MEAN VALUE OF THE ADDITIVE ANALOGUE OF SMARANDACHE FUNCTION *

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Abstract For any positive integer n, let $S(n)$ denotes the Smarandache function, then $S(n)$ is defined the smallest $m \in N^+$, where $n|m!$. In this paper, we study the mean value properties of the additive analogue of $S(n)$, and give an interesting mean value formula for it.

Keywords: Smarandache function; Additive Analogue; Mean Value formula.

§1. Introduction and results

For any positive integer n, let $S(n)$ denotes the Smarandache function, then $S(n)$ is defined the smallest $m \in N^+$, where $n|m!$. In paper [2], Jozsef Sandor defined the following analogue of Smarandache function:

$$S_1(x) = \min\{m \in N : x \leq m!\}, \quad x \in (1, \infty), \tag{1}$$

which is defined on a subset of real numbers. Clearly $S(x) = m$ if $x \in ((m-1)!, m!]$ for $m \geq 2$ (for $m = 1$ it is not defined, as $0! = 1! = 1!$), therefore this function is defined for $x > 1$.

About the arithmetical properties of $S(n)$, many people had studied it before (see reference [3]). But for the mean value problem of $S_1(n)$, it seems that no one have studied it before. The main purpose of this paper is to study the mean value properties of $S_1(n)$, and obtain an interesting mean value formula for it. That is, we shall prove the following:

Theorem. For any real number $x \geq 2$, we have the mean value formula

$$\sum_{n \leq x} S_1(n) = \frac{x \ln x}{\ln \ln x} + O(x).$$

§2. Proof of the theorem

In this section, we shall complete the proof of the theorem. First we need following one simple Lemma. That is,

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Lemma. For any fixed positive integers m and n, if $(m-1)! < n \leq m!$, then we have

$$m = \frac{\ln n}{\ln\ln n} + O(1).$$

Proof. From $(m-1)! < n \leq m!$ and taking the logistic computation in the two sides of the inequality, we get

$$\sum_{i=1}^{m-1} \ln i < \ln n \leq \sum_{i=1}^{m} \ln i. \quad (2)$$

Using the Euler's summation formula, then

$$\sum_{i=1}^{m} \ln i = \int_1^m \ln t dt + \int_1^m (t-[t])(\ln t)' dt = m \ln m - m + O(\ln m) \quad (3)$$

and

$$\sum_{i=1}^{m-1} \ln i = \int_1^{m-1} \ln t dt + \int_1^{m-1} (t-[t])(\ln t)' dt = m \ln m - m + O(\ln m). \quad (4)$$

Combining (2), (3) and (4), we can easily deduce that

$$\ln n = m \ln m - m + O(\ln m). \quad (5)$$

So

$$m = \frac{\ln n}{\ln m - 1} + O(1). \quad (6)$$

Similarly, we continue taking the logistic computation in two sides of (5), then we also have

$$\ln m = \ln\ln n + O(\ln\ln m), \quad (7)$$

and

$$\ln\ln m = O(\ln\ln\ln n). \quad (8)$$

Hence,

$$m = \frac{\ln n}{\ln\ln n} + O(1).$$

This completes the proof of Lemma.

Now we use Lemma to complete the proof of Theorem. For any real number $x \geq 2$, by the definition of $s_1(n)$ and Lemma we have

$$\begin{aligned} \sum_{n\leq x} S_1(n) &= \sum_{\substack{n\leq x \\ (m-1)!<n\leq m!}} m \quad (9) \\ &= \sum_{n\leq x} \left(\frac{\ln n}{\ln\ln n} + O(1) \right) \\ &= \sum_{n\leq x} \frac{\ln n}{\ln\ln n} + O(x). \end{aligned}$$

By the Euler's summation formula, we deduce that

$$\begin{aligned}
&\sum_{n\leq x}\frac{\ln n}{\ln\ln n}\\
=\;& \int_2^x \frac{\ln t}{\ln\ln t}dt+\int_2^x (t-[t])\left(\frac{\ln t}{\ln\ln t}\right)' dt+\frac{\ln x}{\ln\ln x}(x-[x]) \quad (10)\\
=\;& \frac{x\ln x}{\ln\ln x}+O\left(\frac{x}{\ln\ln x}\right).
\end{aligned}$$

So, from (9) and (10) we have

$$\sum_{n\leq x} S_1(n)=\frac{x\ln x}{\ln\ln x}+O(x).$$

This completes the proof of Theorem.

References

[1] F.Smarandache, Only Problems, Not Solutions, Xiquan Publ.House, Chicago, 1993.

[2] Jozsef Sandor, On an additive analogue of the function S, Smaramche Notions Journal, **13** (2002), 266-270.

[3] Jozsef Sandor, On an generalization of the Smarandache function, Notes Numb. Th. Discr. Math. **5** (1999), 41-51.

[4] Tom M Apostol, Introduction to Analytic Number Theory, New York, Springer-Verlag, 1976.