\mu\overline{\epsilon}_1$. Then it is easily to know that

$$\mu(\overline{x}) = \begin{pmatrix} \cos\theta_{\overline{x}} & \sin\theta_{\overline{x}} \\ \sin\theta_{\overline{x}} & -\cos\theta_{\overline{x}} \end{pmatrix}.$$

Now if an s-line L passing through non-Euclidean points $\overline{x}_1, \overline{x}_2, \cdots, \overline{x}_m \in \mathbf{R}^2$, then Theorem 2.4 implies that

$$\begin{pmatrix} \cos\theta_{\overline{x}_1} & \sin\theta_{\overline{x}_1} \\ \sin\theta_{\overline{x}_1} & -\cos\theta_{\overline{x}_1} \end{pmatrix} \begin{pmatrix} \cos\theta_{\overline{x}_2} & \sin\theta_{\overline{x}_2} \\ \sin\theta_{\overline{x}_2} & -\cos\theta_{\overline{x}_2} \end{pmatrix} \cdots \begin{pmatrix} \cos\theta_{\overline{x}_m} & \sin\theta_{\overline{x}_m} \\ \sin\theta_{\overline{x}_m} & -\cos\theta_{\overline{x}_m} \end{pmatrix} = I_{2\times 2}.$$

Thus

$$\mu(\overline{x}) = \begin{pmatrix} \cos(\theta_{\overline{x}_1} + \theta_{\overline{x}_2} + \cdots + \theta_{\overline{x}_m}) & \sin(\theta_{\overline{x}_1} + \theta_{\overline{x}_2} + \cdots + \theta_{\overline{x}_m}) \\ \sin(\theta_{\overline{x}_1} + \theta_{\overline{x}_2} + \cdots + \theta_{\overline{x}_m}) & \cos(\theta_{\overline{x}_1} + \theta_{\overline{x}_2} + \cdots + \theta_{\overline{x}_m}) \end{pmatrix} = I_{2\times 2}.$$

Whence, $\theta_{\overline{x}_1} + \theta_{\overline{x}_2} + \cdots + \theta_{\overline{x}_m} = 2k\pi$ for an integer k. This fact is in agreement with that of Theorem 2.1, only with different disguises.

An *embedded graph* G on $\mathbf{R}^n$ is a $1-1$ mapping $\tau : G \to \mathbf{R}^n$ such that for $\forall e, e' \in E(G)$, $\tau(e)$ has no self-intersection and $\tau(e)$, $\tau(e')$ maybe only intersect at their end points. Such an embedded graph G in $\mathbf{R}^n$ is denoted by $G_{\mathbf{R}^n}$. For example, the n-cube $\mathcal{C}_n$ is such an embedded graph with vertex set $V(\mathcal{C}_n) = \{\ (x_1, x_2, \cdots, x_n) \mid x_i = 0 \ \text{ or } \ 1 \ \ for\ 1 \leq i \leq n\ \}$ and two vertices $(x_1, x_2, \cdots, x_n))$ and $(x_1', x_2', \cdots, x_n')$ are adjacent if and only if they are differ exactly in one entry. We present two n-cubes in Fig.3 for $n = 2$ and $n = 3$.

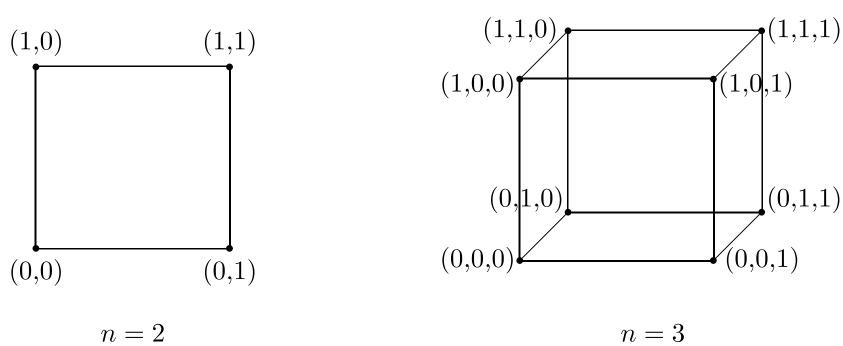

Fig.3

Similarly, an embedded graph $G_{\mathbf{R}^n}$ is called *Smarandachely* if there exists a pseudo-Euclidean space $(\mathbf{R}^n, \mu)$ with a mapping $\mu : \overline{x} \in \mathbf{R}^n \to [\overline{x}]$ such that all of its vertices are non-Euclidean points in $(\mathbf{R}^n, \mu)$. Certainly, these vertices of valency 1 is not important for Smarandachely embedded graphs. We concentrate our attention on embedded 2-connected graphs.

Theorem 2.5 *An embedded* 2*-connected graph* $G_{\mathbf{R}^n}$ *is Smarandachely if and only if there is a mapping* $\mu : \overline{x} \in \mathbf{R}^n \to [\overline{x}]$ *and a directed circuit-decomposition*

$$E_{\frac{1}{2}} = \bigoplus_{i=1}^{s} E(\overrightarrow{C}_i)$$

such that these matrix equations

$$\prod_{\overline{x} \in V(\overrightarrow{C}_i)} X_{\overline{x}} = I_{n\times n} \quad 1 \leq i \leq s$$

are solvable.

Proof By definition, if $G_{\mathbf{R}^n}$ is Smarandachely, then there exists a mapping $\mu : \overline{x} \in \mathbf{R}^n \to [\overline{x}]$ on $\mathbf{R}^n$ such that all vertices of $G_{\mathbf{R}^n}$ are non-Euclidean in $(\mathbf{R}^n, \mu)$. Notice there are only two orientations on an edge in $E(G_{\mathbf{R}^n})$. Traveling on $G_{\mathbf{R}^n}$ beginning from any edge with one orientation, we get a closed s-line $\overrightarrow{C}$, i.e., a directed circuit. After we traveled all edges in $G_{\mathbf{R}^n}$ with the possible orientations, we get a directed circuit-decomposition

$$E_{\frac{1}{2}} = \bigoplus_{i=1}^{s} E(\overrightarrow{C}_i)$$

with an s-line $\overrightarrow{C}_i$ for integers $1 \leq i \leq s$. Applying Theorem 2.4, we get

$$\prod_{\overline{x} \in V(\overrightarrow{C}_i)} \mu(\overline{x}) = I_{n\times n} \quad 1 \leq i \leq s.$$

Thus these equations

$$\prod_{\overline{x} \in V(\overrightarrow{C}_i)} X_{\overline{x}} = I_{n\times n} \quad 1 \leq i \leq s$$

have solutions $X_{\overline{x}} = \mu(\overline{x})$ for $\overline{x} \in V(\overrightarrow{C}_i)$.

Conversely, if these is a directed circuit-decomposition

$$E_{\frac{1}{2}} = \bigoplus_{i=1}^{s} E(\overrightarrow{C}_i)$$

such that these matrix equations

$$\prod_{\overline{x} \in V(\overrightarrow{C}_i)} X_{\overline{x}} = I_{n\times n} \quad 1 \leq i \leq s$$

are solvable, let $X_{\overline{x}} = A_{\overline{x}}$ be such a solution for $\overline{x} \in V(\overrightarrow{C}_i)$, $1 \leq i \leq s$. Define a mapping $\mu : \overline{x} \in \mathbf{R}^n \to [\overline{x}]$ on $\mathbf{R}^n$ by

$$\mu(\overline{x}) = \begin{cases} A_{\overline{x}} & \text{if } \overline{x} \in V(G_{\mathbf{R}^n}), \\ I_{n\times n} & \text{if } \overline{x} \notin V(G_{\mathbf{R}^n}). \end{cases}$$

Then we get a Smarandachely embedded graph $G_{\mathbf{R}^n}$ in the pseudo-Euclidean space $(\mathbf{R}^n, \mu)$ by Theorem 2.4. □

§3. Linear Isometries on Pseudo-Euclidean Space

If all points in a pseudo-Euclidean space $(\mathbb{R}^n, \mu)$ are Euclidean, i.e., the case (a) in Fig.1, then $(\mathbb{R}^n, \mu)$ is nothing but just the Euclidean space $\mathbf{R}^n$. The following results on linear isometries of Euclidean spaces are well-known.

Theorem 3.1 *Let* $\mathbf{E}$ *be an* n*-dimensional Euclidean space with normal basis* $\{\overline{\epsilon}_1, \overline{\epsilon}_2, \cdots, \overline{\epsilon}_n\}$ *and* T *a linear transformation on* $\mathbf{E}$ *determined by* $\overline{Y}^t = [a_{ij}]_{n\times n}\overline{X}^t$, *where* $\overline{X} = (\overline{\epsilon}_1, \overline{\epsilon}_2, \cdots, \overline{\epsilon}_n)$ *and* $\overline{Y} = (T(\overline{\epsilon}_1), T(\overline{\epsilon}_2), \cdots, T(\overline{\epsilon}_n))$. *Then* T *is a linear isometry on* $\mathbf{E}$ *if and only if* $[a_{ij}]_{n\times n}$ *is an orthogonal matrix, i.e.,* $[a_{ij}]_{n\times n}[a_{ij}]^t_{n\times n} = I_{n\times n}$.

Theorem 3.2 *An isometry on a Euclidean space* $\mathbf{E}$ *is a composition of three elementary isometries on* $\mathbf{E}$ *following:*

Translation $\mathbb{T}_{\overline{e}}$. *A mapping that moves every point* $(x_1, x_2, \cdots, x_n)$ *of* $\mathbf{E}$ *by*

$$T_{\overline{e}} : (x_1, x_2, \cdots, x_n) \to (x_1 + e_1, x_2 + e_2, \cdots, x_n + e_n),$$

where $\overline{e} = (e_1, e_2, \cdots, e_n)$.

Rotation $\mathbb{R}_{\overline{\theta}}$. *A mapping that moves every point of* $\mathbf{E}$ *through a fixed angle about a fixed point. Similarly, taking the center* O *to be the origin of polar coordinates* $(r, \phi_1, \phi_2, \cdots, \phi_{n-1})$, *a rotation* $R_{\theta_1,\theta_2,\cdots,\theta_{n-1}} : \mathbf{E} \to \mathbf{E}$ *is*

$$R_{\theta_1,\theta_2,\cdots,\theta_{n-1}} : (r, \phi_1, \phi_2, \cdots, \phi_{n_1}) \to (r, \phi_1 + \theta_1, \phi_2 + \theta_2, \cdots, \phi_{n_1} + \theta_{n-1}),$$

where θ_i *is a constant angle,* $\theta_i \in \mathbf{R}$ $(\mathrm{mod} 2\pi)$ *for integers* $1 \le i \le n-1$.

Reflection $\mathbb{F}$. *A reflection* F *is a mapping that moves every point of* $\mathbf{E}$ *to its mirror-image in a fixed Euclidean subspace* E' *of dimensional* $n-1$, *denoted by* $F = F(E')$. *Thus for a point* P *in* $\mathbf{E}$, $F(P) = P$ *if* $P \in E'$, *and if* $P \notin E'$, *then* $F(P)$ *is the unique point in* $\mathbf{E}$ *such that* E' *is the perpendicular bisector of* P *and* $F(P)$.

Theorem 3.3 *An isometry* $\mathcal{I}$ *on a Euclidean space* $\mathbf{E}$ *is affine, i.e., determined by*

$$\overline{Y}^t = \lambda [a_{ij}]_{n\times n}\overline{X}^t + \overline{e},$$

where λ *is a constant number,* $[a_{ij}]_{n\times n}$ *a orthogonal matrix and* $\overline{e}$ *a constant vector in* $\mathbf{E}$.

Notice that a vector $\overrightarrow{V}$ can be uniquely determined by the basis of $\mathbf{R}^n$. For $\overline{x} \in \mathbf{R}^n$, there are infinite orthogonal frames at point $\overline{x}$. Denoted by $\mathcal{O}_{\overline{x}}$ the set of all normal bases at point $\overline{x}$. Then a *pseudo-Euclidean space* $(\mathbf{R}, \mu)$ is nothing but a Euclidean space $\mathbf{R}^n$ associated with a linear mapping $\mu : \{\overline{\epsilon}_1, \overline{\epsilon}_2, \cdots, \overline{\epsilon}_n\} \rightarrow \{\overline{\epsilon}'_1, \overline{\epsilon}'_2, \cdots, \overline{\epsilon}'_n\} \in \mathcal{O}_{\overline{x}}$ such that $\mu(\overline{\epsilon}_1) = \overline{\epsilon}'_1$, $\mu(\overline{\epsilon}_2) = \overline{\epsilon}'_2$, $\cdots$, $\mu(\overline{\epsilon}_n) = \overline{\epsilon}'_n$ at point $\overline{x} \in \mathbf{R}^n$. Thus if $\overrightarrow{V}_{\overline{x}} = c_1\overline{\epsilon}_1 + c_2\overline{\epsilon}_2 + \cdots + c_n\overline{\epsilon}_n$, then $\mu({}_{\overline{x}}\overrightarrow{V}) = c_1\mu(\overline{\epsilon}_1) + c_2\mu(\overline{\epsilon}_2) + \cdots + c_n\mu(\overline{\epsilon}_n) = c_1\overline{\epsilon}'_1 + c_2\overline{\epsilon}'_2 + \cdots + c_n\overline{\epsilon}'_n$.

Without loss of generality, assume that

$$\begin{aligned}
&\mu(\overline{\epsilon}_1) = x_{11}\overline{\epsilon}_1 + x_{12}\overline{\epsilon}_2 + \cdots + x_{1n}\overline{\epsilon}_n,\\
&\mu(\overline{\epsilon}_2) = x_{21}\overline{\epsilon}_1 + x_{22}\overline{\epsilon}_2 + \cdots + x_{2n}\overline{\epsilon}_n,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,\\
&\mu(\overline{\epsilon}_n) = x_{n1}\overline{\epsilon}_1 + x_{n2}\overline{\epsilon}_2 + \cdots + x_{nn}\overline{\epsilon}_n.
\end{aligned}$$

Then we find that

$$\begin{aligned}
\mu({}_{\overline{x}}\overrightarrow{V}) &= (c_1, c_2, \cdots, c_n)(\mu(\overline{\epsilon}_1), \mu(\overline{\epsilon}_2), \cdots, \mu(\overline{\epsilon}_n))^t\\
&= (c_1, c_2, \cdots, c_n)\begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n}\\ x_{21} & x_{22} & \cdots & x_{2n}\\ \cdots & \cdots & \cdots & \cdots\\ x_{n1} & x_{n2} & \cdots & x_{nn}\end{pmatrix}(\overline{\epsilon}_1, \overline{\epsilon}_2, \cdots, \overline{\epsilon}_n)^t.
\end{aligned}$$

Denoted by

$$[\overline{x}] = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n}\\ x_{21} & x_{22} & \cdots & x_{2n}\\ \cdots & \cdots & \cdots & \cdots\\ x_{n1} & x_{n2} & \cdots & x_{nn}\end{pmatrix} = \begin{pmatrix} \langle\mu(\overline{\epsilon}_1), \overline{\epsilon}_1\rangle & \langle\mu(\overline{\epsilon}_1), \overline{\epsilon}_2\rangle & \cdots & \langle\mu(\overline{\epsilon}_1), \overline{\epsilon}_n\rangle\\ \langle\mu(\overline{\epsilon}_2), \overline{\epsilon}_1\rangle & \langle\mu(\overline{\epsilon}_2), \overline{\epsilon}_2\rangle & \cdots & \langle\mu(\overline{\epsilon}_2), \overline{\epsilon}_n\rangle\\ \cdots & \cdots & \cdots & \cdots\\ \langle\mu(\overline{\epsilon}_n), \overline{\epsilon}_1\rangle & \langle\mu(\overline{\epsilon}_n), \overline{\epsilon}_2\rangle & \cdots & \langle\mu(\overline{\epsilon}_n), \overline{\epsilon}_n\rangle\end{pmatrix},$$

called the *rotation matrix* of $\overline{x}$ in $(\mathbf{R}^n, \mu)$. Then $\mu : \overrightarrow{V}_{\overline{x}} \rightarrow {}_{\overline{x}}\overrightarrow{V}$ is determined by $\mu(\overline{x}) = [\overline{x}]$ for $\overline{x} \in \mathbf{R}^n$. Furthermore, such an rotation matrix $[\overline{x}]$ is orthogonal for points $\overline{x} \in \mathbf{R}^n$ by definition, i.e., $[\overline{x}]\,[\overline{x}]^t = I_{n\times n}$. Particularly, if $\overline{x}$ is Euclidean, then such an orientation matrix is nothing but $\mu(\overline{x}) = I_{n\times n}$. Summing up all these discussions, we know the following result.

Theorem 3.4 *If* $(\mathbf{R}^n, \mu)$ *is a pseudo-Euclidean space, then* $\mu(\overline{x}) = [\overline{x}]$ *is an* $n \times n$ *orthogonal matrix for* $\forall\ \overline{x} \in \mathbf{R}^n$.

By definition, we know that $\mathrm{Isom}(\mathbf{R}^n) = \langle\mathbb{T}_{\overline{e}}, \mathbb{R}_{\overline{\theta}}, \mathbb{F}\rangle$. An isometry τ of a pseudo-Euclidean space $(\mathbf{R}^n, \mu)$ is an isometry on $\mathbf{R}^n$ such that $\mu(\tau(\overline{x})) = \mu(\overline{x})$ for $\forall\overline{x} \in \mathbf{R}^n$. Clearly, all such isometries form a group $\mathrm{Isom}(\mathbf{R}^n, \mu)$ under composition operation with $\mathrm{Isom}(\mathbf{R}^n, \mu) \leq \mathrm{Isom}(\mathbf{R}^n)$. We determine isometries of pseudo-Euclidean spaces in this subsection.

Certainly, if $\mu(\overline{x})$ is a constant matrix $[c]$ for $\forall\overline{x} \in \mathbf{R}^n$, then all isometries on $\mathbf{R}^n$ is also isometries on $(\mathbf{R}^n, \mu)$. Whence, we only discuss those cases with at least two values for $\mu : \overline{x} \in \mathbf{R}^n \rightarrow [\overline{x}]$ similar to that of $(\mathbf{R}^2, \mu)$.

Translation. Let $(\mathbf{R}^n, \mu)$ be a pseudo-Euclidean space with an isometry of translation $T_{\overline{e}}$, where $\overline{e} = (e_1, e_2, \cdots, e_n)$ and $P,\ Q \in (\mathbf{R}^n, \mu)$ a non-Euclidean point, a Euclidean point, respectively. Then $\mu(T_{\overline{e}}^k(P)) = \mu(P)$, $\mu(T_{\overline{e}}^k(Q)) = \mu(Q)$ for any integer $k \geq 0$ by definition. Consequently,

$$P,\ T_{\overline{e}}(P),\ T_{\overline{e}}^2(P),\ \cdots,\ T_{\overline{e}}^k(P), \cdots,$$
$$Q,\ T_{\overline{e}}(Q),\ T_{\overline{e}}^2(Q),\ \cdots,\ T_{\overline{e}}^k(Q), \cdots$$

are respectively infinite non-Euclidean and Euclidean points. Thus there are no isometries of translations if $(\mathbf{R}^n, \mu)$ is finite.

In this case, if there are rotations $R_{\theta_1,\theta_2,\cdots,\theta_{n-1}}$, then there must be $\theta_1, \theta_2, \cdots, \theta_{n-1} \in \{0, \pi/2\}$ and if $\theta_i = \pi/2$ for $1 \leq i \leq l$, $\theta_i = 0$ if $i \geq l+1$, then $e_1 = e_2 = \cdots = e_{l+1}$.

Rotation. Let (R^n, μ) be a pseudo-Euclidean space with an isometry of rotation $R_{\theta_1,\cdots,\theta_{n-1}}$ and $P,\ Q \in (\mathbf{R}^n, \mu)$ a non-Euclidean point, a Euclidean point, respectively. Then

$$\mu(R_{\theta_1,\theta_2,\cdots,\theta_{n-1}}(P)) = \mu(P), \quad \mu(R_{\theta_1,\theta_2,\cdots,\theta_{n-1}}(Q)) = \mu(Q)$$

for any integer $k \geq 0$ by definition. Whence,

$$P,\ R_{\theta_1,\theta_2,\cdots,\theta_{n-1}}(P),\ R_{\theta_1,\theta_2,\cdots,\theta_{n-1}}^2(P),\ \cdots,\ R_{\theta_1,\theta_2,\cdots,\theta_{n-1}}^k(P), \cdots,$$
$$Q,\ R_{\theta_1,\theta_2,\cdots,\theta_{n-1}}(Q),\ R_{\theta_1,\theta_2,\cdots,\theta_{n-1}}^2(Q),\ \cdots,\ R_{\theta_1,\theta_2,\cdots,\theta_{n-1}}^k(Q), \cdots$$

are respectively non-Euclidean and Euclidean points.

In this case, if there exists an integer k such that $\theta_i | 2k\pi$ for all integers $1 \leq i \leq n-1$, then the previous sequences is finite. Thus there are both finite and infinite pseudo-Euclidean space $(\mathbf{R}^n, \mu)$ in this case. But if there is an integer i_0, $1 \leq i_0 \leq n-1$ such that $\theta_{i_0} \not| \ 2k\pi$ for any integer k, then there must be either infinite non-Euclidean points or infinite Euclidean points. Thus there are isometries of rotations in a finite non-Euclidean space only if there exists an integer k such that $\theta_i | 2k\pi$ for all integers $1 \leq i \leq n-1$. Similarly, an isometry of translation exists in this case only if $\theta_1, \theta_2, \cdots, \theta_{n-1} \in \{0, \pi/2\}$.

Reflection. By definition, a reflection F in a subspace E' of dimensional $n-1$ is an involution, i.e., $F^2 = 1_{\mathbf{R}^n}$. Thus if $(\mathbf{R}^n, \mu)$ is a pseudo-Euclidean space with an isometry of reflection F in E' and $P,\ Q \in (\mathbf{R}^n, \mu)$ are respectively a non-Euclidean point and a Euclidean point. Then it is only need that $P,\ F(P)$ are non-Euclidean points and $Q,\ F(Q)$ are Euclidean points. Therefore, a reflection F can be exists both in finite and infinite pseudo-Euclidean spaces $(\mathbf{R}^n, \mu)$.

Summing up all these discussions, we get results following for finite or infinite pseudo-Euclidean spaces.

Theorem 3.5 *Let $(\mathbf{R}^n, \mu)$ be a finite pseudo-Euclidean space. Then there maybe isometries of translations $T_{\overline{e}}$, rotations $R_{\overline{\theta}}$ and reflections on $(\mathbf{R}^n, \mu)$. Furthermore,*

(1) *If there are both isometries $T_{\overline{e}}$ and $R_{\overline{\theta}}$, where $\overline{e} = (e_1, \cdots, e_n)$ and $\overline{\theta} = (\theta_1, \cdots, \theta_{n-1})$, then $\theta_1, \theta_2, \cdots, \theta_{n-1} \in \{0, \pi/2\}$ and if $\theta_i = \pi/2$ for $1 \leq i \leq l$, $\theta_i = 0$ if $i \geq l+1$, then $e_1 = e_2 = \cdots = e_{l+1}$.*

(2) *If there is an isometry $R_{\theta_1,\theta_2,\cdots,\theta_{n-1}}$, then there must be an integer k such that $\theta_i \mid 2k\pi$ for all integers $1 \leq i \leq n-1$.*

(3) *There always exist isometries by putting Euclidean and non-Euclidean points* $\overline{x} \in \mathbf{R}^n$ *with* $\mu(\overline{x})$ *constant on symmetric positions to* E' *in* $(\mathbf{R}^n, \mu)$.

Theorem 3.6 *Let* $(\mathbf{R}^n, \mu)$ *be a infinite pseudo-Euclidean space. Then there maybe isometries of translations* $T_{\overline{e}}$, *rotations* $R_{\overline{\theta}}$ *and reflections on* $(\mathbf{R}^n, \mu)$. *Furthermore,*

(1) *There are both isometries* $T_{\overline{e}}$ *and* $R_{\overline{\theta}}$ *with* $\overline{e} = (e_1, e_2, \cdots, e_n)$ *and* $\overline{\theta} = (\theta_1, \theta_2, \cdots, \theta_{n-1})$, *only if* $\theta_1, \theta_2, \cdots, \theta_{n-1} \in \{0, \pi/2\}$ *and if* $\theta_i = \pi/2$ *for* $1 \leq i \leq l$, $\theta_i = 0$ *if* $i \geq l+1$, *then* $e_1 = e_2 = \cdots = e_{l+1}$.

(2) *There exist isometries of rotations and reflections by putting Euclidean and non-Euclidean points in the orbits* $\overline{x}^{\langle R_{\overline{\theta}} \rangle}$ *and* $\overline{y}^{\langle F \rangle}$ *with a constant* $\mu(\overline{x})$ *in* $(\mathbf{R}^n, \mu)$.

We determine isometries on $(\mathbf{R}^3, \mu)$ with a 3-cube $\mathcal{C}^3$ shown in Fig.9.4.2. Let $[\overline{a}]$ be an 3×3 orthogonal matrix, $[\overline{a}] \neq I_{3\times 3}$ and let $\mu(x_1, x_2, x_3) = [\overline{a}]$ for $x_1, x_2, x_3 \in \{0,\ 1\}$, otherwise, $\mu(x_1, x_2, x_3) = I_{3\times 3}$. Then its isometries consist of two types following:

Rotations:

$R_1,\ R_2,\ R_3$: these rotations through $\pi/2$ about 3 axes joining centres of opposite faces;

$R_4,\ R_5,\ R_6,\ R_7,\ R_8,\ R_9$: these rotations through π about 6 axes joining midpoints of opposite edges;

$R_{10},\ R_{11},\ R_{12},\ R_{13}$: these rotations through about 4 axes joining opposite vertices.

Reflection F: the reflection in the centre fixes each of the grand diagonal, reversing the orientations.

Then $\mathrm{Isom}(\mathbf{R}^3, \mu) = \langle R_i,\ F,\ 1 \leq i \leq 13 \rangle \simeq S_4 \times Z_2$. But if let $\left[\overline{b}\right]$ be another 3×3 orthogonal matrix, $\left[\overline{b}\right] \neq [\overline{a}]$ and define $\mu(x_1, x_2, x_3) = [\overline{a}]$ for $x_1 = 0,\ x_2, x_3 \in \{0,\ 1\}$, $\mu(x_1, x_2, x_3) = \left[\overline{b}\right]$ for $x_1 = 1,\ x_2, x_3 \in \{0,\ 1\}$ and $\mu(x_1, x_2, x_3) = I_{3\times 3}$ otherwise. Then only the rotations R, R^2, R^3, R^4 through $\pi/2,\ \pi,\ 3\pi/2$ and 2π about the axis joining centres of opposite face

$$\{(0,0,0), (0,0,1), (0,1,0), (0,1,1)\} \text{ and } \{(1,0,0), (1,0,1), (1,1,0), (1,1,1)\},$$

and reflection F through to the plane passing midpoints of edges

$$(0,0,0)-(0,0,1),\ (0,1,0)-(0,1,1),\ (1,0,0)-(1,0,1),\ (1,1,0)-(1,1,1)$$

or $$(0,0,0)-(0,1,0),\ (0,0,1)-(0,1,1),\ (1,0,0)-(1,1,0),\ (1,0,1)-(1,1,1)$$

are isometries on $(\mathbf{R}^3, \mu)$. Thus $\mathrm{Isom}(\mathbf{R}^3, \mu) = \langle R_1,\ R_2,\ R_3,\ R_4,\ F \rangle \simeq D_8$.

Furthermore, let $[\overline{a}_i]$, $1 \leq i \leq 8$ be orthogonal matrixes distinct two by two and define $\mu(0,0,0) = [\overline{a}_1]$, $\mu(0,0,1) = [\overline{a}_2]$, $\mu(0,1,0) = [\overline{a}_3]$, $\mu(0,1,1) = [\overline{a}_4]$, $\mu(1,0,0) = [\overline{a}_5]$, $\mu(1,0,1) = [\overline{a}_6]$, $\mu(1,1,0) = [\overline{a}_7]$, $\mu(1,1,1) = [\overline{a}_8]$ and $\mu(x_1, x_2, x_3) = I_{3\times 3}$ if $x_1,\ x_2,\ x_3 \neq 0$ or 1. Then $\mathrm{Isom}(\mathbf{R}^3, \mu)$ is nothing but a trivial group.

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