# ON THE MEAN VALUE OF THE SMARANDACHE DOUBLE FACTORIAL FUNCTION

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**Abstract** For any positive integer $n$, the Smarandache double factorial function $Sdf(n)$ is defined as the least positive integer $m$ such that $m!!$ is divisible by $n$. In this paper, we study the mean value properties of $Sdf(n)$, and give an interesting mean value formula for it.

**Keywords:** F.Smarandache problem; Smarandache function; Mean Value.

## §1. Introduction and results

For any positive integer $n$, the Smarandache double factorial function $Sdf(n)$ is defined as the least positive integer $m$ such that $m!!$ is divisible by $n$, where

$$m!! = \begin{cases} 2 \cdot 4 \cdots m, & \text{if } 2|m; \\ 1 \cdot 3 \cdots m, & \text{if } 2\dagger\text{m}. \end{cases}$$

About the arithmetical properties of $Sdf(n)$, many people had studied it before (see reference [2]). The main purpose of this paper is to study the mean value properties of $Sdf(n)$, and obtain an interesting mean value formula for it. That is, we shall prove the following:

**Theorem.** *For any real number $x \geq 2$, we have the asymptotic formula*

$$\sum_{n \leq x} Sdf(n) = \frac{7\pi^2}{24} \frac{x^2}{\ln x} + O\left(\frac{x^2}{\ln^2 x}\right).$$

## §2. Proof of the theorem

In this section, we shall complete the proof of the theorem. First we need the following two simple Lemmas.

**Lemma 1.** *if $2 \dagger n$ and $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ is the factorization of $n$, where $p_1, p_2, \cdots, p_k$ are distinct odd primes and $\alpha_1, \alpha_2, \cdots, \alpha_k$ are positive integers, then*

$$Sdf(n) = \max(Sdf(p_1^{\alpha_1}), Sdf(p_2^{\alpha_2}), \cdots, Sdf(p_k^{\alpha_k}))$$

**Proof.** Let $m_i = Sdf(p_i^{\alpha_i})$ for $i = 1, 2, \cdots, k$. Then we get $2\dagger m_i (i = 1, 2, \cdots, k)$ and

$$p_i^{\alpha_i} | (m_i)!!, i = 1, 2, \cdots, k.$$

Let $m = \max(m_1, m_2, \cdots, m_k)$. Then we have

$$(m_i)!! | m!!, i = 1, 2, \cdots, k.$$

Thus we get

$$p_i^{\alpha_i} | m!!, i = 1, 2, \cdots, k.$$

Notice that $p_1, p_2, \cdots, p_k$ are distinct odd primes. We have

$$gcd(p_i^{\alpha_i}, p_j^{\alpha_j}) = 1, 1 \le i < j \le k.$$

Therefore, we obtain $n | m!!$. It implies that

$$Sdf(n) \le m.$$

On the other hand, by the definition of m, if $Sdf(n) < m$, then there exists a prime power $p_j^{\alpha_j} (1 \le j \le k)$ such that

$$p_j^{\alpha_j} | Sdf(n)!!.$$

We get $n | Sdf(n)!!$, a contradiction. Therefore, we obtain $Sdf(n) = m$.

This proves Lemma 1.

**Lemma 2.** *For positive integer n( $2\dagger n$), let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ is the prime powers factorization of n and $P(n) = \max_{1 \le i \le k} \{p_i\}$. if there exists $P(n)$ satisfied with $P(n) > \sqrt{n}$, then we have the identity*

$$Sdf(n) = P(n).$$

**Proof.** First we let $Sdf(n) = m$, then $m$ is the smallest positive integer such that $n | m!!$. Now we will prove that $m = P(n)$. We assume $P(n) = p_0$. From the definition of $P(n)$ and lemma 1, we know that $Sdf(n) = \max(p_0, (2\alpha_i - 1)p_i)$. Therefore we get

(I) If $\alpha_i = 1$, then $Sdf(n) = p_0 \ge n^{\frac{1}{2}} \ge (2\alpha_i - 1)p_i$;

(II) If $\alpha_i \ge 2$, then $Sdf(n) = p_0 > 2 \ln n n^{\frac{1}{4}} > (2\alpha_i - 1)p_i$.

Combining (I)-(II), we can easily obtain

$$Sdf(n) = P(n)$$

This proves Lemma 2.

Now we use the above Lemmas to complete the proof of Theorem. First we separate the summation in the Theorem into two parts.

$$\sum_{n \le x} Sdf(n) = \sum_{u \le \frac{x-1}{2}} Sdf(2u + 1) + \sum_{u \le \frac{x}{2}} Sdf(2u), \tag{1}$$

For the first part. we let the sets $\mathcal{A}$ and $\mathcal{B}$ as following:

$$\mathcal{A} = \{2u+1 | 2u+1 \leq x, P(2u+1) \leq \sqrt{2u+1}\}$$

and

$$\mathcal{B} = \{2u+1 | 2u+1 \leq x, P(2u+1) > \sqrt{2u+1}\}.$$

Using the Euler summation formula, we get

$$\sum_{2u+1 \in \mathcal{A}} Sdf(2u+1) \ll \sum_{2u+1 \leq x} \sqrt{2u+1} \ln(2u+1) \ll x^{\frac{3}{2}} \ln x. \quad (2)$$

Similarly, from the Abel's identity we also get

$$\begin{aligned}
& \sum_{2u+1 \in \mathcal{B}} Sdf(2u+1) \\
= & \sum_{\substack{2u+1 \leq x \\ P(2u+1) > \sqrt{2u+1}}} P(2u+1) \\
= & \sum_{1 \leq 2l+1 \leq \sqrt{x}} \sum_{2l+1 \leq p \leq \frac{x}{2l+1}} p + O\left( \sum_{2l+1 \leq \sqrt{x}} \sum_{\sqrt{2l+1} \leq p \leq \frac{x}{2l+1}} \sqrt{x} \right) \\
= & \sum_{1 \leq 2l+1 \leq \sqrt{x}} \left( \frac{x}{2l+1} \pi(\frac{x}{2l+1}) - (2l+1)\pi(2l+1) - \int_{\sqrt{x}}^{\frac{x}{2l+1}} \pi(s) ds \right) \\
& + O\left( x^{\frac{3}{2}} \ln x \right), \quad (3)
\end{aligned}$$

where $\pi(x)$ denotes all the numbers of prime which is not exceeding $x$.

For $\pi(x)$, we have

$$\pi(x) = \frac{x}{\ln x} + O\left(\frac{x}{\ln^2 x}\right)$$

and

$$\begin{aligned}
& \sum_{1 \leq 2l+1 \leq \sqrt{x}} \left( \frac{x}{2l+1} \pi(\frac{x}{2l+1}) - (2l+1)\pi(2l+1) - \int_{\sqrt{x}}^{\frac{x}{2l+1}} \pi(s) ds \right) \\
& = \sum_{1 \leq 2l+1 \leq \sqrt{x}} \left( \frac{1}{2} \frac{x^2}{(2l+1)^2 \ln \frac{x}{(2l+1)}} - \frac{1}{2} \frac{(2l+1)^2}{\ln(2l+1)} \right. \\
& \quad + O\left( \frac{x^2}{(2l+1)^2 \ln^2 \frac{x}{(2l+1)}} \right) + O\left( \frac{(2l+1)^2}{\ln^2(2l+1)} \right) \\
& \quad \left. + O\left( \frac{x^2}{(2l+1)^2 \ln^2 \frac{x}{(2l+1)}} - \frac{(2l+1)^2}{\ln^2(2l+1)} \right) \right). \quad (4)
\end{aligned}$$

Hence

$$\sum_{1\le 2l+1\le\sqrt{x}}\frac{x^2}{(2l+1)^2\ln\frac{x}{2l+1}}=\sum_{0\le l\le\frac{\sqrt{x}-1}{2}}\frac{x^2}{(2l+1)^2\ln\frac{x}{2l+1}}$$

$$=\sum_{0\le l\le\frac{\ln x-1}{2}}\frac{x^2}{(2l+1)^2\ln x}+O\left(\sum_{\frac{\ln x-1}{2}\le l\le\frac{\sqrt{x}-1}{2}}\frac{x^2\ln(2l+1)}{(2l+1)^2\ln^2 x}\right)$$

$$=\frac{\pi^2}{8}\frac{x^2}{\ln x}+O\left(\frac{x^2}{\ln^2 x}\right). \quad (5)$$

Combining (2), (3),( 4) and (5) we obtain

$$\sum_{u\le\frac{x-1}{2}}Sdf(2u+1)=\frac{\pi^2}{8}\frac{x^2}{\ln x}+O\left(\frac{x^2}{\ln^2 x}\right). \quad (6)$$

For the second part, we notice that $2u=2^\alpha n_1$ where $\alpha, n_1$ are positive integers with $2\dagger n_1$, let $S(2u)=\min\{m \mid 2u|m!\}$, from the definition of $Sdf(2u)$, we have

$$\sum_{2u\le x}Sdf(2u)=\sum_{\substack{2^\alpha n_1\le x\\ 2^\alpha>n_1}}Sdf(2^\alpha n_1)\ll\sum_{\alpha\le\frac{\ln x}{\ln 2}}\sqrt{x}\ll\sqrt{x}\ln x, \quad (7)$$

and

$$\sum_{2u\le x}Sdf(2u)=2\sum_{2u\le x}S(2u)+O(\sqrt{x}\ln x)=\frac{\pi^2}{6}\frac{x^2}{\ln x}+O\left(\frac{x^2}{\ln^2 x}\right). \quad (8)$$

Combining (7) and (8) we obtain

$$\sum_{u\le\frac{x}{2}}Sdf(2u)=\frac{\pi^2}{6}\frac{x^2}{\ln x}+O\left(\frac{x^2}{\ln^2 x}\right). \quad (9)$$

From (1), (6) and (9) we obtain the asymptotic formula

$$\sum_{n\le x}Sdf(n)=\frac{7\pi^2}{24}\frac{x^2}{\ln x}+\left(\frac{x^2}{\ln^2 x}\right).$$

This completes the proof of Theorem.

## References

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