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On the mean value of the Near Pseudo Smarandache Function

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Abstract The main purpose of this paper is using the analytic method to study the asymptotic properties of the Near Pseudo Smarandache Function, and give two interesting asymptotic formulae for it.

Keywords Near Pseudo Smarandache Function, mean value, asymptotic formula.

§1. Introduction

In reference [1], David Gorski defined the Pseudo Smarandache function $Z(n)$ as: let n be any positive integer, $Z(n)$ is the smallest integer such that $1+2+3+\ldots+Z(n)$ is divisible by n. In reference [2], A.W.Vyawahare defined a new function $K(n)$ which is a slight modification of $Z(n)$ by adding a smallest natural number k, so this function is called "Near Pseudo Smarandache Function". It is defined as follows: let n be any positive integer, $K(n)=m$, where $m=\sum\limits_{n=1}^{n} n+k$ and k is the smallest natural number such that n divides m. About the mean value properties of the smallest natural number k in Near Pseudo Smarandache function, it seems that none had studied them before, at least we couldn't find any reference about it. In this paper, we use the analytic method to study the mean value properties of $d(k)$ and $\varphi(k)$, and give two interesting asymptotic formulae for it. That is, we shall prove the following:

Theorem 1. *Let k is the smallest natural number such that n divides Near Pseudo Smarandache function $K(n)$, $d(n)$ denotes Dirichlet divisor function. Then for any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{n\leq x} d(k)=\sum_{n\leq x} d\left(K(n)-\frac{n(n+1)}{2}\right)=\frac{3}{4}x\log x+Ax+O(x^{\frac{1}{2}}\log^2 x),$$

where A is a computable constant.

Theorem 2. *For any real number $x \geq 1$, k is the smallest natural number such that n divides Near Pseudo Smarandache function $K(n)$, $\varphi(n)$ denotes the Euler's totient function. Then we have the asymptotic formula*

$$\sum_{n\leq x} \varphi\left(K(n)-\frac{n(n+1)}{2}\right)=\frac{93}{28\pi^2}x^2+O(x^{\frac{3}{2}+\epsilon}),$$

where ϵ denotes any fixed positive number.

§2. Some lemmas

To complete the proof of the theorems, we need the following several simple Lemmas:

Lemma 1. *Let n be any positive integer, then we have*

$$K(n) = \begin{cases} \frac{n(n+3)}{2}, & \text{if } n \text{ is odd}, \\ \frac{n(n+2)}{2}, & \text{if } n \text{ is even}. \end{cases}$$

Proof. (See reference [2]).

Lemma 2. *For any real number $x \geq 1$, we have*

$$\sum_{n \leq x} d(n) = x \log x + (2C - 1)x + O\left(\sqrt{x}\right),$$

where C is the Euler constant,

$$\sum_{n \leq x} \varphi(k) = \frac{3}{\pi^2} x^2 + O(x \log x).$$

Proof. These results can be get immediately from [3].

Lemma 3. *For any real number $x \geq 1$, we have*

$$\sum_{n \leq x} d(2n) = \frac{3}{2} x \log x + \left(\frac{\log 2}{2} - \frac{3}{2}\right) x + O\left(x^{\frac{1}{2}} \log^2 x\right),$$

$$\sum_{n \leq x} \varphi(2n) = \frac{2}{7\zeta(2)} x^2 + O(x^{\frac{3}{2}+\epsilon}).$$

Proof. Firstly, we shall prove the first formula of Lemma 3. Let $s = \sigma + it$ be a complex number and $f(s) = \sum\limits_{n=1}^{\infty} \frac{d(2n)}{n^s}$. Note that $d(2n) \ll n^{\epsilon}$, so it is clear that $f(s)$ is a Dirichlet series absolutely convergent for Re$(s)> 1$, by the Euler product formula [3] and the definition of $d(n)$ we get

$$\begin{aligned} f(s) &= \prod_p \sum_{m=0}^{\infty} \frac{d(2p^m)}{p^{ms}} \\ &= \sum_{m=0}^{\infty} \frac{d(2^{m+1})}{2^{ms}} \cdot \prod_{p>2} \sum_{m=0}^{\infty} \frac{d(2p^m)}{p^{ms}} \\ &= 2\zeta^2(s) \cdot \frac{\left(\prod\limits_{p>2} \sum\limits_{m=0}^{\infty} \frac{d(p^m)}{p^{ms}}\right) \cdot \left(\sum\limits_{m=0}^{\infty} \frac{d(2^{m+1})}{2^{ms}}\right)}{\prod\limits_p \sum\limits_{m=0}^{\infty} \left(\frac{d(p^m)}{p^{ms}}\right)} \\ &= 2\zeta^2(s) \cdot \frac{\sum\limits_{m=0}^{\infty} \frac{d(2^{m+1})}{2^{ms}}}{\sum\limits_{m=0}^{\infty} \frac{d(2^m)}{2^{ms}}} \\ &= \zeta^2(s) \left(2 - \frac{1}{2^s}\right). \end{aligned} \tag{1}$$

where $\zeta(s)$ is the Riemann zeta-function, and $\prod_p$ dentoes the product over all primes.

From (1) and the Perron's formula [4], for $b = 1+\epsilon, T \geq 1$ and $x \geq 1$ we have

$$\sum_{n\leq x} d(2n) = \frac{1}{2\pi i}\int_{b-iT}^{b+iT} f(s)\frac{x^s}{s}ds + O\left|\frac{x^b}{T}\right| + O\left(\frac{xH(2x)\log x}{T}\right). \tag{2}$$

Taking $a = \frac{1}{2}+\epsilon$, we move the integral line in (2). Then

$$\begin{aligned}\sum_{n\leq x} d(2n) &= \operatorname*{Res}_{s=1}\zeta^2(s)\left(2-\frac{1}{2^s}\right)\frac{x^s}{s}\\ &+ \frac{1}{2\pi i}\left|\int_{b-iT}^{a-iT}+\int_{a-iT}^{a+iT}+\int_{a+iT}^{b+iT}\right|\zeta^2(s)\left(2-\frac{1}{2^s}\right)\frac{x^s}{s}ds\\ &+ O\left|\frac{x^b}{T}\right| + O\left|\frac{xH(2x)\log x}{T}\right|,\end{aligned}$$

where

$$\left|\int_{b-iT}^{a-iT}+\int_{a+iT}^{b+iT}\right|\zeta^2(s)(2-\frac{1}{2^s})\frac{x^s}{s}ds \ll \frac{x}{T},$$
$$\int_{a-iT}^{a+iT}\zeta^2(s)(2-\frac{1}{2^s})\frac{x^s}{s}ds \ll x^{\frac{1}{2}}\log^2 T.$$

Hence, we have

$$\begin{aligned}\sum_{n\leq x} d(2n) &= \operatorname*{Res}_{s=1}\zeta^2(s)(2-\frac{1}{2^s})\frac{x^s}{s} + O\left|\frac{x}{T}\right|\\ &+ O\left(x^{\frac{1}{2}}\log^2 T\right) + O\left|\frac{x^b}{T}\right| + O\left|xH(2x)\frac{\log x}{T}\right|\\ &= \operatorname*{Res}_{s=1}\zeta^2(s)(2-\frac{1}{2^s})\frac{x^s}{s} + O\left|\frac{x}{T}\right|\\ &+ O\left(x^{\frac{1}{2}}\log^2 T\right) + O\left|x^{1+\varepsilon}\frac{\log x}{T}\right|.\end{aligned} \tag{3}$$

Taking $T = x^{\frac{1}{2}+\epsilon}$ in (3), then

$$\begin{aligned}\sum_{n\leq x} d(2n) &= \operatorname*{Res}_{s=1}\zeta^2(s)(2-\frac{1}{2^s})\frac{x^s}{s} + O\left(x^{\frac{1}{2}-\varepsilon}\right) + O\left(x^{\frac{1}{2}}\log^2 x\right)\\ &= \operatorname*{Res}_{s=1}\zeta^2(s)\left(2-\frac{1}{2^s}\right)\frac{x^s}{s} + O\left(x^{\frac{1}{2}}\log^2 x\right).\end{aligned} \tag{4}$$

Now we can easily get the residue of the function $\zeta^2(s)\left(2-\frac{1}{2^s}\right)\cdot\frac{x^s}{s}$ at second order pole point $s=1$ with

$$\operatorname*{Res}_{s=1}\zeta^2(s)\left(2-\frac{1}{2^s}\right)\frac{x^s}{s} = \frac{3}{2}x\log x + \left(\frac{\log 2}{2}-\frac{3}{2}\right)x. \tag{5}$$

Combining (4) and (5), we may immediately get

$$\sum_{n\leq x} d(2n) = \frac{3}{2}x\log x + \left(\frac{\log 2}{2}-\frac{3}{2}\right)x + O\left(x^{\frac{1}{2}}\log^2 x\right).$$

This completes the proof of the first formula of Lemma 3.

Let $h(s) = \sum_{n=1}^{\infty} \frac{\varphi(2n)}{n^s}$. From Euler product formula [2] and the definition of $\varphi(n)$ we also have

$$\begin{aligned} h(s) &= \prod_p \left(1 + \sum_{m=1}^{\infty} \frac{\varphi(2p^m)}{p^{ms}}\right) \\ &= \left(1 + \sum_{m=1}^{\infty} \frac{\varphi(2^{m+1})}{2^{ms}}\right) \cdot \prod_{p>2} \left(1 + \sum_{m=1}^{\infty} \frac{\varphi(2p^m)}{p^{ms}}\right) \\ &= \frac{\zeta(s-1)}{\zeta(s)} \cdot \frac{\prod_{p>2}\left(1 + \sum_{m=1}^{\infty} \frac{\varphi(p^m)}{p^{ms}}\right) \cdot \left(1 + \sum_{m=1}^{\infty} \frac{\varphi(2^{m+1})}{2^{ms}}\right)}{\prod_p \left(1 + \sum_{m=1}^{\infty} \frac{\varphi(p^m)}{p^{ms}}\right)} \\ &= \frac{\zeta(s-1)}{\zeta(s)} \cdot \frac{\left(1 + \sum_{m=1}^{\infty} \frac{\varphi(2^{m+1})}{2^{ms}}\right)}{\left(1 + \sum_{m=1}^{\infty} \frac{\varphi(2^m)}{2^{ms}}\right)} \\ &= \frac{\zeta(s-1)}{\zeta(s)} \cdot \frac{2^s}{2^s+3}. \end{aligned}$$

By Perron formula [4] and the method of proving the first formula of Lemma 3, we can obtain the second formula of Lemma 3.

§3. Proof of the theorems

In this section, we will complete the proof of the Theorems. From the first formula of Lemma 3 we can obtain

$$\sum_{n \le \frac{x}{2}} d(2n) = \frac{3}{4} x \log x - \left(\frac{\log 2}{2} - \frac{3}{8}\right) x + O\left(x^{\frac{1}{2}} \log^2 x\right).$$

Let $f(n) = K(n) - \frac{n(n+1)}{2} = k$, then from Lemma 1 and the first formula of Lemma 2 we have

$$\begin{aligned} \sum_{n \le x} d(k) &= \sum_{n \le x} d\left(K(n) - \frac{n(n+1)}{2}\right) \\ &= \sum_{\substack{n \le x \\ 2|n}} d(\frac{n}{2}) + \sum_{\substack{n \le x \\ 2\dagger n}} d(n) \\ &= \sum_{n \le \frac{x}{2}} d(n) + \sum_{n \le x} d(n) - \sum_{n \le \frac{x}{2}} d(2n) \\ &= \frac{3}{4} x \log x + Ax + O\left(x^{\frac{1}{2}} \log^2 x\right), \end{aligned}$$

where A is a computable constant.

This complets the proof of Theorem 1.

Now we complete the proof of Theorem 2. Noting that $\zeta(2)=\frac{\pi^2}{6}$, from the second formula of Lemma 3 we can obtain

$$\sum_{n\le\frac{x}{2}}\varphi(2n)=\frac{3}{7\pi^2}x^2+O(x^{\frac{3}{2}+\epsilon}).$$

Then from Lemma 1 and the second formula of Lemma 2 we have

$$\begin{aligned}\sum_{n\le x}\varphi(k) &= \sum_{n\le x}\varphi\left(K(n)-\frac{n(n+1)}{n}\right)\\ &= \sum_{\substack{n\le x\\ 2|n}}\varphi(\frac{n}{2})+\sum_{\substack{n\le x\\ 2\dagger n}}\varphi(n)\\ &= \sum_{n\le\frac{x}{2}}\varphi(n)+\sum_{n\le x}\varphi(n)-\sum_{n\le\frac{x}{2}}\varphi(2n)\\ &= \frac{93}{28\pi^2}x^2+O\left(x^{\frac{3}{2}+\epsilon}\right),\end{aligned}$$

where ϵ is any fixed positive number.

This complets the proof of Theorem 2.

References

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